

# Introduction To Machine Learning

David Sontag

New York University

Lecture 21, April 14, 2016

# Expectation maximization

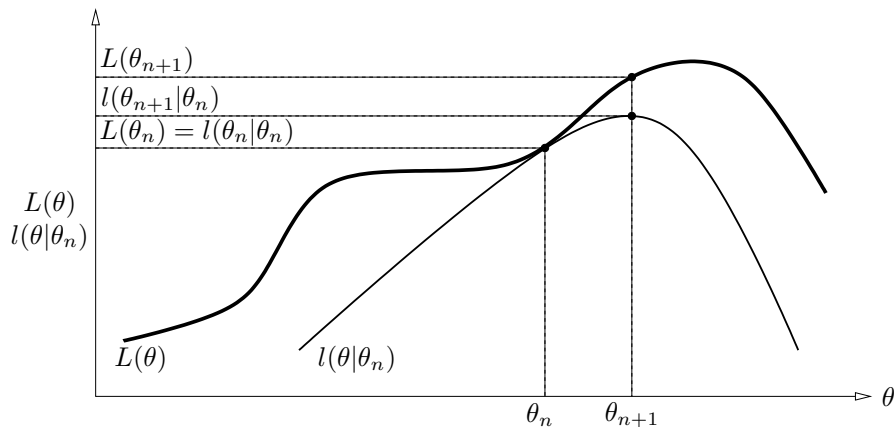
Algorithm is as follows:

- 1 Write down the **complete log-likelihood**  $\log p(\mathbf{x}, \mathbf{z}; \theta)$  in such a way that it is linear in  $\mathbf{z}$
- 2 Initialize  $\theta_0$ , e.g. at random or using a good first guess
- 3 Repeat until convergence:

$$\theta_{t+1} = \arg \max_{\theta} \sum_{m=1}^M E_{p(\mathbf{z}_m | \mathbf{x}_m; \theta_t)} [\log p(\mathbf{x}_m, \mathbf{Z}; \theta)]$$

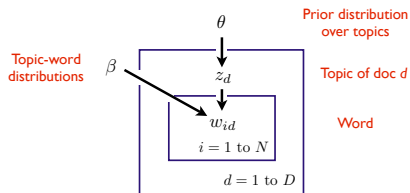
- Notice that  $\log p(\mathbf{x}_m, \mathbf{Z}; \theta)$  is a random function because  $\mathbf{Z}$  is unknown
- By linearity of expectation, objective decomposes into expectation terms and data terms
- “E” step corresponds to computing the objective (i.e., the **expectations**)
- “M” step corresponds to **maximizing** the objective

# Derivation of EM algorithm



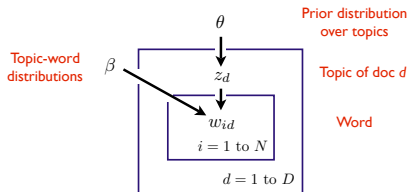
(Figure from tutorial by Sean Borman)

# Application to mixture models



- This model is a type of (discrete) **mixture model**
  - Called *multinomial* naive Bayes (a word can appear multiple times)
  - Document is generated from a single topic

# EM for mixture models



- The complete likelihood is  $p(\mathbf{w}, \mathbf{Z}; \theta, \beta) = \prod_{d=1}^D p(\mathbf{w}_d, Z_d; \theta, \beta)$ , where

$$p(\mathbf{w}_d, Z_d; \theta, \beta) = \theta_{Z_d} \prod_{i=1}^N \beta_{Z_d, w_{id}}$$

- Trick #1: re-write this as

$$p(\mathbf{w}_d, Z_d; \theta, \beta) = \prod_{k=1}^K \theta_k^{1[Z_d=k]} \prod_{i=1}^N \prod_{k=1}^K \beta_{k, w_{id}}^{1[Z_d=k]}$$

# EM for mixture models

- Thus, the complete log-likelihood is:

$$\log p(\mathbf{w}, \mathbf{z}; \theta, \beta) = \sum_{d=1}^D \left( \sum_{k=1}^K 1[Z_d = k] \log \theta_k + \sum_{i=1}^N \sum_{k=1}^K 1[Z_d = k] \log \beta_{k, w_{id}} \right)$$

- In the “E” step, we take the expectation of the complete log-likelihood with respect to  $p(\mathbf{z} | \mathbf{w}; \theta^t, \beta^t)$ , applying linearity of expectation, i.e.

$$E_{p(\mathbf{z} | \mathbf{w}; \theta^t, \beta^t)}[\log p(\mathbf{w}, \mathbf{z}; \theta, \beta)] =$$

$$\sum_{d=1}^D \left( \sum_{k=1}^K p(Z_d = k | \mathbf{w}; \theta^t, \beta^t) \log \theta_k + \sum_{i=1}^N \sum_{k=1}^K p(Z_d = k | \mathbf{w}; \theta^t, \beta^t) \log \beta_{k, w_{id}} \right)$$

- In the “M” step, we maximize this with respect to  $\theta$  and  $\beta$

# EM for mixture models

- Just as with complete data, this maximization can be done in closed form
- First, re-write expected complete log-likelihood from

$$\sum_{d=1}^D \left( \sum_{k=1}^K p(Z_d = k | \mathbf{w}; \theta^t, \beta^t) \log \theta_k + \sum_{i=1}^N \sum_{k=1}^K p(Z_d = k | \mathbf{w}; \theta^t, \beta^t) \log \beta_{k, w_{id}} \right)$$

to

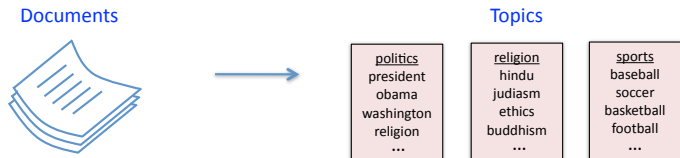
$$\sum_{k=1}^K \log \theta_k \sum_{d=1}^D p(Z_d = k | \mathbf{w}_d; \theta^t, \beta^t) + \sum_{k=1}^K \sum_{w=1}^W \log \beta_{k,w} \sum_{d=1}^D N_{dw} p(Z_d = k | \mathbf{w}_d; \theta^t, \beta^t)$$

- We then have that

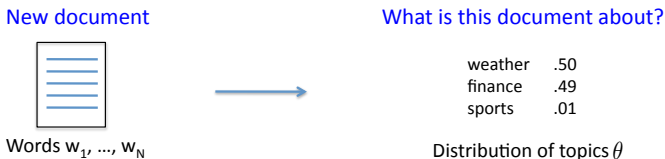
$$\theta_k^{t+1} = \frac{\sum_{d=1}^D p(Z_d = k | \mathbf{w}_d; \theta^t, \beta^t)}{\sum_{\hat{k}=1}^K \sum_{d=1}^D p(Z_d = \hat{k} | \mathbf{w}_d; \theta^t, \beta^t)}$$

# Latent Dirichlet allocation (LDA)

- **Topic models** are powerful tools for exploring large data sets and for making inferences about the content of documents



- Many applications in information retrieval, document summarization, and classification



- LDA is one of the simplest and most widely used topic models



# Generative model for a document in LDA

- 1 Sample the document's **topic distribution**  $\theta$  (aka topic vector)

$$\theta \sim \text{Dirichlet}(\alpha_1:T)$$

where the  $\{\alpha_t\}_{t=1}^T$  are fixed hyperparameters. Thus  $\theta$  is a distribution over  $T$  topics with mean  $\theta_t = \alpha_t / \sum_{t'} \alpha_{t'}$

- 2 For  $i = 1$  to  $N$ , sample the **topic**  $z_i$  of the  $i$ 'th word

$$z_i | \theta \sim \theta$$

- 3 ... and then sample the actual **word**  $w_i$  from the  $z_i$ 'th topic

$$w_i | z_i \sim \beta_{z_i}$$

where  $\{\beta_t\}_{t=1}^T$  are the *topics* (a fixed collection of distributions on words)

# Generative model for a document in LDA

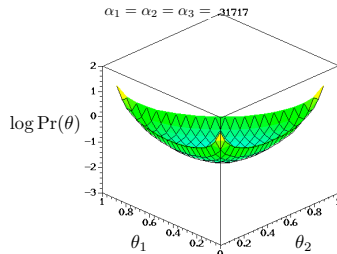
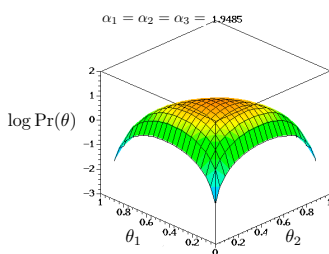
- 1 Sample the document's **topic distribution**  $\theta$  (aka topic vector)

$$\theta \sim \text{Dirichlet}(\alpha_{1:T})$$

where the  $\{\alpha_t\}_{t=1}^T$  are hyperparameters. The Dirichlet density, defined over  $\Delta = \{\vec{\theta} \in \mathbb{R}^T : \forall t \theta_t \geq 0, \sum_{t=1}^T \theta_t = 1\}$ , is:

$$p(\theta_1, \dots, \theta_T) \propto \prod_{t=1}^T \theta_t^{\alpha_t - 1}$$

For example, for  $T=3$  ( $\theta_3 = 1 - \theta_1 - \theta_2$ ):

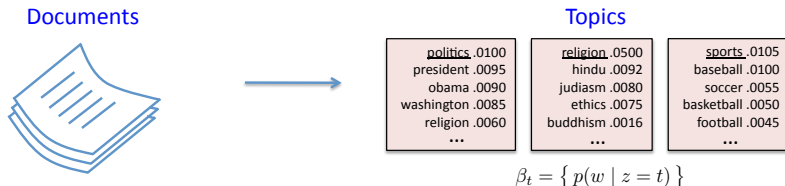


# Generative model for a document in LDA

- 3 ... and then sample the actual **word**  $w_i$  from the  $z_i$ 'th topic

$$w_i | z_i \sim \beta_{z_i}$$

where  $\{\beta_t\}_{t=1}^T$  are the *topics* (a fixed collection of distributions on words)



# Example of using LDA

 $\beta_1$ 

| Topic   | Proportion |
|---------|------------|
| gene    | 0.04       |
| dna     | 0.02       |
| genetic | 0.01       |
| ...     |            |

|          |      |
|----------|------|
| life     | 0.02 |
| evolve   | 0.01 |
| organism | 0.01 |
| ...      |      |

|        |      |
|--------|------|
| brain  | 0.04 |
| neuron | 0.02 |
| nerve  | 0.01 |
| ...    |      |

 $\beta_T$ 

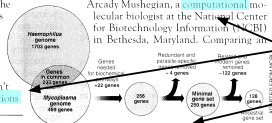
|          |      |
|----------|------|
| data     | 0.02 |
| number   | 0.02 |
| computer | 0.01 |
| ...      |      |

Documents

## Seeking Life's Bare (Genetic) Necessities

COLD SPRING HARBOR, NEW YORK—How many **genes** does an **organism** need to **survive**? Last week at the genome meeting here,\* two genome researchers with radically different approaches presented complementary views of the basic genes needed for **life**. One research team, using **computer** analyses to compare known **genomes**, concluded that today's **organisms** can be sustained with just 250 genes, and that the earliest life forms required a mere 128 **genes**. The other researcher mapped genes in a simple parasite and estimated that for this organism, 800 genes are plenty to do the job—but that anything short of 100 wouldn't be enough.

Although the numbers don't match precisely, those **predictions** are not all that far apart," especially in comparison to the 75,000 **genes** in the human genome, notes Siv Andersson at Uppsala University in Sweden, who arrived at the 800 number. But coming up with a consensus answer may be more than just a **genetic** numbers game, particularly as more and more **genomes** are completely mapped and sequenced. "It may be a way of organizing any newly **sequenced genome**," explains Arcady Mushegian, a **computational** molecular biologist at the National Center for Biotechnology Information (NCBI) in Bethesda, Maryland. Comparing an



Stripping down. Computer analysis yields an estimate of the minimum modern and ancient genomes.

\* Genome Mapping and Sequencing, Cold Spring Harbor, New York, May 8 to 12.

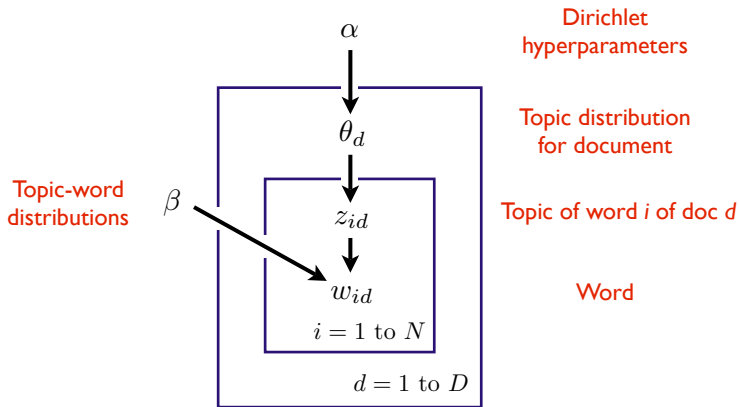
SCIENCE • VOL. 272 • 24 MAY 1996

Topic proportions and assignments

 $z_{1d}$ 
 $z_{2d}$ 
 $z_{3d}$ 
 $z_{4d}$ 
 $z_{5d}$ 
 $z_{6d}$ 
 $z_{7d}$ 
 $z_{8d}$ 
 $z_{Nd}$ 
 $\theta_d$ 

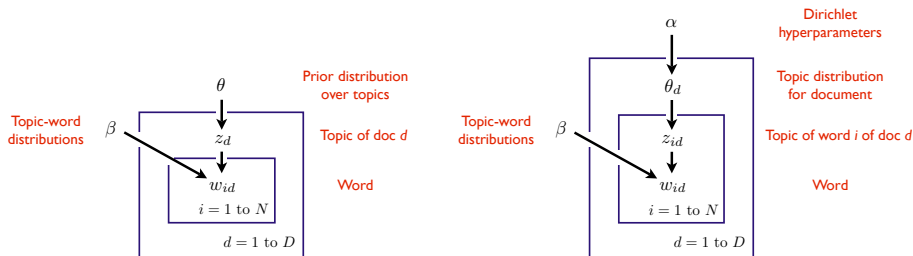
(Blei, *Introduction to Probabilistic Topic Models*, 2011)

# “Plate” notation for LDA model



Variables within a plate are replicated in a conditionally independent manner

# Comparison of mixture and admixture models



- Model on left is a **mixture model**
  - Called *multinomial* naive Bayes (a word can appear multiple times)
  - Document is generated from a single topic
- Model on right (LDA) is an **admixture model**
  - Document is generated from a distribution over topics