Today:

Testing Δ-freeness

An application of the SRL:

Property testing of a graph: is it triangle-free?

Given: graph G, adjacency matrix format

Desired Behavior: if G is Δ-free, output PASS

if G is $\underbrace{\varepsilon\text{-far}}$ from Δ-free then $\Pr[\text{output FAIL}] > 1/2$

must delete $\geq \varepsilon n^2$ edges to make it Δ-free

1-sided error

How much time does this require?

trivial $O(n^3)$, $\underbrace{O(n^\omega)}_{\text{matmult}}$, ..., $O(1)$?

Algorithm

← constant time!

Do $O(\delta^{-1})$ times

Pick v_1, v_2, v_3

if Δ reject & halt

Accept

Thm $\forall \varepsilon, \exists \delta$ ← fctn of ε only s.t. $\forall G$ s.t. $|V| = n$

$\forall$ s.t. G is ε-far from Δ-free, then G has $\geq \delta \binom{n}{3}$ distinct Δ's.

← note, this thm is specific to notion of ε-far from Δ-free defined above ("Adjacency matrix model")

Corollary Algorithm has desired behavior

Pf of Corr (Given Thm)

if Δ-free, accepts ✓

if ε-far,

$\geq \delta \binom{n}{3}$ Δ's

each loop passes with prob $\leq 1-\delta$

$\Pr[\text{don't find } \Delta] \leq (1-\delta)^{c/\delta}$

$\leq e^{-c} < 1/3$ ∎

} ← for proper choice of const c in "O" notation

Proof of Thm

Use regularity to get equipartition $\{V_1 \cdots V_k\}$

s.t. $\frac{5}{\varepsilon} \leq k \leq T(5\varepsilon^{-1}, \varepsilon')$

equivalently: $\frac{\varepsilon n}{5} \geq \frac{n}{k} \geq \frac{n}{T(5\varepsilon^{-1}, \varepsilon')}$ (# nodes per partition: $\frac{n}{k}$)

(do this by starting with arbitrary equipartition into $5/\varepsilon$ sets as $\mathcal{A}$)

for $\varepsilon' = \min\{\frac{\varepsilon}{5}, \gamma^{\Delta}(\frac{\varepsilon}{5})\}$

s.t. $\leq \varepsilon' \binom{k}{2}$ pairs not ε'-regular

Need: # of partitions fairly large s.t. # edges inside a partition not too big

Assume n/k is integer

$G' \equiv$ take G and

1) delete edges of G internal to any V_i

how many?

$$\le \underbrace{\frac{n}{k}}_{\text{deg w/in } V_i} \cdot \underbrace{n}_{\text{sum over all nodes}} \underbrace{\le}_{\text{choice of } k} \frac{\varepsilon n^2}{5}$$

2) delete edges between ε'-nonregular pairs

how many?

$$\le \underbrace{\varepsilon' \binom{k}{2}}_{\#\text{ non-regular pairs}} \underbrace{\left(\frac{n}{k}\right)^2}_{\text{max \# edges per pair}} \le \frac{\varepsilon}{5} \cdot \frac{k^2}{2} \cdot \frac{n^2}{k^2} \le \frac{\varepsilon}{10} \cdot n^2$$

max # edges per pair - here we use equipartition ⇒ max size of V_i is $\approx \frac{n}{k}(+1)$?

3) delete edges between $\underbrace{\text{low density}}_{< \frac{\varepsilon}{5}}$ pairs

how many?

$$\le \sum_{\text{low density}} \frac{\varepsilon}{5} \left(\frac{n}{k}\right)^2 \qquad \text{note } \sum \left(\frac{n}{k}\right)^2 \le \binom{n}{2}$$

$$\le \frac{\varepsilon}{5} \binom{n}{2} \approx \frac{\varepsilon n^2}{10}$$

Total deleted edges from G: $< \varepsilon n^2$

But, G was ε-far from Δ-free

so G' must still have a Δ !!!

(the Δ must be 1) in 3 distinct $V_i\ V_j\ V_k$ since removed inter partition edges

2) regular pairs - since removed non-regular pairs

3) high density pairs - since removed low density pairs

by construction of G')

$\therefore \exists\ i, j, k$ distinct s.t. $x \in V_i,\ y \in V_j,\ z \in V_k$

$V_i\ V_j\ V_k$ all $\geq \frac{\varepsilon}{5}^{\eta}$ density pairs $\Longleftarrow$ not just a Δ, but a high density Δ!!

+ $\geq \underbrace{\gamma^{\Delta}\left(\frac{\varepsilon}{5}\right)}_{\geq \frac{\eta}{2} \geq \frac{\varepsilon}{10}}$-regular

Δ-counting Lemma $\Rightarrow$

$\geq \delta^{\Delta}\left(\frac{\varepsilon}{5}\right)|V_i|\cdot|V_j|\cdot|V_k|$ triangles in G'

where $\delta^{\Delta} = (1-\eta)\frac{\eta^3}{8}$

$\geq \frac{1}{8}\cdot\frac{\varepsilon^3}{8000} = \frac{\varepsilon^3}{16000}$

$\geq \dfrac{\delta^{\Delta}\left(\frac{\varepsilon}{5}\right)n^3}{\left(T(5\varepsilon^{-1}, \varepsilon')\right)^3}$ Δ's

$\geq \delta' \binom{n}{3}$ Δ's in G' (and thus in G)

for $\delta' = 6\,\delta^{\Delta}\left(\frac{\varepsilon}{5}\right)\left(T\left(\frac{5}{\varepsilon}, \varepsilon'\right)\right)^{-3}$

$\blacksquare$

Extensions

- Komlos–Simonovits holds for all const sized subgraphs
- almost "as is" can use method to test

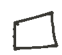
  all 1st-order graph properties

  $\exists u_1 u_2 u_3 \ldots u_k \ \forall v_1 \ldots v_\ell \ R(\underbrace{u_1 \ldots u_k v_1 \ldots v_\ell}_{\text{defined by } \vee, \wedge, \neg \text{ nbrs}})$

  i.e. $\forall u_1 u_2 u_3 \ R(\underbrace{u_1 u_2 u_3}_{\text{encodes}})$

  $\neg(u_1 \sim u_2,\ u_2 \sim u_3,\ u_1 \sim u_3)$

  H-freeness for const size H

  Induced  vs  forbidden

  not induced

- 1-sided const time ≈ hereditary graph props [Alon Shapira]

  closed under vertex removal (not necessarily edges)

  includes monotone graph props

  chordal

  perfect

  interval graph

  difficulty: infinite set of forbidden subgraphs also forbidden as induced.

- 2-sided const time ≈ regular partition is hardest testing problem

  property testable iff can reduce to testing [Alon Fisher Newman Shapira]

  if satisfies one of finitely many Szemerédi partitions.