Lecture 9

Szemerédi's Regularity Lemma

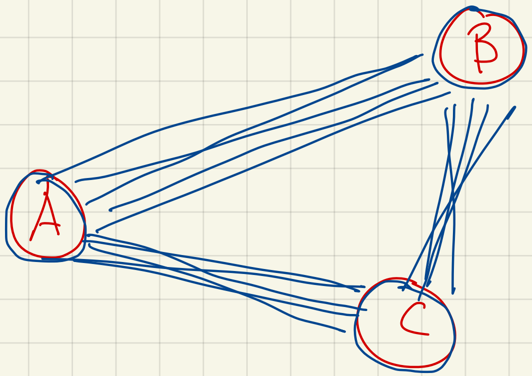
Testing dense graph properties via SRL:

Δ-freeness

Graphs with "random" properties:

Example question:

How many triangles in a random tripartite graph?



density η

$\forall\ u \in A,\ v \in B,\ w \in C$:

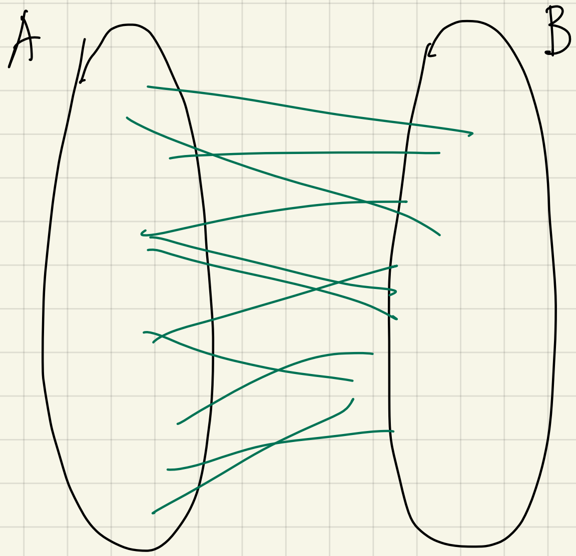
$$\Pr[u \sim v \sim w] = \eta^3 \qquad \sigma_{u,v,w} = \begin{cases} 1 & \text{if } u \sim v \sim w \\ 0 & \text{o.w.} \end{cases}$$

$$E[\sigma_{u,v,w}] = \eta^3$$

$$E[\#\text{ triangles}] = E\Big[\sum_{\substack{u \in A \\ v \in B \\ w \in C}} \sigma_{u,v,w}\Big] = \eta^3 \cdot |A| \cdot |B| \cdot |C|$$

Can we make weaker assumptions + still get reasonable bounds?

Density & Regularity of set pairs:



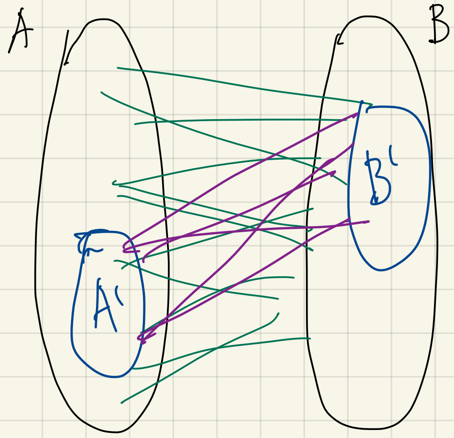
<u>def.</u> For $A, B \subseteq V$ s.t.

(1) $A \cap B = \emptyset$

(2) $|A|, |B| > 1$

Let $e(A,B)$ = # edges between $A + B$

& density $d(A,B) = \dfrac{e(A,B)}{|A| \cdot |B|}$



Say A, B is γ-regular if $\forall\, A' \subseteq A, B' \subseteq B$

s.t. $|A'| \geq \gamma |A|$

$|B'| \geq \gamma |B|$

$$|d(A', B') - d(A,B)| < \gamma$$

behaves like a "random graph"

Lemma

density

regularity parameter, depends only on η

$\forall \eta > 0 \quad \exists \gamma = \frac{1}{2}\eta \equiv \gamma^{\Delta}(\eta)$

$\delta = (1-\eta)\frac{\eta^3}{8} \geq \frac{\eta^3}{16} \equiv \delta^{\Delta}(\eta)$

if $\eta < 1/2$

triangles, depends only on η

$d(A,B) = \frac{e(A,B)}{|A|\cdot|B|}$

A, B is γ-regular if $\forall A' \subseteq A, B' \subseteq B$

s.t. $|A'| \geq \gamma|A|$

$|B'| \geq \gamma|B|$

$|d(A',B') - d(A,B)| < \gamma$

s.t. if A, B, C disjoint subsets of V s.t. each pair is γ-regular with density $> \eta$

then G contains $\geq \delta \cdot |A| \cdot |B| \cdot |C|$ distinct Δ's with node in each of A, B, C.

if A, B, C disjoint subsets of V s.t. each pair is γ-regular with density $> \eta$ then G contains $\geq \delta \cdot |A| \cdot |B| \cdot |C|$ distinct Δ's

$$d(A,B) = \frac{e(A,B)}{|A| \cdot |B|}$$

A, B is γ-regular if $\forall\, A' \subseteq A,\ B' \subseteq B$

s.t. $|A'| \geq \gamma |A|$

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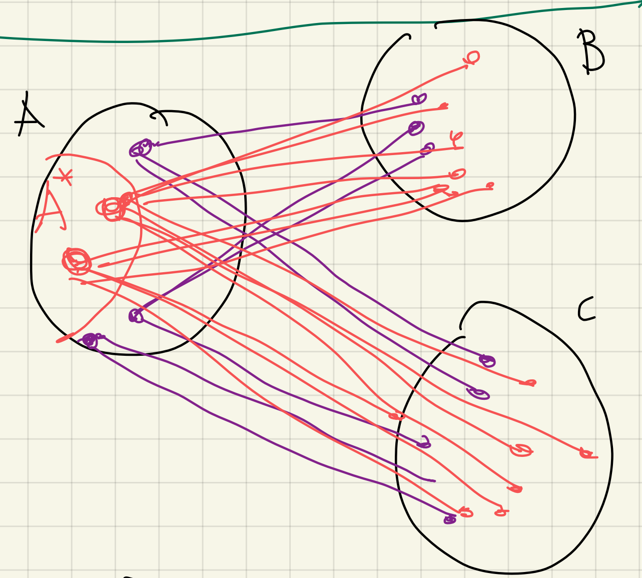
$$|d(A', B') - d(A,B)| < \gamma$$



<u>Proof</u>:

$A^* \leftarrow$ nodes in A with $\geq |\eta - \gamma| \cdot |B|$ nbrs in B

$\geq |\eta - \gamma| \cdot |C|$ nbrs in C

<u>Claim</u> $|A^*| \geq (1 - 2\gamma) |A|$

<u>why? (Pf of claim)</u>

$A' \leftarrow$ "bad" nodes w.r.t. B ($< |\eta - \gamma| \cdot |B|$ nbrs in B)

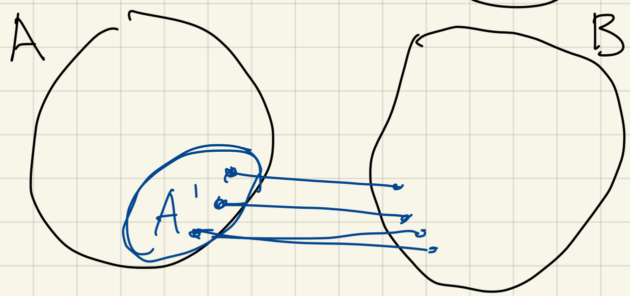
$A'' \leftarrow$ " " " C (" " " " C)

then $|A'| \leq \gamma |A|$ $(+\ |A''| \leq \gamma |A|)$

<u>why?</u> consider pair A', B.

def of A'

$$d(A', B) < \frac{|A'| \cdot |\eta - \gamma| \cdot |B|}{|A'| \cdot |B|} = \eta - \gamma$$



but $d(A,B) > \eta$

so $|d(A', B) - d(A,B)| > \gamma$

$+\ |B| \geq \gamma |B|$

so if $|A'| \geq \gamma |A|$ then (A,B) is <u>not</u> γ-regular

Let $A^* = A \setminus (A' \cup A'')$ then $|A^*| \geq |A| - |A'| - |A''| \geq |A| - 2\gamma|A| = (1-2\gamma)\cdot|A|$ ▢

if A, B, C disjoint subsets of V st. each pair is γ-regular with density $> \eta$

then G contains $\geq \delta \cdot |A| \cdot |B| \cdot |C|$ distinct Δ's

$$d(A,B) = \frac{e(A,B)}{|A| \cdot |B|}$$

A, B is γ-regular if $\forall A' \subseteq A, B' \subseteq B$

s.t. $|A'| \geq \gamma |A|$

$|B'| \geq \gamma |B|$

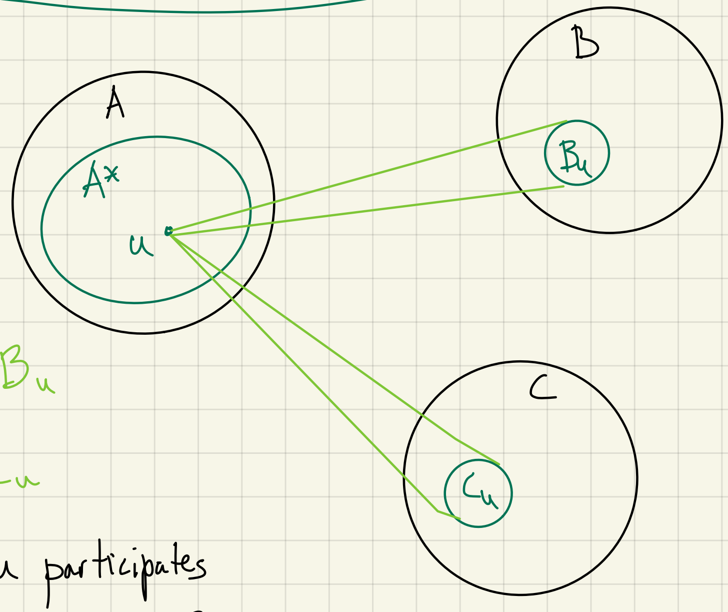
$$|d(A', B') - d(A,B)| < \gamma$$

Proof:

$A^* \leftarrow$ nodes in A with $\geq |\eta - \gamma| \cdot |B|$ nbrs in B

$\geq |\eta - \gamma| \cdot |C|$ nbrs in C

Claim $|A^*| \geq (1 - 2\gamma)|A|$



For each $u \in A^*$: define $B_u \equiv$ nbrs of u in B

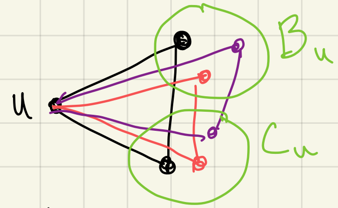
$C_u \equiv$ nbrs of u in C

both pretty big by def of A^*

Find lots of distinct Δ's

since $\gamma < \frac{\eta}{2}$, $(\eta - \gamma > \gamma)$ $|B_u| \geq (\eta - \gamma)|B| \geq \gamma |B|$

$|C_u| \geq (\eta - \gamma)|C| \geq \gamma |C|$



#edges between $B_u + C_u \Rightarrow$ lower bnd on # distinct Δ's in which u participates

$$d(B,C) \geq \eta \Rightarrow d(B_u, C_u) \geq \eta - \gamma \Rightarrow e(B_u, C_u) \geq (\eta - \gamma)|B_u||C_u| \geq (\eta - \gamma)^3 \cdot |B||C|$$

$+ B_u, C_u$ big enough $+ (B,C)$ is γ regular

So total # Δ's $\geq (1 - 2\gamma)|A| \cdot (\eta - \gamma)^3 |B||C| \geq (1 - \eta)(\eta/2)^3 |A||B||C|$

choose $\gamma = \eta/2$

∎

Do interesting graphs have regularity properties?

Yes in some sense, all graphs do "can be approximated as small collection of random graphs"

Szemerédi's Regularity Lemma

would like it to say:

"one can equipartition nodes of V into $V_1 \dots V_k$ (for const k) s.t.

all pairs (V_i, V_j) are ε- regular"

only most $\leq \varepsilon \binom{k}{2}$ are not

Sometimes need $k > m$ for some m ($k=1$, $k=n$ trivial)

Szemerédi's Regularity Lemma: (especially useful version)

$\forall\ m,\ \varepsilon > 0 \quad \exists\ T = T(m, \varepsilon)$ s.t. given $G = (V, E)$ s.t. $|V| > T$

& $\mathcal{A}$ an equipartition of V into sets

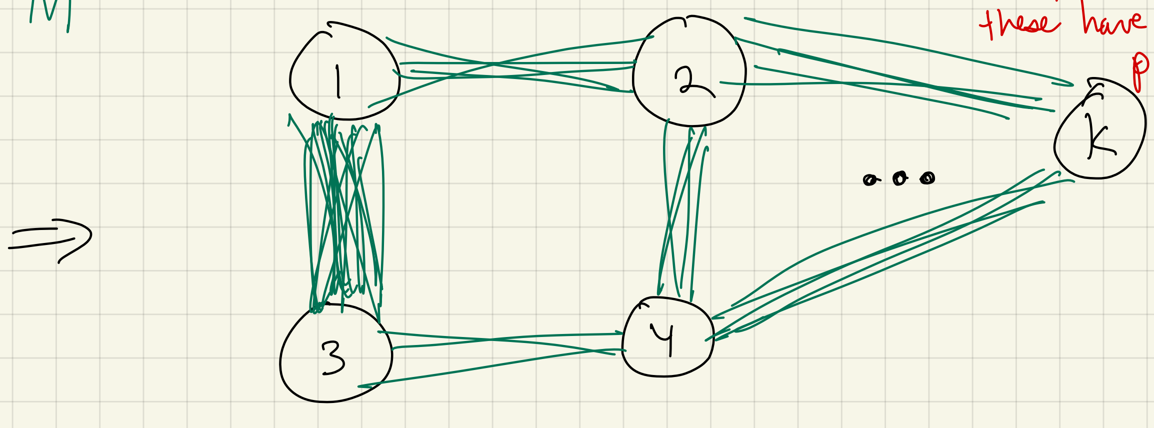
then exists equipartition $\mathcal{B}$ into k sets which refines $\mathcal{A}$

s.t. $m \le k \le T$

& $\le \varepsilon \binom{k}{2}$ set pairs $\underline{\text{not}}$ ε-regular

const # partitions s.t. each pair behaves like $\underbrace{\text{random graph}}_{\text{these have nice properties!}}$

Note: T does $\underline{\text{not}}$ depend on $|V|$



Why was SRL first studied?

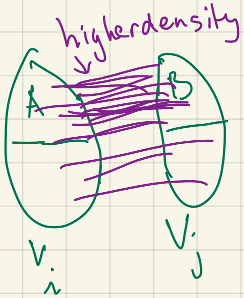
to prove conjecture of Erdös + Turán: Sequences of ints have long arithmetic progressions

Very rough idea of proof:

same densities

"expectation of $d^2(V_i, V_j)$" $\longrightarrow$ $\text{ind}(V_1 \ldots V_k) = \frac{1}{k^2} \sum_{i=1}^{k} \sum_{j=i+1}^{k} d^2(V_i, V_j) \leq \frac{1}{2}$ "variance of d"

note: $E[d(V_i, V_j)] = \frac{|E|}{|V|^2}$



issue: what if split V_i for many V_j?
$\Rightarrow$ split into exponential subsets

if a partition violates, can refine s.t. $\text{ind}(V_1', \ldots, V_{k'}')$ grows significantly (i.e. by $\approx \varepsilon^c$) so in less than $\frac{1}{\varepsilon^c}$ refinements, have good partition

note, if refine, Cauchy Schwartz $\Rightarrow$ ind can't decrease

How big is k? u.b. Tower of size $\frac{1}{\varepsilon^c}$

l.b. " " " $\frac{1}{\varepsilon^{c'}}$

$\left. 2^{2^{2^{2^{\cdot^{\cdot^{\cdot}}}}}} \right\} \frac{1}{\varepsilon^c}$