6.889 Sub-linear Time Algorithms

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Lecture 1

Topics:

- Overview
- Diameter of point set
- #Connected Components
- MST

Overview

- Big data – how do you access it?
  too much to view it all?
  can change by the time you make a decision?

  e.g. what is "diameter" of facebook graph?

- Compromise:

  "exactly", "for all", "there exists" statements
  $\Rightarrow$ "approximately", "there is a large group"...

- Models:
  1) Random access queries – can access word of input in step
     - adj matrix vs. adj list
     - locn i
  2) Samples – get sample of distribution/input in step

Course requirements:

- Scribing 25%
  must be in latex
- Psets 35%
- Project 25%
- Class participation 15%
  (includes grading)

Projects:

- groups of 1-3
- Possible ideas:
  - solve new problem (or try ideas + explain why they fail)
  - read + survey some papers
  - implement an algorithm (or two or three)

A first Example: Diameter of a point set

note input size n is m^2

<u>Input</u> m points described by distance matrix D

s.t. D_{ij} is distance from i, j

+ D is: 1) symmetric

2) satisfies $\Delta \neq$

ie. $D_{ij} \leq D_{ik} + D_{kj}$ $\forall\, i,j,k$

<u>Output</u>

let i, j be s.t. D_{ij} is max $\Leftarrow$ D_{ij} is "diameter"

Output k, l s.t. $D_{kl} \geq \frac{D_{ij}}{2}$ $\Leftarrow$ 2-approximation

<u>Algorithm</u>

- Pick k arbitrarily
- Pick l to maximize D_{kl}
- Output k, l

runtime: $O(m) = O(\sqrt{n})$

Why does it work?

$$
\begin{aligned}
D_{ij} &\le D_{ik} + D_{k\ell} && \Delta\neq \\
&= D_{ki} + D_{k\ell} && \text{symmetry} \\
&\le D_{k\ell} + D_{k\ell} && \text{choice of } \ell \\
&\le 2 D_{k\ell}
\end{aligned}
$$

How many connected components in G?

<u>Input</u> $G = (V, E)$, ε — adjacency list representation

max degree d

$|V| = n$

$|E| = m \le d \cdot n$

<u>Output</u> y s.t. if $C = \#$ conn comp

then $C - \varepsilon \cdot n \le y \le C + \varepsilon n$ ← "additive approx to w/in εn"

A different characterization of # conn components:

<u>notation</u>: $\forall v$ let $n_v \equiv \#$ nodes in v's conn comp.

<u>observation</u>: $\forall$ connected component $A \subseteq V$

$$\sum_{u \in A} \frac{1}{n_u} = \sum_{u \in A} \frac{1}{|A|} = 1$$

So new characterization of # conn comp:

$$C = \sum_{u \in V} \frac{1}{n_u}$$

Why is this better?

Computing $\frac{1}{n_u}$ needs $O(n)$ time?

sum $O(n)$ terms? ⇐ will estimate!

Estimating $c = \sum_{u \in V} \frac{1}{n_u}$:

Two ideas:

1) estimate $\frac{1}{n_u}$ quickly

2) estimate $\sum_u \frac{1}{n_u}$ via sampling bounds

Estimating $\frac{1}{n_u}$:

<u>def.</u> $\hat{n}_u \equiv \min\{n_u, 2/\varepsilon\}$

$$\hat{c} \equiv \sum_{u \in V} \frac{1}{\hat{n}_u}$$

<u>Lemma</u> $\forall u \quad \left| \frac{1}{n_u} - \frac{1}{\hat{n}_u} \right| \leq \varepsilon/2$

<u>Corr</u> $|c - \hat{c}| \leq \frac{\varepsilon n}{2}$ ← so if can compute $\hat{c}$ faster, it is useful!

<u>Pf of lemma</u>

if $n_u \leq 2/\varepsilon$ then $\hat{n}_u = n_u$ so $\left|\frac{1}{n_u} - \frac{1}{\hat{n}_u}\right| = 0$

else $n_u > 2/\varepsilon$ so $\hat{n}_u = 2/\varepsilon < n_u$

$$\Rightarrow \quad 0 \leq \frac{1}{n_u} \leq \frac{1}{\hat{n}_u} = \frac{\varepsilon}{2}$$

since $n_u > 0$

$$\Rightarrow \quad \left|\frac{1}{\hat{n}_u} - \frac{1}{n_u}\right| \leq \varepsilon/2 \quad \blacksquare$$

How long to compute $\hat{n}_u$?

<u>Algorithm</u> compute $\hat{n}_u$

Do BFS starting from u until
- visit whole component of u
- or visit $2/\varepsilon$ distinct nodes

Output # visited nodes

<u>runtime</u>

$O(d \cdot 1/\varepsilon)$

↑ time per step of BFS

How to estimate $\sum_u \frac{1}{\hat{n}_u}$?

<u>Algorithm</u> estimate $\hat{c}$

$r \leftarrow b/\varepsilon^3$

Choose $U = \{u_1, \dots u_r\}$ random nodes

$\forall\ u \in U$

Compute $\hat{n}_u$ via above algorithm

Output $\tilde{c} = \frac{n}{r} \sum_{u \in U} \frac{1}{\hat{n}_u}$

<u>runtime</u> $O((d/\varepsilon) \cdot 1/\varepsilon^3) = O(d/\varepsilon^4)$

Why is it good?

Thm $\Pr\left[|\tilde{c} - \hat{c}| \le \varepsilon n/2 \right] \ge 3/4$

Pf

$$\boxed{\begin{array}{l} \text{Chernoff Bnd: } X_1 \cdots X_r \text{ iid} \quad X_i \in [0,1] \\ S = \sum_{i=1}^{r} X_i \qquad p = E[X_i] = E[S]/r \\ \text{Then: } \Pr\left[\left| \frac{S}{r} - p \right| \ge \delta p \right] \le e^{-\Omega(rp\delta^2)} \end{array}}$$

here $X_i = \dfrac{1}{\hat{n}_{u_i}}$

$$p = E\left[\frac{1}{\hat{n}_{u_i}}\right] = \frac{1}{n} \cdot \sum_{u \in V} \frac{1}{\hat{n}_{u_i}} = \frac{\hat{c}}{n}$$

$$\delta = \frac{\varepsilon}{2}$$

$$\frac{S}{r} = \underbrace{\frac{1}{r} \sum_{i=1}^{r} \frac{1}{\hat{n}_{u_i}}}_{\frac{\tilde{c}}{n}}$$

so $\Pr\left[\left| \frac{\tilde{c}}{n} - \frac{\hat{c}}{n} \right| \ge \frac{\varepsilon}{2} \cdot \frac{\hat{c}}{n} \right] = \Pr\left[|\tilde{c} - \hat{c}| \ge \frac{\varepsilon}{2} \hat{c} \right]$

$$\le e^{-\left(\frac{b}{\varepsilon^{\cancel{3}}} \cdot \frac{\hat{c}}{n} \cdot \frac{\cancel{\varepsilon^2}}{4} \right)}$$

want this to be ≥ 2 so that probability $\le e^{-2} \le \frac{1}{4}$

since $\frac{\varepsilon}{2} \le \frac{1}{\hat{n}_u} \le 1$

summing over u: $\frac{\varepsilon n}{2} \le \hat{c} \le n$

so $\frac{\varepsilon}{2} \le \frac{\hat{c}}{n} \le 1$

$$\le e^{-\left(\frac{b}{\cancel{\varepsilon}} \cdot \frac{\cancel{\varepsilon}}{2} \cdot \frac{1}{4} \right)}$$

pick $b \ge 16$

Now we are done:

<u>Corr.</u> $\Pr[|c-\tilde{c}| \le \varepsilon n] \ge 3/4$

<u>Pf.</u> $|c-\tilde{c}| \le |c-\hat{c}| + |\hat{c}-\tilde{c}|$ by $\Delta \neq$

$|c-\hat{c}|$: always $\le \frac{\varepsilon n}{2}$ by corr

$|\hat{c}-\tilde{c}|$: $\le \frac{\varepsilon n}{2}$ by thm with prob $\ge 3/4$

Approximating Min Spanning Tree (MST)

<u>Input</u> $G=(V,E)$ adj list representation $n=|V|$

max degree d

each edge has weight $w_{uv} \in \{1..W\} \cup \{\infty\}$

ie. $w_{uv} \notin E$

G connected

ε

<u>Output</u>

let $M = \min\limits_{T \text{ spans } G} \{ w(T) \}$

tree

tree touches every node

$\sum\limits_{(i,j)\in T} w_{ij}$

output $\hat{M}$ s.t. $(1-\varepsilon)M \le \hat{M} \le (1+\varepsilon)M$

Assumption on wts $\Rightarrow$ $n-1 \le w(T) \le W(n-1)$

A different characterization of MST:

<u>def</u> $G^{(i)} = (V, E^{(i)})$ where $E^{(i)} = \{(u,v) \mid w_{uv} \in \{1..i\}\}$

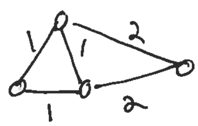
$c^{(i)} = \#$ conn comp of $G^{(i)}$

Some examples before characterization:

1) $w=1$ only size 1 weights + connected by assumption

here $M = n-1$

2) $w=2$ weights $\in \{1,2\}$



G

$G^{(1)}$

$C^{(1)} = 2$

idea of Kruskal:

use as many wt 1 edges as you can

only need wt 2 edges to connect the components

$\Rightarrow$ need $C^{(1)} - 1$ wt 2 edges in MST

(recall that total # of edges is $n-1$)

Total wt of MST:

$$M = (n-1) + (C^{(1)} - 1) = n - 2 + C^{(1)}$$

↑ 1 for each edge

↑ additional 1 for wt 2 edges

Claim $M = n - w + \sum_{i=1}^{w-1} C^{(i)}$

Pf.

let α_i = # edges of wt i in any MST of G

Kruskal's tells us that all MST's have same value of α_i

why?

$$\sum_{i \geq l} \alpha_i = \# \text{ conn comp of } G^{(l)} - 1$$

$$= C^{(l)} - 1$$

where $C^{(0)} = n$ (no edges in $G^{(0)}$)

$$M = \sum_{i=1}^{w} i \cdot \alpha_i$$

$$= \sum_{i=1}^{w} \alpha_i + \sum_{i=2}^{w} \alpha_i + \sum_{i=3}^{w} \alpha_i + \ldots + \underbrace{\sum_{i=w}^{w} \alpha_i}_{= \alpha_w}$$

$$= (n-1) + (C^{(1)} - 1) + (C^{(2)} - 1) + \ldots + (C^{(w-1)} - 1)$$

$$= n - w + \sum_{i=1}^{w-1} C^{(i)}$$

$\blacksquare$

MST 4
2017

Approximation Algorithm:

For $i = 1$ to $w-1$

$\hat{c}^{(i)}$ = approx # c.c. of $G^{(i)}$ to within $\underbrace{\frac{\varepsilon}{2w}}_{\varepsilon'} \cdot n$ (additive error)

adds poly(log $\frac{w}{\varepsilon}$) factor to runtime

Output $\hat{M} = n - w + \sum_{i=1}^{w-1} \hat{c}^{(i)}$

Runtime:

$\tilde{O}(d/(\varepsilon')^4) = \tilde{O}(d w^4/\varepsilon^4)$ for each call to approx # c.c.

how do you recompute $G^{(i)}$? ignore edges of wt $> i$

Total $\tilde{O}(d w^5/\varepsilon^4)$

(Can improve to $O(\frac{dw}{\varepsilon^2} \log \frac{dw}{\varepsilon})$

+ need $\Omega(dw/\varepsilon^2)$)

Approximation guarantee:

"error" if approx error of approx #cc is too big ($> \varepsilon'$)

how? HW0!

Call approx # cc with error probability $\leq \frac{1}{4 \cdot w}$

Pr [all calls to approx # cc give output that is ε' additive approx] $\geq 1 - \frac{w}{4w}$ ← union bound

$= 3/4$

If $\uparrow$ happens: $|M - \hat{M}| \leq w \cdot \frac{\varepsilon n}{2w} = \frac{\varepsilon n}{2}$ ← small additive error

this is where we use lower bnd on edge wts → since $M \geq n-1 \geq n/2$, $|M - \hat{M}| \leq \varepsilon M$ ← small multiplicative error

$\boxed{\text{Conclusion:}}$ runtime depends only on $d, w, 1/\varepsilon$
gives additive / multiplicative error