

## Problems for Recitation 23

**Theorem 1.** Let  $E_1, \dots, E_n$  be events, and let  $X$  be the number of these events that occur. Then:

$$\text{Ex}(X) = \text{Pr}(E_1) + \text{Pr}(E_2) + \dots + \text{Pr}(E_n)$$

**Theorem 2 (Markov's Inequality).** Let  $X$  be a nonnegative random variable. If  $c > 0$ , then:

$$\text{Pr}(X \geq c) \leq \frac{\text{Ex}(X)}{c}$$

**Theorem 3 (Chebyshev's Inequality).** For all  $x > 0$  and any random variable  $R$ ,

$$\text{Pr}(|R - \text{Ex}(R)| \geq x) \leq \frac{\text{Var}[R]}{x^2}$$

**Theorem 4 (Union Bound).** For events  $E_1, \dots, E_n$ :

$$\text{Pr}(E_1 \cup \dots \cup E_n) \leq \text{Pr}(E_1) + \dots + \text{Pr}(E_n)$$

**Theorem 5 ("Murphy's Law").** If events  $E_1, \dots, E_n$  are mutually independent and  $X$  is the number of these events that occur, then:

$$\text{Pr}(E_1 \cup \dots \cup E_n) \geq 1 - e^{-\text{Ex}(X)}$$

**Theorem 6 (Chernoff Bounds).** Let  $E_1, \dots, E_n$  be a collection of mutually independent events, and let  $X$  be the number of these events that occur. Then:

$$\text{Pr}(X \geq c \text{Ex}(X)) \leq e^{-(c \ln c - c + 1) \text{Ex}(X)} \quad \text{when } c \geq 1$$

**Problem 1.** Sometimes I forget a few items when I leave the house in the morning.

(a) For example, here are probabilities that I forget various pieces of footwear:

left sock	0.2
right sock	0.1
left shoe	0.1
right shoe	0.3

Let  $X$  be the number of these that I forget. What is  $E_X(X)$ ?

(b) Upper bound the probability that I forget one or more items. Make no independence assumptions.

(c) Use the Markov Inequality to upper bound the probability that I forget 3 or more items.

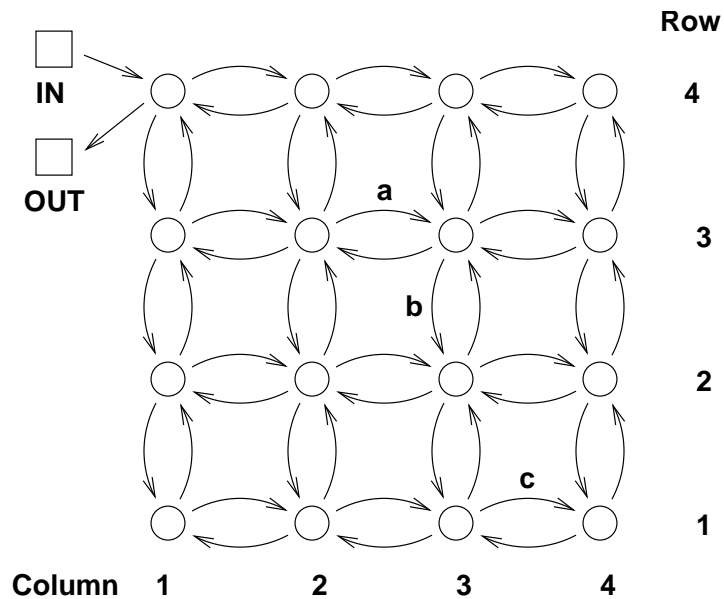
(d) Now suppose that I forget each item of footwear independently. Use Chebyshev's Inequality to upper bound the probability that I forget two or more items.

(e) Use Theorem 5 to lower bound the probability that I forget one or more items.

(f) I'm supposed to remember many other items, of course: clothing, watch, backpack, notebook, pencil, kleenex, ID, keys, etc. Let  $X$  be the total number of items I remember. Suppose I remember items mutually independently and  $E_X(X) = 36$ . Use Chernoff's Bound to give an upper bound on the probability that I remember 48 or more items.

(g) Give an upper bound on the probability that I remember 108 or more items.

**Problem 2.** A routing network called an  $n \times n$  array is shown below for  $n = 4$ . There is an input terminal and an output terminal attached to every node. But for clarity only one such pair of terminals is shown.



A packet travels between two nodes by first moving horizontally to the correct column and then vertically to the correct row. Suppose that each input sends one packet to an output selected uniformly and independently at random. (So zero, one, two, or more packets can be sent to a single output.) The goal of this problem is analyze congestion in an array using probability tools.

- (a) What is the expected number of packets that cross edge  $a$  in the  $4 \times 4$  array shown above? Also, compute the expected number of packets that cross edges  $b$  and  $c$ .

(b) Now consider an  $n \times n$  array. Number the rows from 1 to  $n$  and number the columns from 1 to  $n$ . What is the expected number of packets that cross an upward edge  $e$  from row  $k$  to row  $k + 1$ ?

(c) What is the expected number of packets that cross a rightward edge  $f$  from column  $k$  to column  $k + 1$ ?

(d) Let  $\mu$  be the expected number of packets crossing one of the edges with the greatest expected congestion. What is  $\mu$ ? (Assume  $n$  is even.)

(e) Let  $X$  be the number of packets that cross a particular edge. We know that  $\text{Ex}(X) \leq \mu$ . But if we're unlucky and something weird happens, then  $X$  might be much greater, which means that the edge is especially congested. Give an upper bound on:

$$\Pr\left(X \geq \mu + \sqrt{2n \ln n}\right)$$

Assume that  $n$  is large enough that the second Chernoff inequality applies. (Note that  $\sqrt{2n \ln n}$  is quite small compared to  $\mu$  for large  $n$ ; so we're trying to show that even small deviations above the average congestion are unlikely.)

(f) We've now shown that *one* particular edge is not likely to be congested. However, an  $n \times n$  array contains a *lot* of edges. So there are a lot of different places where something could go wrong. Compute the number of edges in an  $n \times n$  array and then use the Union Bound to upper bound the probability that  $\mu + \sqrt{2n \ln n}$  or more packets cross some edge.