Today:

Undirected S-T Connectivity revisited

(deterministic logspace)

Given: undir G
nodes s,t

Question: are s,t in same component?

an easy case:

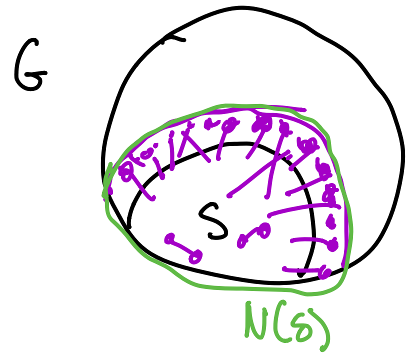
def. (N, D, λ)-graph

nodes, degree, upper bnd on λ_2 of transition matrix

a well known fact: Tanner, Alon-Millman

$\forall \lambda < 1$, $\exists \varepsilon > 0$ s.t. $\forall$ (N, D, λ)-graph
$+ \forall S$ s.t. $|S| < \frac{N}{2}$ $\quad |N(S)| \geq (1+\varepsilon)|S|$

in cludes S



$|N(S) \setminus S| \geq \varepsilon \cdot |S|$

why?

$\lambda_2 < 1 \implies G$ has $O(\log n)$ diameter

Idea for "low diameter" ($O(\log n)$) + const degree
(each component is low diameter)

starting at s:
- enumate all paths of length $O(\log n) = \ell$

  $\#\text{paths} = D^{\ell} = D^{O(\log n)} = N^{O(1)}$

  since $D = O(1)$
- if ever see t, output "connected"

  o.w. "disconnected"

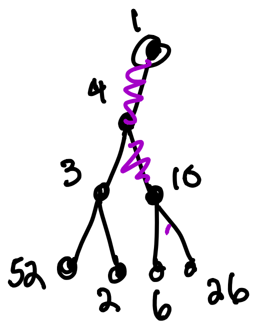
Correct? ✓

Space: Keep track of DFS

const # bits for each step

$O(\log n)$ length

Total $O(\log^2 n)$



$s = 1 \to 4 \to 3 \to 52$
$\leftarrow 3 \to 2$
$\leftarrow 3 \leftarrow 4 \to 10 \to 6$
$\leftarrow 10 \to 26$
$\leftarrow 10 \leftarrow 4 \leftarrow 1$

$O((\log n)^2)$ solution: keep track of nodes on DFS stack

$O(\log n)$ solution; keep track of <u>choices</u> on DFS stack. Can find parents by starting at root & following choices

$S = 1$

L L L
L L R
L R L

L R L
$1 \to 4 \to 10 \to 6$

$\log n$ for start $+ (\log n) \cdot O(1)$ to find previous

Problem:

not all graphs
- (N, D, λ) for $\lambda < 1$
- $O(\log N)$ diam
- Const deg

For general graphs:

Thm $\forall$ connected, non-bipartite $\lambda(G) \leq 1 - \frac{1}{DN^2}$

not too good

What about powering?

G is $(N, D, \lambda) \rightarrow G^t$ is (N, D^t, λ^t)

good/bad?

+ same soln
+ reduce λ_2
- increased Degree

will power but will add operation
which reduces degree
w/o increasing λ_2 by too much

"Base graph"

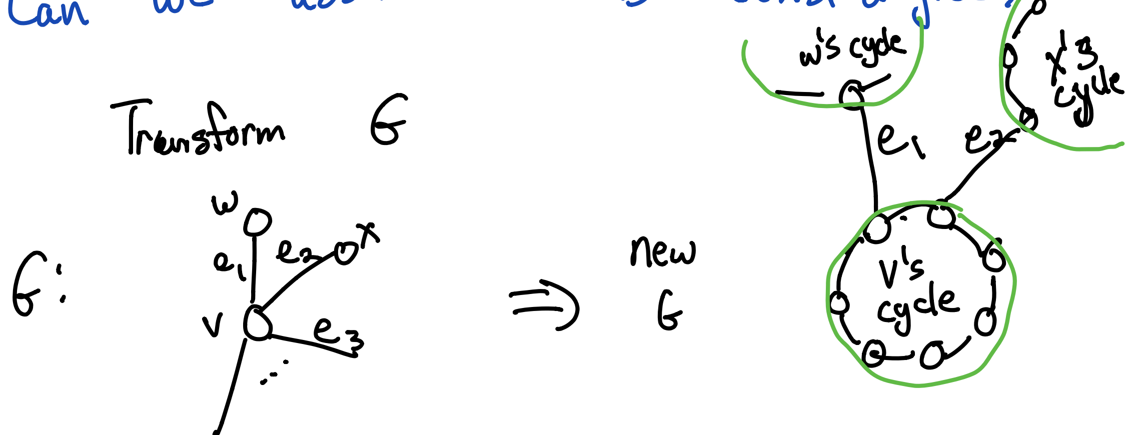
Thm 1 $\exists$ const D_e & $((D_e)^{16}, D_e, \frac{1}{2})$-graph

$\uparrow N \quad \uparrow D \quad \uparrow \lambda$

- Const size graph that has small λ_2
- can use it for any input
- can find via enumeration

Can we assume G is const degree?

Transform G



degree $\leq N$

nodes $= N$

degree ≤ 3

nodes $\leq N^2$

Same connectivity properties

Representing graphs:

Rotation map: $Rot_G : [N] \times [D] \to [N] \times [D]$

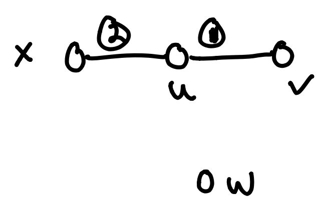
$Rot_G(v,i) = (w,j)$ if

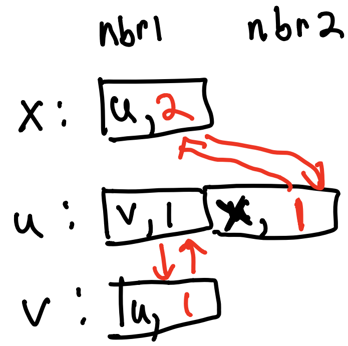
i^{th} edge of v leads to w

& j^{th} edge of w leads to v

allows back & forth on same edge

G:





Replacement Product $G \circledR H$

Given	G	D-reg	N nodes	G'	$N \cdot D$ nodes
	H	d-reg	D nodes		degree $d+1 \ll D$

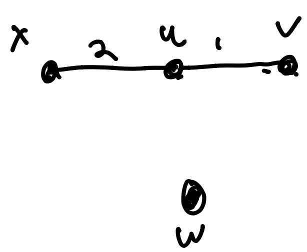
reduces degree, what does it do to λ_2?

nodes: $v \in G$ replaced by copy H

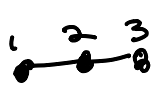
edges: - each vertex in H_v connected to nbrs in H_v

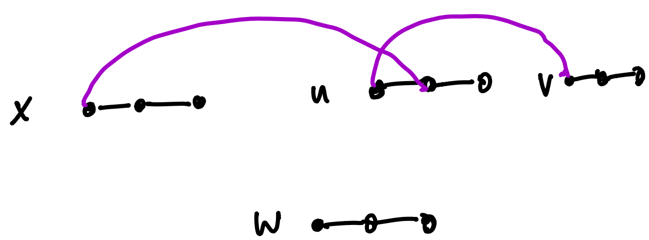
- if u is i^{th} nbr of v in G
  & v is j^{th} nbr of u
  add edge from i^{th} node of H_v
  to j^{th} " " H_u

G



H





$\Rightarrow$ G ⓡ H

Zig Zag Product $G ⓩ H$

Given G D-reg N nodes, H d-reg D nodes $\}$ G'' with $N \cdot D$ nodes, deg d^2

nodes: as in G'

each $v \in G$ replaced by copy of H

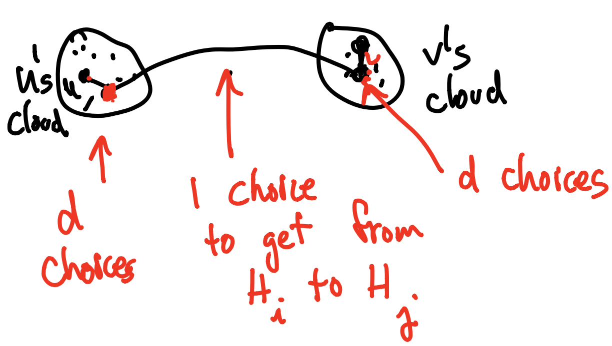
edges: path of length 3 in G'

$(u,v) \in G''$ iff $u \in H_i$ "cloud i"

$\exists\, w \in H_i$ s.t. $(u,w) \in E(H_i)$

$(w,z) \in G ⓡ H$

$(z,v) \in E(H_j)$ where $v \in H_j$



new degree d^2

Thm for $\alpha \le \frac{1}{2}$

G an (N, D, λ)- graph & H a $\underbrace{(D, d, \alpha)\text{-graph}}^{\text{guaranteed by Thm 1}}$

$G ⓩ H$ is $(ND, d^2, \lambda_{Gⓩ H})$-graph

s.t. $\frac{1}{2}(1-\alpha^2)(1-\lambda) \le 1 - \lambda_{Gⓩ H}$

So $\lambda_{Gⓩ H} \le 1 - \frac{1}{2}\underbrace{(1-\alpha^2)}_{\ge 3/4}(1-\lambda)$ $\swarrow \alpha^2 < 1/4$

$\le 1 - 3/8(1-\lambda)$

$\le \frac{2}{3} + \frac{\lambda}{3}$ ← still < 1

so degree drops & λ_2 isn't so bad

How to use?

Main transformation:

Given: G D^{16}-reg on N nodes

H D-reg on D^{16} nodes

Transformation:

$\ell \leftarrow$ smallest int s.t. $\left(1-\frac{1}{DN^2}\right)^{2^\ell} < \frac{1}{2}$

$G_0 \leftarrow G$

$G_i \leftarrow (\underbrace{G_{i-1} \textcircled{z} H}_{\text{deg reduction}})^8$ ← powering

Output: G_ℓ

Properties of G_ℓ:

$\#$ nodes $= N(D^{16})^\ell$ degree is $O(1)$
$= \text{poly}(N)$

Lemma: $\lambda(G_\ell) \leq 1/2$ so diameter is small

Use alg in beginning of class on G_ℓ