LATTICE POINT METHODS FOR COMBINATORIAL GAMES

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Abstract. We encode arbitrary finite impartial combinatorial games in terms of lattice points in rational convex polyhedra. Encodings provided by these *lattice games* can be made particularly efficient for octal games, which we generalize to *squarefree games*. These encompass all heap games in a natural setting where the Sprague–Grundy theorem for normal play manifests itself geometrically. We provide an algorithm to compute normal play strategies.

The setting of lattice games naturally allows for misère play, where 0 is declared a losing position. Lattice games also allow situations where larger finite sets of positions are declared losing. Generating functions for sets of winning positions provide data structures for strategies of lattice games. We conjecture that every lattice game has a *rational strategy*: a rational generating function for its winning positions. Additionally, we conjecture that every lattice game has an *affine stratification*: a partition of its set of winning positions into a finite disjoint union of finitely generated modules for affine semigroups. This conjecture is true for normal-play squarefree games and every lattice game with finite misère quotient.

1. Introduction

Combinatorial games involve two players, both with complete information, taking turns moving on a fixed game tree. The games considered here are impartial, meaning that both players have the same available moves from each position, and finite, meaning that the game tree is finite, although we are interested in families of games in which the totality of the available positions is infinite. The quintessential example of such (a family of) games is Nim, in which each node of the tree corresponds to a finite set of heaps of given sizes, and a move is accomplished by removing any number of beans from a single heap.

The *normal play* version of Nim, where the last person to move is the winner, was solved over a century ago [Bou02]. A complete structure theory for normal play games, known as the Sprague–Grundy theorem from the late 1930s [Spr36, Gru39], builds on Bouton's solution by reducing all finite impartial games to it: every impartial game under normal play is equivalent to a single Nim heap of some size. Background and details can be found in [ANW07, BCG82]. Readers coming from commutative algebra and desiring an accessible, efficient introduction to the concepts and standard notations of combinatorial game theory and misère games (most of which are not used here) should see Siegel's lecture notes [Sie06].

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In contrast, in *misère play* the last player to move is the loser, as in Dawson's chess [Daw34]. Misère games are much more complex than normal play games, as observed by Conway (see [Con01]), after inherent difficulties effectively stymied all progress. In particular, Dawson's chess remains elusive, despite exciting recent advances in misère theory by Plambeck and Siegel [Pla05, PlSi07], about which we say more shortly.

Our goal is to provide a setting in which algorithmic and theoretical techniques concerning lattice points in polyhedra can be brought to bear on computational and abstract periodicity questions from misère combinatorial game theory. Our approach is to rephrase the language of heap-based games, especially the historically popular *octal games* (of which Dawson's chess is an example), with the consequence of placing them in a certain natural generality, that of *squarefree games* (see Definition 6.3, and Section 6 in general). The lattice encoding allows for both normal and misère play, as well as generalizations in which finitely many positions are declared to be losing positions (Definition 2.9). The lattice encoding of squarefree games is also efficient; that said, it turns out that our lattice point language captures arbitrary finite impartial combinatorial games (Theorem 5.1), although the algorithmic efficiency—which is key to the goal of solving octal games—is lost in translation.

Generally speaking, any useful notion of a "solution" or "strategy" for a game should be a data structure with at least two fundamental properties:

- it can be efficiently stored, and
- it can be efficiently processed to compute a winning move from any position.

Ideally, the data structure should also be efficiently computable, given the game board and rule set, but that confronts a separate issue of finding a strategy, as opposed to simply recognizing one.

In the above sense, the *misère quotients* invented and mined by Plambeck and Siegel [Pla05, PlSi07] constitute excellent data structures for solutions of misère play impartial combinatorial games (and partizan ones [All09], too), when the quotients are finite. This has been the basis for substantial advances in computations involving specific misère games, such as some previously unyielding octal games, including some with infinite quotients; see [PlSi07, Wei09], for example. However, even when an infinite misère quotient is given as a finitely presented monoid, say by generators and relations, there remains a need to record the winning and losing positions—that is, the *bipartition* of the misère quotient into P-positions and N-positions.

The *lattice games* that we develop are played on *game boards* constituting sets of lattice points in polyhedra (Definition 2.9). For example, when the polyhedron is a cone, the game board is an *affine semigroup*. The allowed moves constitute a finite set of vectors called a *rule set* (Definition 2.3), satisfying some conditions of compatibility with the game board. A rule set uniquely determines the sets of winning and losing positions on a given game board (Theorem 4.6).

Methods from combinatorial commutative algebra, as it relates to sets of lattice points, provide clues as to how to express the presence of structure in the sets of winning positions of games. We conjecture that every lattice game has

- a *rational strategy*: the generating function for its winning positions is a ratio of polynomials with integer coefficients (Conjecture 8.5); and
- an *affine stratification*: an expression of its winning positions as a finite disjoint union of finitely generated modules for affine semigroups (Conjecture 8.9).

The second conjecture is stronger, in that it implies the first in an efficient algorithmic sense (Theorem 8.12), but it is the first that posits a data structure exhibiting the two fundamental properties listed above for a successful strategy (Theorem 8.4). That said, affine stratifications reflect real, observed phenomena more subtle than mere rationality of generating functions. We intend affine stratifications to provide vehicles for producing rational strategies, and they have already been useful as such in examples.

The above conjectures are true for squarefree games under normal play (Corollary 6.12 and Example 8.3), and for any game whose misère quotient is finite [KM10]; see Remark 8.11. Additionally, for squarefree normal play games, we present an algorithm for computing the winning positions (Theorem 7.4).

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We feel it is most appropriate for this paper to appear in a volume dedicated to Dennis Stanton. In fact, the mere thought of submitting this work to his volume benefited us greatly: it brought to mind Dennis's work on the combinatorics of various kinds of generating functions, which led us quickly to rational functions as finite data structures for solutions of combinatorial games. This path took us straight to the formulation of Conjecture 8.5 as a consequence of the affine stratifications in Definition 8.6 and Conjecture 8.9.

2. Lattice games

General abstract lattice games are played on game boards consisting of lattice points in polyhedra. Readers unfamiliar with polyhedra can find an elementary exposition of the necessary facts in Ziegler's book [Zie95, Lectures 0–2]; we limit ourselves to recalling a few core notions.

A *polytope* is the convex hull of a finite set in $\mathbb{R}^d$. A *polyhedron* is the intersection of finitely many closed affine halfspaces, each one bounded by a hyperplane that need not pass through the origin. Here, an *affine subspace* is a translate of a vector subspace—i.e., an affine subspace need not pass through the origin. A polyhedron is *rational* if it has an expression in terms of halfspaces defined by linear inequalities with rational coefficients.

Every bounded polyhedron is a polytope; this fact is intuitively true but not trivial to prove [Zie95, Theorem 1.1]. Every polyhedron Π is a Minkowski sum

$$\Pi = P + C = \{p + c \mid p \in P \text{ and } c \in C\}$$

of a polytope P and a cone C, where a *cone* is an intersection of closed halfspaces each bounded by a linear hyperplane—that is, passing through the origin. Equivalently, a cone is a subset of $\mathbb{R}^d$ closed under sums and under scaling by nonnegative real numbers. The cone C in the expression $\Pi = P + C$ is well-defined, and is called the *recession cone* of Π. In contrast, there is no unique choice of polytope P in general; but if Π is *pointed*, meaning that Π possesses at least one vertex (or equivalently, that C possesses a vertex—necessarily unique and lying at the origin) then a minimal choice for P would be the convex hull of the vertices of Π. Note that, for a nonnegative real number $\mu \in \mathbb{R}_+$, the scaled polyhedron $\mu\Pi = \{\mu\pi \mid \pi \in \Pi\}$ is another polyhedron, usually not equal to Π itself, although $\mu C = C$ for any cone C if $\mu > 0$.

Much of the framework here will depend on the geometry and algebra of affine semigroups for which a general reference is [MiSt05, Chapter 7]. For now we recall some basic definitions. A *semigroup* is a set with an associative binary operation. If the operation has an identity element, then the semigroup is a *monoid*. An *affine semigroup* is a monoid that is isomorphic to a finitely generated subsemigroup of $\mathbb{Z}^d$ for some d. (The use of the term "affine" is different from the one above; here, it is related more closely to "affine algebraic variety" than to "affine subspace".) An affine semigroup is *pointed* if the identity is its only unit (i.e., invertible element). An affine semigroup A more or less look like $C \cap \mathbb{Z}^d$ for the rational polyhedral cone $C = \mathbb{R}_+A$ generated by A, but with some points deleted; the pointed condition occurs precisely when $\mathbb{R}_+A$ is pointed.

Fix a pointed rational convex polyhedron $\Pi \subset \mathbb{R}^d$ with recession cone C of dimension d. Write $\Lambda = \Pi \cap \mathbb{Z}^d$ for the set of integer points in Π.

Example 2.1. The case of primary interest is $\Pi = C = \mathbb{R}^d_+$, so $\Lambda = \mathbb{N}^d$, in which lattice points with nonnegative coordinates represent positions in the game. The class of *heap games* is subsumed in this context: from an initial finite set of heaps of beans, the players take turns changing a heap—whichever they select—into some number of heaps of smaller sizes. The rules of a heap game specify the allowed smaller sizes. The game of Nim follows this pattern: a player is allowed to remove beans from any single heap, thus either creating one heap of strictly smaller size or deleting the heap entirely. In terms of lattice games, a position $p = (\pi_1, \ldots, \pi_d) \in \mathbb{N}^d$ represents π_i

heaps of size i for $i = 1, \ldots, d$. Octal games, quaternary games, hexadecimal games, and so on, are heap games; we will examine these later (under normal play) in the wider context of *squarefree games*, to be defined and analyzed in Section 6.

Moves in lattice games will require some hypotheses in order to guarantee that positions can reach a suitable neighborhood of the zero game. The geometric condition implying this behavior involves the following notion.

Definition 2.2. Given an extremal ray ρ of a cone C, the *negative tangent cone* of C along ρ is $-T_\rho C = -\bigcap_{H \supset \rho} H_+$ where $H_+ \supseteq C$ is the positive closed halfspace bounded by a supporting hyperplane H for C. Equivalently, $-T_\rho C = \bigcap_{H \supset \rho} -H_+ = \bigcap_{H \supset \rho} H_-$.

Throughout this paper, set $\mathbf{0} = (0, \ldots, 0) \in \mathbb{Z}^d$.

Definition 2.3. A finite subset $\Gamma \subset \mathbb{Z}^d \smallsetminus \{\mathbf{0}\}$ is a *rule set* if

1. there exists a linear function on $\mathbb{R}^d$ that is positive on $C \smallsetminus \{\mathbf{0}\}$ and on Γ; and
2. for each ray ρ of C, some vector $\gamma_\rho \in \Gamma$ lies in the negative tangent cone $-T_\rho C$.

Example 2.4. With notation as in Example 2.1, the positions of the game Nim with heaps of size at most 2 correspond to $\mathbb{N}^2$. Each move either removes a 1-heap, removes a 2-heap, or turns a 2-heap into a 1-heap. Hence the rule set Γ consists of $(1, 0)$, $(0, 1)$, and $(-1, 1)$, respectively. The options of $p = (\pi_1, \pi_2)$ are the elements of the set $(p - \Gamma) \cap \mathbb{N}^2$. We verify that Γ is a rule set: for condition 1, the function $\ell : \mathbb{Z}^2 \to \mathbb{Z}$ defined by $\ell(x, y) = x + 2y$ is positive on $\mathbb{N}^2 \smallsetminus \{\mathbf{0}\}$ and on Γ; condition 2 is satisfied by the basis vectors in Γ.

Example 2.5 (Heap games)**.** In the situation of Example 2.1, the rule set of a heap game is, by definition, a finite set of vectors γ each having the property that all of the nonzero entries of γ are negative, except the last nonzero entry of γ, which equals 1. Therefore, any linear function $\ell = (\ell_1, \ldots, \ell_d)$ is positive on $\mathbb{N}^d \smallsetminus \{\mathbf{0}\}$ and on Γ as long as ℓ_i is positive and sufficiently bigger than ℓ_{i-1} for each i. The tangent cone axiom is satisfied by definition for heaps of a given size i as long as there is a way to act on a heap of that size; that is, as long as some $\gamma \in \Gamma$ has $\gamma_i = 1$.

Lemma 2.6. *The affine semigroup* $\mathbb{N}\Gamma$ *generated by any rule set* Γ *is pointed.*

Proof. The nonzero vectors in $\mathbb{R}_+\Gamma$ all lie on the same side of the hyperplane given by the vanishing of the linear function. ☐

Remark 2.7. Condition 1 in Definition 2.3 implies more than Lemma 2.6: it implies also that $\mathbb{N}\Gamma$ and Λ point in the same direction. That is, moves, which are elements of $-\Gamma$, bring positions closer to $\mathbf{0}$.

Lemma 2.8. *Any rule set* Γ *induces a partial order* $\preceq$ *on* Λ *with* $p \preceq q$ *if* $q - p \in \mathbb{N}\Gamma$.

Proof. This follows immediately from the definitions of poset and Lemma 2.6. ☐

Definition 2.9 (Lattice games)**.** Given the polyhedral set $\Lambda = \Pi \cap \mathbb{Z}^d$, fix a rule set Γ.

- A *game board* $\mathcal{B}$ is the complement in Λ of a finite Γ-order ideal in Λ called the set of *defeated positions*.
- A *lattice game* is defined by a game board and a rule set.
- A position $p \in \Lambda$ *has a move to* $q \in \Lambda$ if $p - q \in \Gamma$.
- A move from a position p to q is *legal* if q lies on the game board $\mathcal{B}$.
- The *options* of a position are the positions to which it has legal moves.

Example 2.10. The game board for Nim in Example 2.1 is $\mathcal{B} = \mathbb{N}^d$ in normal play. On the other hand, in misère play, we declare $\mathbf{0}$ to be a defeated position—the only defeated position—so the game board is $\mathcal{B} = \mathbb{N}^d \smallsetminus \{\mathbf{0}\}$.

3. Geometry of rule sets

The axioms for rule sets in Definition 2.3 and game boards in Definition 2.9 were chosen to define what we suspect is the widest class of games satisfying the conjectures in Section 8, among games played on lattice points in this manner. This particular choice of axioms, however, resulted from numerous discussions about the properties of more inclusive and more restrictive classes of games. The purpose of this section is to explain our rationale, including consequences of the axioms in Section 2, while introducing some potentially useful extra conditions to put on rule sets.

3.1. **Why should the rule set be pointed?** If rule sets were allowed to generate cones with nontrivial lineality, then any position far from the game board boundary would have a loop of moves back to itself, violating the blanket finiteness condition. (This is the same reason $\mathbf{0}$ is not allowed as a move.) Indeed, since a subset of Γ generates the lineality group as a monoid, some positive combination of moves along the lineality equals $\mathbf{0}$. Thus nontrivial lineality is ruled out, so we must require the cone $\mathbb{R}_+\Gamma$ to be pointed in Definition 2.3. Consequently, just as abstract finite combinatorial games allow induction on options, the pointed hypothesis allows for induction on options in lattice games, by choosing a linear function ℓ on the game board that is positive on Γ, since $\ell(p) < \ell(p - \gamma)$ for any position p.

3.2. **Why arbitrary polyhedra?** Given the interpretation of $\mathbb{N}^d$ in terms of heaps, it seemed natural, at first, to play all lattice games on $\mathbb{N}^d$, after disallowing $\mathbf{0}$ for the purposes of misère play. However, we saw no a priori reason to require that a rule set must span $\mathbb{R}^d$. But without such a "fullness" hypothesis on Γ, the positions reachable from a given initial position p are restricted to lie in the coset through p of the lattice $\mathbb{Z}\Gamma$ generated by Γ. Thus we were led to arbitrary polyhedral game boards, because $(p + \mathbb{Z}\Gamma) \cap \mathbb{N}^d$ comprises the $\mathbb{Z}\Gamma$-lattice points in the polyhedron $\Pi = (p + \mathbb{R}\Gamma) \cap \mathbb{R}^d_+$. Although all of the examples that currently interest us are played on the polyhedron $\mathbb{R}^d_+$, we have no reason to believe that the conjectures in Section 8 fail for arbitrary polyhedral game boards.

3.3. **Why the tangent cone axiom?** Suppose, for the moment, that we are to play a game on the board $\mathbb{N}^d$. If the cone $\mathbb{R}_+\Gamma$ generated by the rule set Γ fails to contain the nonnegative orthant $\mathbb{R}^d_+$, then the positions in $\mathbb{R}^d_+ \setminus \mathbb{R}_+\Gamma$ can certainly never reach the winning position $\mathbf{0}$ by any sequence of moves. Thus, when the polyhedron Π is the nonnegative orthant $\mathbb{R}^d_+$, we considered using the condition $\mathbb{R}_+\Gamma \supseteq \mathbb{R}^d_+$ as a rule set axiom. The generalization of this requirement to arbitrary polyhedral game boards is $\mathbb{R}_+\Gamma \supseteq C$: the rule set cone must contain the recession cone of the game board. Indeed, for general polyhedral boards, writing $\Pi = \Pi_0 + C$ for a polytope Π_0, we get

$$\Pi_0 + \mathbb{R}_+\Gamma \supseteq \Pi_0 + C = \Pi,$$

so the bounded convex set Π_0 occupies the role for arbitrary polyhedral boards that the position $\mathbf{0}$ occupies for $\mathbb{N}^d$. Note that it is unreasonable to expect a sequence of moves to reach a single goal position on an arbitrary polyhedral board (which position would it be?), and this becomes especially true in generalized misère play, where a finite subset of the lattice points are declared to be defeated positions.

As natural and simple as it may be, the condition $\mathbb{R}_+\Gamma \supseteq C$ guarantees only that every position has a sequence of moves to near the boundary of the game board, not necessarily to the origin or anywhere near it (or, for arbitrary polyhedra, near Π_0).

Example 3.1. The set $\Gamma = \{(1,0,0),(0,1,0),(1,-1,1),(-1,1,1)\}$ generates a real pointed cone containing the nonnegative octant $\mathbb{R}^3_+$. If Γ were allowed as a rule set, then no position along the third axis would have any legal moves.

The example demonstrates an elementary observation: if every position is to have a sequence of legal moves that ends in a neighborhood of the origin, then every position sufficiently far along every extreme ray of the game board must have at least one legal move. The tangent cone axiom is precisely what guarantees this; and once it does, every position has a sequence of moves to a bounded neighborhood. Let us be precise.

Definition 3.2. Given a finite set $\Gamma \subset \mathbb{Z}^d \setminus \{\mathbf{0}\}$, a Γ-*path* is a sequence $p_0, \ldots, p_r$ with $p_{k-1} - p_k \in \Gamma$ for $k = \{1, \ldots, r\}$. For any set $S \subseteq \mathbb{Z}^d$ of lattice points (such as a game board or the lattice points in a polyhedron), this Γ-path *goes from p to q in S* if $p = p_0$ and $p_r = q$ and $p_0, p_1, \ldots, p_r$ all lie in S.

Theorem 3.3. *In any lattice game, there is a finite set $F \subset \mathcal{B}$ of game board positions such that every position in $\mathcal{B}$ has a Γ-path in $\mathcal{B}$ to F. Equivalently, the set of* victorious positions *(those from which there are no legal moves) is finite.*

Proof. The equivalence is straightforward, so we prove only the second claim.

There is a finite set $\Lambda_0 \subset \Lambda$ of lattice points such that $\Lambda = \Lambda_0 + (C \cap \mathbb{Z}^d)$; indeed, one can take for Λ_0 the set of all lattice points in a sufficiently large neighborhood (in Π) of the polytope Π_0 from Section 3.3. Since the legal moves in $\mathcal{B}$ between positions within $\lambda + (C \cap \mathbb{Z}^d)$ are exactly the legal moves in $\lambda + (C \cap \mathbb{Z}^d)$, it suffices to treat the case $\Pi = \lambda + C$. After translating by $-\lambda$ if necessary, we may assume that $\Pi = C$.

There are two types of victorious positions in C: those from which all moves to positions in C land in $\mathcal{D}$, and those from which no moves land in C. As the first of these sets is finite, because $\mathcal{D}$ and Γ are finite, we can and do assume that $\mathcal{D} = \varnothing$.

Let R be the set of extremal rays of C. It is enough to show that $\bigcup_{\rho \in R}(\gamma_\rho + C)$ covers all but a bounded subset of C, because every position in $\gamma_\rho + C$ has the legal move γ_ρ. Write C_ρ for the union of the facets of C that do not contain ρ, and let ε be the maximum of the lengths of the vectors γ_ρ. Then $\gamma_\rho + C$ contains (perhaps properly) the set of all points of C lying outside of the ε-neighborhood of C_ρ.

Lemma 3.4. *Let X be a polytope. For each vertex v of X, let X_v be the union of the facets of X not meeting v. For $\mu \in \mathbb{R}_+$, let $N_v(\mu) \subset \mu X$ be the open subset consisting of all points that do not lie within ε of μX_v. Then $\mu X = \bigcup_v N_v(\mu)$ for all $\mu \gg 0$.*

Proof. In the barycentric subdivision of X, if U_v denotes the union of the closed simplices containing the vertex v, then $X = \bigcup_v U_v$. Choose μ big enough so that the barycenter of each positive-dimensional face Y of μX lies at distance greater than ε from the affine spans of all of the facets of Y. Then μU_v is contained in $N_v(\mu)$, so $\bigcup_v N_v(\mu) \supseteq \bigcup_v \mu U_v = X$, completing the proof of the lemma. □

Using the lemma on any transverse hyperplane section X of C proves the theorem. □

We stress that the conclusion of Theorem 3.3 is equivalent to the tangent cone axiom in Definition 2.3, because of the elementary observation following Example 3.1. Since many rule sets naturally satisfy this alternate condition (or a stronger one: the heap rule sets in Example 2.5 always result in Γ-paths to $\mathbf{0}$), we record the remaining implication for future reference.

Lemma 3.5. *Suppose Γ satisfies the positivity axiom of Definition 2.3, but not necessarily the tangent cone axiom. If there is finite set $F \subset \Lambda$ such that every position in Λ has a Γ-path in Λ to F, then Γ is a rule set (i.e., Γ satisfies the tangent cone axiom).*

Infinitude of the set of victorious positions (prevented by Theorem 3.3) could have the potential to break Conjecture 8.5 on the existence of rational strategies. That is one of our key reasons for using the tangent cone axiom instead of the (seemingly more) natural cone containment condition. With that in mind, let us prove that the tangent cone axiom is indeed stronger.

Proposition 3.6. *The cone generated by any rule set contains the recession cone of the game board: $\mathbb{R}_+\Gamma \supseteq C$.*

Proof. If C' is a cone containing C, and if ρ is an extremal ray of C that remains extremal in C', then automatically $-T_\rho C' \supseteq -T_\rho C$. It is therefore enough to show that adding a new generating ray to C or replacing an extremal ray ρ of C by a ray γ_ρ in $-T_\rho C$ yields a cone C' containing C. This is obvious when a new generating ray is added, or when γ_ρ lies along ρ. In the other case, the segment from γ_ρ through any point on ρ extends to pass through a point β_ρ in some boundary face of C not meeting ρ. By construction, γ_ρ plus some (uniquely defined) multiple of β_ρ lies along ρ; the positivity axiom guarantees that it lies along ρ and not $-\rho$. Since the vector β_ρ is a positive combination of extremal rays of C' none of which is ρ, the ray ρ remains in the new cone C' generated by γ_ρ along with the other rays of C, whence $C' \supseteq C$. □

3.4. **The index of the rule set lattice.** In the general polyhedral setting, if we are interested in a fixed starting position, then it imposes no restriction to assume that Γ is saturated, in the following sense, for otherwise we may simply choose a smaller lattice to call $\mathbb{Z}^d$.

Definition 3.7. A rule set Γ is *saturated* if it spans $\mathbb{Z}^d$ as a group: $\mathbb{Z}\Gamma = \mathbb{Z}^d$.

In particular, although the index of $\mathbb{Z}\Gamma$ in $\mathbb{Z}^d$ contributes to the computational complexity, it has no effect on the rationality or stratification conjectures in Section 8, since any lattice game breaks up into a disjoint union of $|\mathbb{Z}^d/\mathbb{Z}\Gamma|$ many lattice games.

That said, many natural lattice games on $\mathbb{N}^d$—where replacing the polyhedron or the lattice with new ones is undesirable—have saturated rule sets. Such is the case with the heap games in Example 2.5, for instance, by the following general criterion.

Proposition 3.8. *Fix a lattice game with game board $\mathbb{N}^d$ and rule set Γ. If every position in $\mathbb{N}^d$ has a Γ-path to $\mathbf{0}$ in $\mathbb{N}^d$, then Γ is saturated.*

Proof. The hypothesis implies that all points in $\mathbb{N}^d$ lie in the same coset of $\mathbb{Z}\Gamma$ as $\mathbf{0}$. □

4. Uniqueness of winning and losing positions

In this section, we show that the sets of winning and losing positions of a lattice game are well-defined. This result, and all of the others in this section, hold in full without the tangent cone axiom for rule sets in Definition 2.3. To begin, here is an algebraic—and seemingly non-recursive—definition of winning and losing positions.

Definition 4.1. If G is a lattice game with game board $\mathcal{B}$ and rule set Γ, then $\mathcal{P}$ is the set of *winning positions* of G, and $\mathcal{N}$ is the set of *losing positions* of G, if $\mathcal{P}$ and $\mathcal{N}$ partition $\mathcal{B}$ and $(\mathcal{P} + \Gamma) \cap \mathcal{B} = \mathcal{N}$.

In game-theoretic terms, the equation $(\mathcal{P} + \Gamma) \cap \mathcal{B} = \mathcal{N}$ says that every position on the game board with a move to a winning position is a losing position. This implies other game-theoretic statements about winning and losing positions, such as the following, which looks similar but is strictly weaker.

Proposition 4.2. *If $\mathcal{B}$ is a game board with winning positions $\mathcal{P}$, losing positions $\mathcal{N}$, and rule set Γ, then $(\mathcal{P} - \Gamma) \cap \mathcal{P} = \varnothing$.*

Proof. Suppose $p \in (\mathcal{P} - \Gamma) \cap \mathcal{P}$. Then $p = p' - \gamma$ for some $p' \in \mathcal{P}$ and some $\gamma \in \Gamma$. Therefore $p' = p + \gamma \in (\mathcal{P} + \Gamma) \cap \mathcal{B} = \mathcal{N}$, a contradiction. □

Proposition 4.2 is weaker than Definition 4.1 because it does not force each losing position to possess a move to some winning position. For example, it is possible to change all but finitely many P-positions to N-positions without violating Proposition 4.2, but Definition 4.1 guarantees the existence of infinitely many P-positions.

Next we explore consequences of the compatibility of the rule set and the game board dictated by the positivity axiom in Definition 2.3.

Lemma 4.3. *Every sequence in Λ decreasing with respect to a rule set Γ is finite.*

Proof. Fix an integer linear function ℓ on $\mathbb{Z}^d$ that is positive on Γ. Then $\ell(q)-\ell(p) = \ell(q-p) \geq 0$ whenever $p \preceq q$, with equality if and only if $p = q$. Therefore ℓ induces a bijection from each Γ-decreasing sequence in Λ to some decreasing sequence in $\mathbb{Z}$. Therefore, it is enough to show that $\ell(\Lambda)$ is bounded below. This can be accomplished using Proposition 3.6 (or simply by assuming that ℓ is nonnegative on C, using Definition 2.3.1, if one wishes to avoid invoking the tangent cone axiom). $\square$

Definition 4.4. Let $T \subseteq \mathbb{Z}^d$. An element $p \in T$ is Γ-*minimal* if $(p - \Gamma) \cap T = \varnothing$. An element $p \in T$ is $\mathbb{N}\Gamma$-*minimal* (or simply *minimal*) if $(p - \mathbb{N}\Gamma) \cap T = \varnothing$.

Example 4.5. In misère play, we assume $\Pi = C = \mathbb{R}^d_+$, $\Lambda = \mathbb{N}^d$, and $\mathcal{D} = \{\mathbf{0}\}$. In this case, every position has a Γ-path to $\mathbf{0}$ in Λ if and only if every Γ-minimal element of $\mathcal{B}$ lies in Γ. Indeed, if a Γ-minimal element $p \in \mathcal{B}$ does not lie in Γ, then $p-\gamma \notin \mathbb{N}^d$ for every $\gamma \in \Gamma$, and hence p does not have a Γ-path to $\mathbf{0}$. Conversely, suppose every Γ-minimal element of $\mathcal{B}$ lies in Γ. By Lemma 4.3, every $p \in \mathcal{B}$ has a Γ-path in $\mathbb{N}^d$ to a Γ-minimal element, and hence to $\mathbf{0}$.

Theorem 4.6. *Given a rule set $\Gamma \subset \mathbb{Z}^d$ and a game board $\mathcal{B}$, there exist unique sets $\mathcal{P}$ and $\mathcal{N}$ of winning and losing positions for $\mathcal{B}$.*

Proof. By Lemma 2.8 and Lemma 4.3, $\mathcal{B}$ has Γ-minimal elements; define $\mathcal{P}_1$ to be the set of these elements. Let $\mathcal{N}_1 = (\mathcal{P}_1+\Gamma)\cap\mathcal{B}$. Inductively, having defined $\mathcal{P}_1, \ldots, \mathcal{P}_{n-1}$ and $\mathcal{N}_1, \ldots, \mathcal{N}_{n-1}$ for some $n \geq 2$, let $\mathcal{P}_n$ consist of the Γ-minimal elements of $\mathcal{B} \smallsetminus \mathcal{P}_{n-1}$, and set $\mathcal{N}_n = (\mathcal{P}_n + \Gamma) \cap \mathcal{B}$. In other words,

- $\mathcal{P}_n$ is the set of all positions $p \in \mathcal{B}$ for which $(p - \Gamma) \cap \mathcal{B}$ is contained in $\mathcal{N}_{n-1}$;
- $\mathcal{N}_n$ is the set of all positions $p \in \mathcal{B}$ such that $p - \gamma \in \mathcal{P}_n$ for some $\gamma \in \Gamma$.

Note that $\mathcal{N}_{n-2} \subseteq \mathcal{N}_{n-1} \Rightarrow \mathcal{P}_{n-1} \subseteq \mathcal{P}_n \Rightarrow \mathcal{N}_{n-1} \subseteq \mathcal{N}_n$, so it follows by induction on n, starting from $\mathcal{N}_0 = \varnothing$, that these containments all hold.

Lemma 4.7. *Let $\mathcal{P} = \bigcup_{k=1}^\infty \mathcal{P}_k$ and $\mathcal{N} = \bigcup_{k=1}^\infty \mathcal{N}_k$. Then*

$$\mathcal{P} = \{p \in \mathcal{B} \mid (p - \Gamma) \cap \mathcal{B} \subseteq \mathcal{N}\}, \text{ and}$$
$$\mathcal{N} = \{p \in \mathcal{B} \mid p - \gamma \in \mathcal{P} \text{ for some } \gamma \in \Gamma\}.$$

Proof. If $p \in \mathcal{P}$, then $p \in \mathcal{P}_k$ for some k, so $(p - \Gamma) \cap \mathcal{B} \subseteq \mathcal{N}_{k-1} \subseteq \mathcal{N}$. On the other hand, if $(p - \Gamma) \cap \mathcal{B} \subseteq \mathcal{N}$ then $(p - \Gamma) \cap \mathcal{B} \subseteq \mathcal{N}_{k-1}$ for some k, since Γ is finite, so $p \in \mathcal{P}_k \subseteq \mathcal{P}$.

If $p \in \mathcal{N}$ then $p \in \mathcal{N}_k$ for some k, so $p - \gamma \in \mathcal{P}_k \subseteq \mathcal{P}$ for some $\gamma \in \Gamma$. On the other hand, if $p - \gamma \in \mathcal{P}$ for some $\gamma \in \Gamma$, then $p - \gamma \in \mathcal{P}_k$ for some k, so $p \in \mathcal{N}_k \subseteq \mathcal{N}$. □

For the existence claimed by the theorem, we check that the sets $\mathcal{P}$ and $\mathcal{N}$ from the lemma satisfy the axioms for sets of winning and losing positions, respectively.

Lemma 4.8. *With $\mathcal{P}$ and $\mathcal{N}$ as in Lemma 4.7, we have $\mathcal{P} \cap \mathcal{N} = \varnothing$.*

Proof. If $p \in \mathcal{P} \cap \mathcal{N}$ then $p \in \mathcal{P}_n \cap \mathcal{N}_{n-1}$ for some n, since the unions defining $\mathcal{P}$ and $\mathcal{N}$ are increasing. But $\mathcal{P}_n \subseteq \mathcal{B} \smallsetminus \mathcal{N}_{n-1}$ by definition. □

Lemma 4.9. *With $\mathcal{P}$ and $\mathcal{N}$ as in Lemma 4.7, we have $\mathcal{P} \cup \mathcal{N} = \mathcal{B}$.*

Proof. Suppose p is Γ-minimal in $\mathcal{B} \smallsetminus (\mathcal{P} \cup \mathcal{N})$. Then $(p - \Gamma) \cap \mathcal{B} \subseteq \mathcal{P} \cup \mathcal{N}$. Therefore p must lie in $\mathcal{P}$ or in $\mathcal{N}$ by Lemma 4.7. □

By Lemma 4.7, it follows immediately that $(\mathcal{P} + \Gamma) \cap \mathcal{B} = \mathcal{N}$.

To prove uniqueness, suppose $\mathcal{B} = \mathcal{P} \cup \mathcal{N} = \mathcal{P}' \cup \mathcal{N}'$, where $\mathcal{P}, \mathcal{N}$ and $\mathcal{P}', \mathcal{N}'$ are pairs of winning and losing positions. First, suppose $\mathcal{P} \cap \mathcal{N}' = \varnothing = \mathcal{P}' \cap \mathcal{N}$. The first equality implies $\mathcal{N}' \subseteq \mathcal{N}$ while the second equality implies $\mathcal{N} \subseteq \mathcal{N}'$. Hence $\mathcal{N} = \mathcal{N}'$ and thus $\mathcal{P} = \mathcal{P}'$. Now suppose, by symmetry, that $\mathcal{P} \cap \mathcal{N}' \neq \varnothing$. Let $p \in \mathcal{P} \cap \mathcal{N}'$. If p is not Γ-minimal in $\mathcal{B}$, then since $p \in \mathcal{N}'$, there is some $\gamma \in \Gamma$ such that $p - \gamma \in \mathcal{P}'$; and since $p \in \mathcal{P}$, we have $p - \gamma \in \mathcal{N}$ (note that $p - \gamma \in \mathcal{B}$ since $\mathcal{P}' \subseteq \mathcal{B}$). Thus $q = p - \gamma \in \mathcal{P}' \cap \mathcal{N}$. Continuing in this manner and applying Lemma 4.3, we reduce to the case where p is Γ-minimal in $\mathcal{B}$. But then $p \in \mathcal{P}'$, contradicting $p \in \mathcal{N}'$. Therefore $\mathcal{P} \cap \mathcal{N}' = \varnothing$ and hence $\mathcal{P} = \mathcal{P}'$ and $\mathcal{N} = \mathcal{N}'$. This completes the proof. □

5. Arbitrary impartial games as lattice games

As it turns out, lattice games can be rigged to encode arbitrary games. To make a precise statement, we briefly recall the definitions of closed sets of games; see [PlSi07] and its references for more details.

Formally speaking, a finite impartial combinatorial game is a set consisting of *options*, each of which is, recursively, a finite impartial combinatorial game. The *disjunctive sum* of two games G and H is the game $G + H$ whose options are the union of $\{G' + H \mid G' \text{ is an option of } G\}$ and $\{G + H' \mid H' \text{ is an option of } H\}$. A set of games is *closed* if it is closed under taking options and under disjunctive sum. In particular, the *closure* of a single game G is the free commutative monoid on G and its *followers*, meaning the games obtained recursively as an option, or an option of an option, etc. The *birthday* of a game is the length of its longest chain of followers.

The theory of lattice games we develop here is, in the following sense, universal.

Theorem 5.1. *The closure of an arbitrary finite impartial combinatorial game, in normal or misère play, can be encoded as a lattice game played on $\mathbb{N}^d$.*

Proof. Let the game have d distinct followers $G_2, \dots, G_d$, and set $G_1 = G$. The disjunctive sum $G_{i_1} + \cdots + G_{i_r}$ corresponds to the position vector $e_{i_1} + \cdots + e_{i_r} \in \mathbb{N}^d$, where $e_1, \dots, e_d$ are the standard basis vectors. The rule set Γ consists of the vectors $e_i - e_j$ for each i and j such that G_j is an option of G_i. Normal play is encoded by setting $\mathcal{D} = \varnothing$, and misère play is encoded by setting $\mathcal{D} = \{\mathbf{0}\}$.

It remains to verify the axioms for a rule set (Definition 2.3). The positivity axiom holds because the function $\{1, \dots, d\} \to \mathbb{N}$ of the indices that sends i to the birthday of G_i is positive on Γ. The tangent cone axiom holds by Lemma 3.5 because every position has a sequence of moves to $\mathbf{0}$. □

Remark 5.2. If any game can be encoded using $\mathbb{N}^d$, why allow arbitrary polyhedral game boards? Beyond Section 3.2, there are at least two more answers.

1. For an arbitrary game, the encoding on $\mathbb{N}^d$ via its game tree is inefficient (exponentially so) in the sense of complexity theory; polyhedral game boards allow efficient encodings of wider classes of games. (Even for a game on a board of the form $\mathbb{N}^d$ for some d, there is almost surely a better encoding as a lattice game than the one provided in the proof of Theorem 5.1.)
2. Our conjectures in Section 8 concerning stratifications in the polyhedral context attempt to place certain kinds of periodicity in their natural generality, retaining only those hypotheses we believe essential for the regularity to arise; there is no reason, at present, to think that the simplicity of $\mathbb{N}^d$ has any bearing.

6. Squarefree games in normal play

The notion of *octal game* encompasses quite a broad range of examples, but it can sometimes feel contrived, such as when coincidences between the rules and certain heap sizes cause positions with nonempty collections of heaps that nonetheless have no options (see Example 6.4). Lattice games suggest a natural common generalization of octal games, as well as hexadecimal games and indeed arbitrary heap games that, in particular, automatically avoids the no-option phenomenon. The *squarefree games* we define here are precisely those lattice games played on $\mathbb{N}^d$ such that the Sprague–Grundy theorem for normal play finite impartial games is commensurate with the coordinates placed on positions by virtue of the game being on $\mathbb{N}^d$.

In this section, we assume that $\Pi = C = \mathbb{R}^d_+$ and hence $\Lambda = \mathbb{N}^d$. The following notation will come in handy a few times.

Definition 6.1. Given $v \in \mathbb{R}^d$, let v_+ and v_- be the nonnegative vectors with disjoint support such that $v = v_+ - v_-$. That is, $v_+ = v \wedge \mathbf{0}$ and $v_- = -(v \vee \mathbf{0}) = v_+ - v$.

Proposition 6.2. *For a rule set Γ, the following are equivalent.*

1. *For each $\gamma \in \Gamma$ and $p \in \mathbb{N}^d$, if $2p - \gamma \in \mathbb{N}^d$ then $p - \gamma \in \mathbb{N}^d$.*
2. *If $p + p'$ is an option of $p + p$, then p' is an option of p.*
3. *The maximum entry of each $\gamma \in \Gamma$ is equal to 1.*

4. *The positive part* γ_+ *is a* 0-1 *vector for all* $\gamma \in \Gamma$.
5. *Each move takes away at most one heap of each size.*

Proof. $1 \Leftrightarrow 2$: Assume 1 holds. Suppose $p + p'$ is an option of $p + p$, so $p' = p - \gamma$ for some $\gamma \in \Gamma$. Then $p' \in \mathbb{N}^d$ and hence is an option of p. Conversely, assume 2 holds. Suppose $2p - \gamma \in \mathbb{N}^d$. Let $p' = p - \gamma$. Then $p + p'$ is an option of $p + p$, so p' is an option of p, hence $p' \in \mathbb{N}^d$.

$1 \Rightarrow 3$: Suppose there exists $\gamma \in \Gamma$ with an entry greater than 1. Let $M = \max\{\gamma_1, \ldots, \gamma_d\}$, and let $p = \lceil \frac{M}{2} \rceil \mathbb{1}$ where $\mathbb{1}$ is the vector with all entries equal to 1. Then the minimum of the entries of $2p - \gamma$ is 1 for odd M and 0 for even M, and hence $2p - \gamma \in \mathbb{N}^d$. However, the minimum of the entries of $p - \gamma$ is $\lceil \frac{M}{2} \rceil - M$ which is negative since $M > 1$. Hence $p - \gamma \notin \mathbb{N}^d$.

$3 \Rightarrow 1$: Suppose 3 holds. Let $\gamma \in \Gamma$ and let $p \in \mathbb{N}^d$ with $2p - \gamma \in \mathbb{N}^d$. For all i such that $p_i = 0$, we must have $\gamma_i \leq 0$, and hence $(p - \gamma)_i \geq 0$. For all j such that $p_j > 0$, we have $(p - \gamma)_j = p_j - \gamma_j \geq 1 - 1 = 0$. Therefore $p - \gamma \in \mathbb{N}^d$.

$3 \Leftrightarrow 4$: This is elementary, since every move γ must possess a strictly positive entry.

$4 \Leftrightarrow 5$: In the notation of Examples 2.1 and 2.5, condition 5 is the translation of condition 4 into the language of heaps. □

Definition 6.3. We say a rule set Γ is *squarefree* if it satisfies any of the equivalent conditions in Proposition 6.2.

Example 6.4. The historically popular *octal games*, invented by Guy and Smith [GuSm56], are heap games in which every move consists of selecting a single heap and, depending on the heap's size and the game's rules, either

1. removing the entire heap;
2. removing some beans from the heap, making it smaller; or
3. removing some beans from the heap, splitting it into two smaller heaps.

A problem arises when there is a heap of size k but the rules do not allow the removal of j beans from a heap, for any $j \leq k$. In this case, we may simply ignore heaps of size k (treat them as heaps of size 0), and octal games naturally become a special class of squarefree games.

Definition 6.5. The *normal play* game board in $\mathbb{N}^d$ for a given rule set is the one with no defeated positions: $\mathcal{D} = \varnothing$.

Our results on normal play squarefree games are best stated in the following terms.

Definition 6.6. Two positions $p, q \in \mathcal{B}$ are *congruent*, written $p \cong q$, if

$$(p + C) \cap \mathcal{P} = p - q + (q + C) \cap \mathcal{P}.$$

In other words, $p + r \in \mathcal{P} \Leftrightarrow q + r \in \mathcal{P}$ for all r in the recession cone C of $\mathcal{B}$.

It is elementary to verify that congruence is an equivalence relation, and that it is additive, in the sense that $p \cong q \Rightarrow p + r \cong q + r$ for all $r \in C \cap \mathbb{Z}^d$. Thus, when $\mathcal{B} = \Lambda = C \cap \mathbb{Z}^d$ is a monoid, the quotient of $\mathcal{B}$ modulo congruence is again a monoid.

Throughout the remainder of this section, we assume that $\mathcal{B}$ is the normal play game board $\mathbb{N}^d$ with winning positions $\mathcal{P}$, losing positions $\mathcal{N}$, and rule set Γ.

Proposition 6.7. *If $p \in \mathcal{B}$, then $p \in \mathcal{P} \Leftrightarrow p \cong \mathbf{0}$.*

Proof. Suppose $p \in \mathcal{P}$. Let $q \in \mathcal{P}$. We claim that $p + q \in \mathcal{P}$. Clearly this is true for $p = \mathbf{0}$. Now assume $p \succ \mathbf{0}$, and suppose that $\hat{p} \in \mathcal{P} \Rightarrow \hat{p} + q \in \mathcal{P}$ for all $\hat{p} \prec p$. Let $\gamma \in \Gamma$ such that $p - \gamma \in \mathcal{B}$. Then $p - \gamma \in \mathcal{N}$, so there is $\gamma' \in \Gamma$ such that $p - \gamma - \gamma' \in \mathcal{P}$. By our induction hypothesis, $(p + q - \gamma) - \gamma' = (p - \gamma - \gamma') + q \in \mathcal{P}$, so $p + q - \gamma \in \mathcal{N}$. Since γ was arbitrary, $p + q \in \mathcal{P}$. Hence $q \in \mathcal{P} \Rightarrow p + q \in \mathcal{P}$.

Now suppose $q \in \mathcal{N}$. Then there is $\gamma \in \Gamma$ such that $q - \gamma \in \mathcal{P}$. Hence $p + q - \gamma \in \mathcal{P}$, so $p + q \in \mathcal{N}$. Therefore $p \cong \mathbf{0}$.

Conversely, suppose $p \cong \mathbf{0}$. Then $p \in \mathcal{P}$ since $\mathbf{0} \in \mathcal{P}$. □

Proposition 6.8. *If Γ is squarefree and $p \in \mathcal{B}$, then $2p \cong \mathbf{0}$.*

Proof. By Proposition 6.7, it suffices to show that $2p \in \mathcal{P}$. It is clearly true for $p = \mathbf{0}$. Now suppose $p \succ \mathbf{0}$ and $2\hat{p} \in \mathcal{P}$ for all $\hat{p} \prec p$. Let $\gamma \in \Gamma$ such that $2p - \gamma \in \mathcal{B}$. Since Γ is squarefree, $p - \gamma \in \mathcal{B}$, hence $2(p - \gamma) \in \mathcal{B}$. By our induction hypothesis, $2p - \gamma - \gamma = 2(p - \gamma) \in \mathcal{P}$, hence $2p - \gamma \in \mathcal{N}$. Since γ was arbitrary, $2p \in \mathcal{P}$. □

Corollary 6.9. *If $p, q \in \mathcal{B}$, then $p \cong q \Leftrightarrow p + q \in \mathcal{P}$.*

Lemma 6.10. *Let n be a positive integer. If $p \in \mathbb{Z}^d$, then there exist unique $q \in \mathbb{Z}^d$ and $r \in \{0, \ldots, n-1\}^d$ such that $p = nq + r$.*

Proof. Let $1 \leq i \leq d$. There are unique $q_i \in \mathbb{Z}$ and $r_i \in \{1, \ldots, n-1\}$ with $p_i = nq_i + r_i$ by the division algorithm. Thus $q = (q_1, \ldots, q_d)$ and $r = (r_1, \ldots, r_d)$ do the job. □

Theorem 6.11. *Let Γ be a squarefree rule set. If $\mathcal{P}_0 = \mathcal{P} \cap \{0, 1\}^d$ then*

$$\mathcal{P} = \mathcal{P}_0 + 2\mathbb{N}^d.$$

Proof. Let $w \in \mathcal{P}_0$. If $p \in 2\mathbb{N}^d$, by Propositions 6.7 and 6.8, $w + p \cong w \cong \mathbf{0}$ and hence $w + p \in \mathcal{P}$. On the other hand, let $w \in \mathcal{P}$. By Lemma 6.10, we may write $w = 2p + q$ for some $p \in \mathcal{B}$ and $q \in \{0, 1\}^d$. By Propositions 6.7 and 6.8, we have $\mathbf{0} \cong w \cong 2p + q \cong q$, hence $q \in \mathcal{P}_0$. □

Corollary 6.12. *In normal play, if the rule set is squarefree, then the set of winning positions is a finite disjoint union of translates of an affine semigroup.*

Remark 6.13. Without the squarefree hypothesis, Theorem 6.11 can fail. As an example, consider the rule set $\Gamma = \{(1, 0), (0, 2)\}$. It is straightforward to check that $(0, 0), (0, 1) \in \mathcal{P}$ but $(0, 2), (0, 3) \in \mathcal{N}$. In fact, the set of P-positions is

$$\mathcal{P} = \{(0, 0), (0, 1), (1, 2), (1, 3), (2, 0), (2, 1), (3, 2), (3, 3)\} + 4\mathbb{N}^2.$$

One might suspect that in general normal play, the P-positions have the form

$$\mathcal{P} = \mathcal{P}_0 + m\mathbb{N}^d.$$

However, more exotic behavior arises for the rule set $\Gamma = \{(1,0),(0,1),(2,2)\}$.

7. Algorithm for normal play

In this section, we examine a method of computing the set $\mathcal{P}_0$ in Theorem 6.11. To do this, we introduce the notion of a pattern.

Definition 7.1. For any set $S \subset \mathbb{Z}^d$, let S_2 denote the set of elements of S with entries modulo 2. A *pattern* is a subset of $\mathbb{Z}_2^d$. A subset $\Gamma \subset \mathbb{Z}^d$ *sustains* a pattern P if $P + \Gamma_2 = N$, where $N = \mathbb{Z}_2^d \smallsetminus P$. When convenient, we will view $\mathbb{Z}_2^d$ as a subset of $\mathbb{Z}^d$ via the embedding $\mathbb{Z}_2^d \hookrightarrow \mathbb{Z}^d$ as 0-1 vectors.

Proposition 7.2. *Let $\Gamma \subseteq \mathbb{Z}^d$. If $\mathbf{0} \notin \Gamma_2$, then Γ sustains a pattern P. Furthermore, there is an algorithm for finding P.*

Proof. Let $p_1 \in \mathbb{Z}_2^d$. For each $k > 1$, having defined p_j for $1 \leq j < k$, we choose $p_k \in \mathbb{Z}_2^d \smallsetminus \bigcup_{j=1}^{k-1}(p_j + \Gamma_2)$. Since $\mathbb{Z}_2^d$ is finite, this algorithm must terminate after some p_n. We claim that Γ sustains $P = \{p_1, \ldots, p_n\}$. Let $N = \mathbb{Z}_2^d \smallsetminus P$. Suppose $p_i = p_j + \gamma$ for some $i < j$ and some $\gamma \in \Gamma_2$. Then $p_j = p_i - \gamma = p_i + \gamma \in p_i + \Gamma_2$, a contradiction. Hence $P + \Gamma_2 \subseteq N$. Now suppose $p \in N$. Then there is some k such that $p \in p_k + \Gamma_2 \subseteq P + \Gamma_2$. □

Remark 7.3. Note that the pattern P we obtain from the algorithm in Proposition 7.2 is not necessarily unique, since it depends on the choice of p_1.

Theorem 7.4. *If $\mathcal{B}$ is a normal play game board with squarefree rule set Γ and winning positions $\mathcal{P} = \mathcal{P}_0 + 2\mathbb{N}^d$, then there is an algorithm for computing $\mathcal{P}_0$.*

Proof. Since Γ is squarefree, $\mathbf{0} \notin \Gamma_2$. By the algorithm in Proposition 7.2, we may obtain a pattern P sustained by Γ such that $\mathbf{0} \in P$. We claim that $P = \mathcal{P}_0$. By Theorem 6.11, whether a position p lies in $\mathcal{P}$ or $\mathcal{N}$ depends solely on its coordinates modulo 2. Therefore $p \in \mathcal{P}$ if and only if $p - \Gamma_2 \in \mathcal{N}$. Hence any pattern sustained by Γ is a viable candidate for $\mathcal{P}_0$. In particular, P works because it contains $\mathbf{0}$. By Theorem 4.6, P is the only pattern that works. Hence $\mathcal{P}_0 = P$. □

8. Rational strategies and affine stratifications

As with any subset of $\mathbb{N}^d$, the set $\mathcal{P}$ of P-positions in a lattice game can be recorded via its *generating function*: the power series $\sum_{p\in\mathcal{P}} \mathbf{t}^p$ in the indeterminates $\mathbf{t} = (t_1, \ldots, t_d)$, where $\mathbf{t}^p = t_1^{\pi_1} \cdots t_d^{\pi_d}$ for $p = (\pi_1, \ldots, \pi_d)$. For particularly well-behaved subsets of $\mathbb{N}^d$, the generating function is *rational*, meaning that it equals the Taylor series expansion of a ratio of polynomials in $\mathbf{t}$ with integer coefficients. These notions make sense as well for generating functions supported on pointed polyhedra, but the series are Laurent, not Taylor.

Definition 8.1. A *rational strategy* for a lattice game is a rational generating function for the set of P-positions.

Example 8.2. Consider the game of NIM with heaps of size at most 2. In normal play, a rational strategy is

$$f(\mathcal{P}; a, b) = \frac{1}{(1-a^2)(1-b^2)},$$

the rational generating function for the affine semigroup $2\mathbb{N}^2$. In misère play, a rational strategy is

$$f(\mathcal{P}; a, b) = \frac{a}{1-a^2} + \frac{b^2}{(1-a^2)(1-b^2)},$$

with the first term enumerating every other lattice point on the first axis, and the second enumerating the normal play P-positions that lie off of the first axis.

Classically [Bou02] in NIM, a disjunctive sum of heaps of sizes $a_1, \ldots, a_n$ is a P-position if and only if $a_1 \oplus \cdots \oplus a_n = 0$, where $\oplus$ is the binary operation of taking the bitwise XOR of the binary representations of the summands; that is, if the sum of the i-th digits of each summand is even for all i.

Example 8.3. A squarefree game in normal play has a rational strategy

$$f(\mathcal{P}; \mathbf{t}) = \sum_{p \in \mathcal{P}_0} \frac{\mathbf{t}^p}{(1-t_1^2)\cdots(1-t_d^2)},$$

in the notation from Theorem 6.11.

Of what use is a rational strategy? When one exists, it constitutes a desirable data structure for representing and manipulating sets of lattice points.

Theorem 8.4 ([GM10])**.** *Any rational strategy for a lattice game, presented as a ratio of two polynomials with integer coefficients, produces algorithms for*

- *determining whether a position is P-position or an N-position, and*
- *computing a legal move to a P-position, given any N-position.*

These algorithms are efficient when the rational strategy is a short *rational function, in the sense of Barvinok and Woods* [BaWo03].

Theorem 8.4 is proved by straightforward application of the algorithms of Barvinok and Woods [BaWo03]. That said, the subtlety lies more in phrasing the statement precisely, particularly when it comes to complexity; see [GM10].

Lattice games need not a priori possess rational strategies, but examples and heuristic arguments lead to the following assertion.

Conjecture 8.5. *Every lattice game possesses a rational strategy.*

Conjecture 8.5 is precise, but it allows for such a wide array of generating functions that it fails to capture the regularities pervading all examples to date. In fact, we were led to Conjecture 8.5 only after observing the existence of certain decompositions.

Definition 8.6. An *affine stratification* of a subset $\mathcal{W} \subseteq \mathbb{Z}^d$ is a partition

$$\mathcal{W} = \biguplus_{i=1}^{r} W_i$$

of $\mathcal{W}$ into a disjoint union of sets W_i, each of which is a finitely generated module for an affine semigroup $A_i \subset \mathbb{Z}^d$; that is, $W_i = F_i + A_i$, where $F_i \subset \mathbb{Z}^d$ is a finite set. An affine stratification of a lattice game is an affine stratification of its set of P-positions.

Example 8.7. Consider again the game of Nim with heaps of size at most 2. An affine stratification for this game is $\mathcal{P} = 2\mathbb{N}^2$; that is, the entire set of P-positions forms an affine semigroup. In misère play, $\mathcal{P} = \big((1,0) + \mathbb{N}(2,0)\big) \uplus \big((0,2) + 2\mathbb{N}^2\big)$ is the disjoint union of $W_1 = 1 + 2\mathbb{N}$ (along the first axis) and W_2, which equals the translate by twice the second basis vector of the affine semigroup $2\mathbb{N}^2$.

Example 8.8. Every normal play squarefree game has an affine stratification; this is Corollary 6.12.

Conjecture 8.9. *Every lattice game possesses an affine stratification.*

Remark 8.10. Conjecture 8.9 is equivalent to the same statement with the extra hypothesis that the rule set is saturated. Indeed, $\mathbb{Z}\Gamma$ has finite index in $\mathbb{Z}^d$, whence the game board is a disjoint union of finitely many games each of whose rule sets is saturated in its ambient lattice.

Remark 8.11. For any lattice game with finite misère quotient, Conjecture 8.9 holds. The proof [KM10] relies on interactions of congruences in commutative monoid theory with the combinatorial commutative algebra of binomial primary decomposition.

The importance of Conjecture 8.9 stems from its computational consequences.

Theorem 8.12 ([GM10])**.** *A rational strategy can be efficiently computed from any affine stratification.*

Again, the proof comes down to the algorithms of Barvinok and Woods [BaWo03], but the notion of "efficiency" must be made precise, and that is even more subtle than Theorem 8.4; see [GM10].

Example 8.13. The misère lattice game on $\mathbb{N}^5$ whose rule set forms the columns of

$$\Gamma = \begin{bmatrix} 1 & 0 & 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & -1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 \end{bmatrix}$$

has infinite misère quotient [PlSi07, Section A.7]. The illustration of the winning positions in this lattice game provided by Plambeck and Siegel [PlSi07, Figure 12] was one of the motivations for the definitions in this paper, particularly Definition 8.6, because it possesses an interesting description as an affine stratification. Indeed, for this lattice game, $\mathcal{P} = W_1 \uplus \cdots \uplus W_7$ for modules $W_k = F_k + A_k$ over the afffine semigroups

$$\begin{aligned}
A_1 &= \mathbb{N}\{(2,0,0,0,0),(0,2,0,0,0),(0,0,1,0,0),(0,0,0,2,0),(0,0,0,0,2)\}\\
A_2 &= \mathbb{N}\{(2,0,0,0,0),(0,2,0,0,0),(0,0,0,2,0),(0,0,0,0,2)\}\\
A_3 &= \mathbb{N}\{(2,0,0,0,0),(0,2,0,0,0),(0,0,0,0,2),(0,0,0,2,2)\}\\
A_4 &= \mathbb{N}\{(2,0,0,0,0),(0,2,0,0,0),(0,0,0,2,0),(0,0,0,2,2)\}\\
A_5 &= \mathbb{N}\{(2,0,0,0,0),(0,2,0,0,0)\}\\
A_6 &= \mathbb{N}\{(2,0,0,0,0),(0,2,0,0,0),(0,0,0,2,2)\}\\
A_7 &= \mathbb{N}\{(2,0,0,0,0)\},
\end{aligned}$$

where the finite generating sets F_k consist of the columns of the following:

$$F_1\colon \begin{bmatrix} 0\\0\\3\\0\\0 \end{bmatrix} \qquad F_2\colon \begin{bmatrix} 0&1&1&1&1\\ 0&1&1&1&1\\ 2&2&2&2&2\\ 0&3&3&3&1\\ 0&1&3&5&5 \end{bmatrix} \qquad F_3\colon \begin{bmatrix} 1&0&0&1&0&1&0&1&0&1\\ 0&1&0&0&1&1&0&0&1&1\\ 2&2&1&1&1&1&0&0&0&0\\ 0&1&0&0&1&1&0&0&1&1\\ 9&12&9&8&9&8&10&9&12&13 \end{bmatrix}$$

$$F_4\colon \begin{bmatrix} 1&1&1&1&0&0&0&0&0&0&1&1&1&0&0&0&1&1&0&0&0&0&0&1&1&1&1&0&0&0&1&1&1\\ 0&0&0&0&1&1&1&0&0&0&0&0&0&1&1&1&1&1&0&0&0&0&0&0&0&0&0&1&1&1&1&1&1\\ 2&2&2&2&2&2&2&1&1&1&1&1&1&1&1&1&1&1&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\ 1&1&1&1&0&0&0&0&2&0&1&3&1&2&2&2&3&1&2&4&4&4&4&3&5&5&3&2&4&2&3&3&3\\ 0&2&4&6&1&5&7&0&4&6&0&4&6&1&5&7&1&5&0&2&4&6&8&0&2&4&6&1&5&7&1&5&7 \end{bmatrix}$$

$$F_5\colon \begin{bmatrix} 1&0&1&1&1&1&0&1&1&1&1&0&0&0&0&0&0&1&1&1&1&1&1&0&0&0&0&0&1&1\\ 1&0&0&0&0&0&1&1&1&1&1&2&0&0&0&0&0&2&0&0&0&0&0&1&1&1&1&1&3&1\\ 2&1&1&1&1&1&1&1&1&1&1&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\ 0&0&0&1&0&1&1&1&2&0&0&0&1&2&0&1&2&0&2&3&0&1&2&1&0&1&0&1&1&2\\ 3&3&2&3&5&6&4&1&3&4&7&0&2&3&4&5&6&1&2&3&3&4&5&0&2&3&5&6&1&4 \end{bmatrix}$$

$$F_6\colon \begin{bmatrix} 0&0&1&1&0&0&1\\ 0&0&0&0&1&1&1\\ 0&0&0&0&0&0&0\\ 0&1&0&1&0&1&1\\ 7&8&6&7&8&9&10 \end{bmatrix} \qquad F_7\colon \begin{bmatrix} 0&1&1\\ 0&0&0\\ 0&0&0\\ 0&0&1\\ 1&0&1 \end{bmatrix}.$$

We are hopeful that this affine stratification, as a mode for presenting the P-positions, will lead to an algorithmic verification of the presentation for the misère quotient.

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