# Preventing Arithmetic Overflows in Alloy

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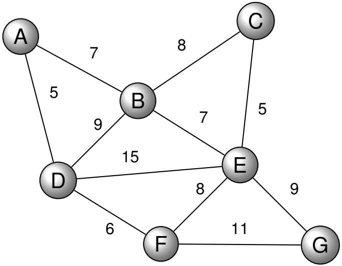
International Conference of Alloy, ASM, B, VDM, and Z Users
Pisa, Italy, June 2012

# Checking Prim's Algorithm

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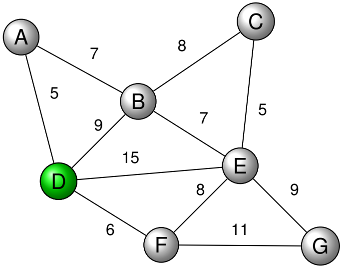


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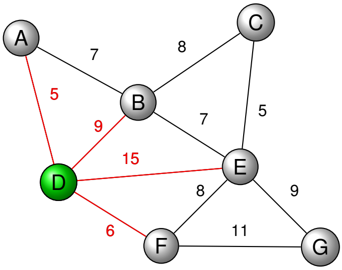


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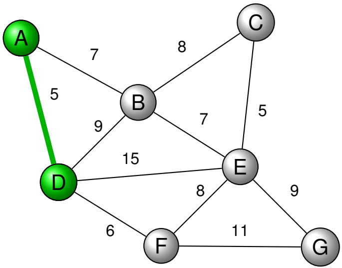


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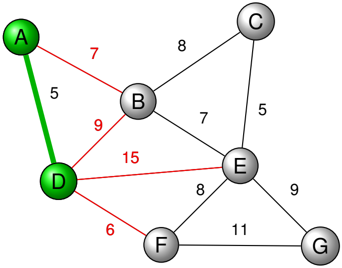


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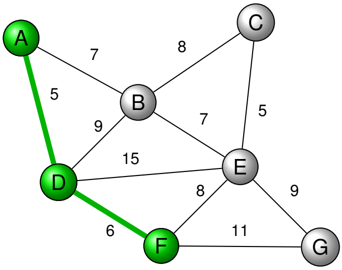


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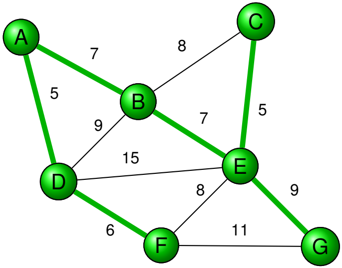


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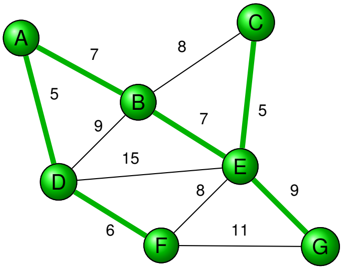


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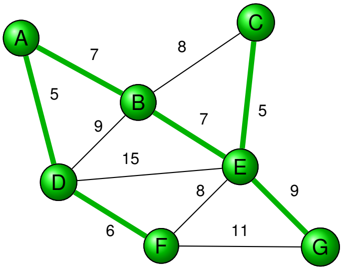


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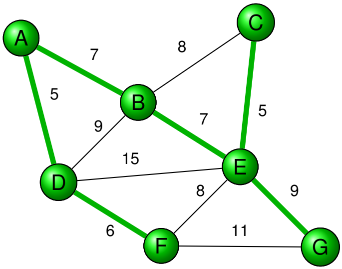


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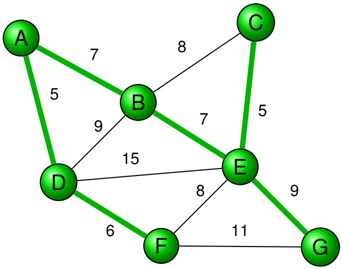


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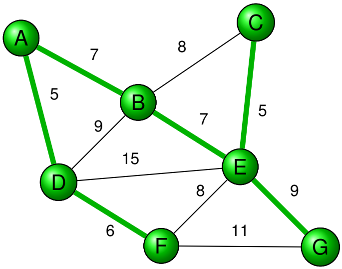


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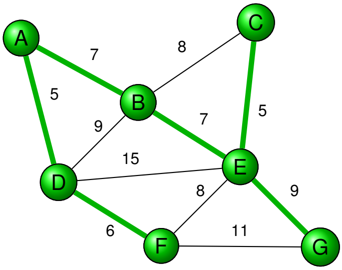


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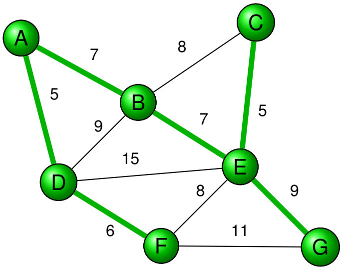


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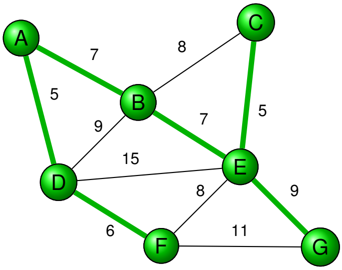


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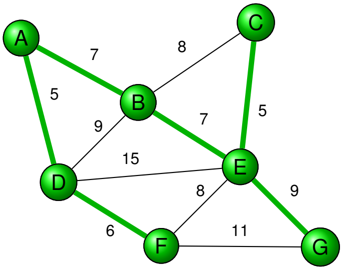


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causes arithmetic overflows!

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  - → arithmetic operations can overflow ⇒ spurious counterexamples
- sound w.r.t. counterexamples if no integers are used
  - → i.e., if a counterexample is found, the property does not hold
  - → **reason**: relational operators are closed under finite universe

# Goal & Approach

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$$\texttt{true} \wedge \perp = \perp, \quad \texttt{false} \wedge \perp = \texttt{false}, \quad \ldots$$

- change the semantics of quantifiers

$$[\![\texttt{all}\ \ x\texttt{:}\ \texttt{Int}\ \texttt{|}\ p(x)]\!] \ = \forall x \in \texttt{Int} \bullet (p(x) = \perp) \vee p(x)$$
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**challenge**: translation to existing SAT-based engine

# Example Using the New Semantics

## semantics of quantifiers

$$[\![\texttt{all}\ \ x\texttt{: Int | } p(x)]\!] = \forall x \in \texttt{Int} \bullet (p(x) = \bot) \vee p(x)$$
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pred p[x, y: Int] {
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check { all x, y: Int | p[x, y] }
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**scope**

`Int = {-4, -3, ..., 2, 3}`

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$p(x,y) = \perp$ : ✗

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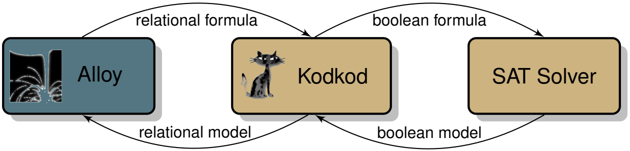
  - → likely to adversely affect models that don't involve integers

## alloy architecture



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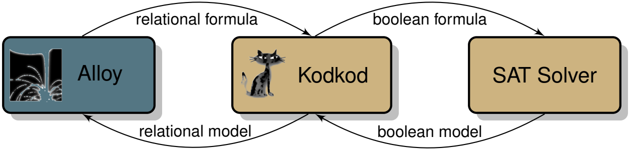

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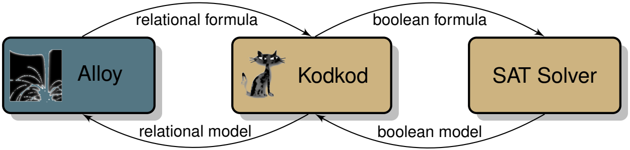

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## approach

- **only** integer functions can result in an undefined integer value ($\perp$)
  - $\rightarrow$ use textbook overflow circuits to detect such cases

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- every boolean formula must denote (evaluate to `true` or `false`)

## consequence

- a truth value must be assigned to predicates involving undefined terms [Farmer'95]

```
all x: Int |
  x > 0 => x.plus[x] > x
```

```
some x: Int |
  x > 0 && x.plus[x] < x
```

$\longrightarrow$ x=3: $[\![$ x.plus[x] > x $]\!]$ = true

$\longrightarrow$ x=3: $[\![$ x.plus[x] < x $]\!]$ = false

## approach

- **only** integer functions can result in an undefined integer value ($\perp$)
  - $\rightarrow$ use textbook overflow circuits to detect such cases
- **single link** from integers to boolean formulas: comparison predicates
  - $\rightarrow$ adjust the semantics of integer comparison predicates
  - $\rightarrow$ when either term is $\perp$, evaluate to make the outer binding irrelevant

# Translation to Classical Logic (2)

## definition

$$[\![x < y]\!]\sigma = \begin{cases} x < y \ \wedge \ x \neq \bot \ \wedge \ y \neq \bot, & \textit{if } \sigma{=}\sigma_\exists \quad \text{(in existential context)} \\ x < y \ \vee \ x = \bot \ \vee \ y = \bot, & \textit{if } \sigma{=}\sigma_\forall \quad \text{(in universal context)} \end{cases}$$

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compositional

$$[\![\neg(x<y)]\!]\sigma_\exists = \neg[\![x<y]\!]\sigma_\exists$$

$$= \neg(x<y \;\wedge\; x\neq\bot \;\wedge\; y\neq\bot)$$

$$= x\geq y \;\vee\; x=\bot \;\vee\; y=\bot$$

$$\neq [\![x\geq y]\!]\sigma_\exists$$

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$$\cancel{\begin{aligned} [\![\neg(x<y)]\!]\sigma_\exists &= \neg[\![x<y]\!]\sigma_\exists \\ &= \neg(x<y \ \wedge \ x \neq \bot \ \wedge \ y \neq \bot) \\ &= x \geq y \ \vee \ x = \bot \ \vee \ y = \bot \\ &\neq [\![x \geq y]\!]\sigma_\exists \end{aligned}}$$

semantics preserving

$$\begin{aligned} [\![\neg(x<y)]\!]\sigma_\exists &= [\![x \geq y]\!]\sigma_\exists \\ &= x \geq y \ \wedge \ x \neq \bot \ \wedge \ y \neq \bot \end{aligned}$$

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# Translation to Classical Logic (2)

**definition**

$$[\![\rho(x,y)]\!]\sigma = \begin{cases} \rho(x,y) \;\wedge\; x \neq \bot \;\wedge\; y \neq \bot, & \text{if } \sigma = \sigma_\exists \quad \text{(in existential context)} \\ \rho(x,y) \;\vee\; x = \bot \;\vee\; y = \bot, & \text{if } \sigma = \sigma_\forall \quad \text{(in universal context)} \end{cases}$$

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**rule for negation:**

$$\begin{aligned} [\![\neg p]\!]\sigma_\exists &= \neg[\![p]\!]\sigma_\forall \\ [\![\neg p]\!]\sigma_\forall &= \neg[\![p]\!]\sigma_\exists \end{aligned}$$

# Evaluation

**how does the new encoding affect performance?**

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  - → (only when arithmetic operations are used)
- no extra primary variables are used

# Evaluation

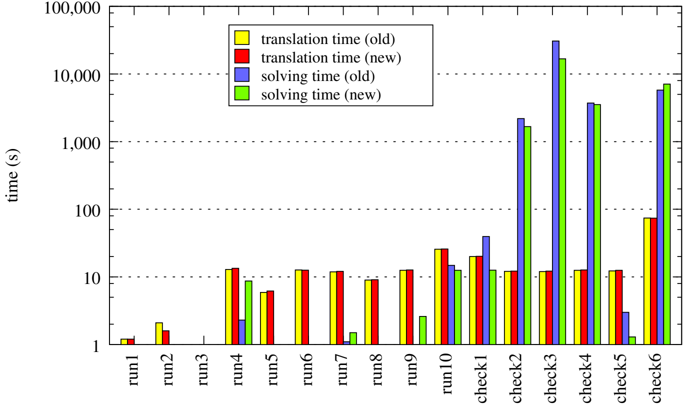
## how does the new encoding affect performance?

- extra clauses are generated to detect and prevent overflows
  - → (only when arithmetic operations are used)
- no extra primary variables are used
- possible effects of extra clauses on solving time:
  - → **speedup**: because search space smaller (more constrained)
  - → **slowdown**: SAT solver can get stuck more easily

# Experiment

## flash filesystem [Kang, ABZ'08]

- heavy use of arithmetic (for computing memory addresses)
- we ran 10 simulations and 6 checks
- **total time** decreased from 12 hours to 8 hours
- **this result is not meant to be conclusive!**



# Summary

## summary

- alloy made sound with respect to counterexamples

## applications that can benefit

- program verifications
- test case generation
- specification execution

## ideas for future work

- user-defined partial functions

CSAIL

# Thank You!

http://alloy.mit.edu

ALLOY

# Spurious Counterexamples due to Overflows

**reason for overflows**

- wraparound semantics for arithmetic operations
  - → `Int = {-4, -3, ..., 2, 3}` $\Longrightarrow$ `3 + 1 = -4`

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## prototypical anomalies

- sum of two positive integers is not necessarily positive!

```
check {
  all x, y: Int |
    x > 0 && y > 0 => x.plus[y] > 0
} for 3 Int
```

**counterexample**

```
Int = {-4, -3, ..., 2, 3}
a = 3; b = 1;
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**counterexample**

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Int = {-4, -3, ..., 2, 3}
a = 3; b = 1;
a.plus[b] = -4
```

- cardinality of a non-empty set is not necessarily positive!

```
check {
  all s: set univ |
    some s iff #s > 0
} for 4 but 3 Int
```

**counterexample**

```
Int = {-4, -3, ..., 2, 3}
s = {S0, S1, S2, S3}
#s = -4
```

# Example

## rules

$$[\![\rho(x,y)]\!]\sigma \quad = \quad \begin{cases} \rho(x,y) \ \wedge \ x \neq \bot \ \wedge \ y \neq \bot, & \textit{if } \ \sigma{=}\sigma_\exists \\ \rho(x,y) \ \vee \ x = \bot \ \vee \ y = \bot, & \textit{if } \ \sigma{=}\sigma_\forall \end{cases}$$

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## example

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all x: Int |
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$\longrightarrow x = 3:$

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some x: Int |
  x > 0 && !(x.plus[x] > x)
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## example

```
all x: Int |
  x > 0 => x.plus[x] > x
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$\longrightarrow x = 3:$

$$[\![x + x > x]\!]\sigma_\forall$$
$$= 3 + 3 > 0 \;\vee\; 3 + 3 = \bot \;\vee\; 0 = \bot$$
$$= \texttt{false} \;\vee\; \texttt{true} \;\vee\; \texttt{false}$$
$$= \texttt{true}$$

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$$\llbracket x + x > x \rrbracket \sigma_\forall$$
$$= 3 + 3 > 0 \ \vee \ 3 + 3 = \bot \ \vee \ 0 = \bot$$
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```
some x: Int |
  x > 0 && !(x.plus[x] > x)
```

$\longrightarrow x = 3:$

$$\llbracket !(x + x > x) \rrbracket \sigma_\exists$$
$$= \neg \llbracket x + x > x \rrbracket \sigma_\forall$$
$$= \neg \texttt{true}$$
$$= \texttt{false}$$

# Law of the Excluded Middle

**is law of the excluded middle still preserved?**

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  - → all integers when multiplied by 2 are either negative or non-negative?

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  all x: Int | x.mul[2] < 0 or !(x.mul[2] < 0)
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check {
  all x: Int | x.mul[2] < 0 or !(x.mul[2] < 0)
} for 4 Int
```

✓

*all integers x <u>such that x times 2 does not overflow</u>,*
*x times 2 is either negative or non-negative*

- violation of the law is still observable

```
check { 4.plus[5]  = 6.plus[3] } for 4 Int
```

✓

```
check { 4.plus[5] != 6.plus[3] } for 4 Int
```

✓

  - → the violation is visible if truth is associated with a check yields a counterexample at all

# Partial Functions in Logic

## overflows in alloy

- instance of a more general problem: handling partial functions in logic

## existing solutions/approaches

- logic of partial functions (LPF) [C. B. Jones]
  - → both integer functions and boolean formulas may be undefined
  - → uses a 3-valued logic
- traditional approach [Farmer'95]
  - → functions may be partial, but formulas must be denoting
  - → if any term is undefined, formula evaluates to `false`
  - → leaves open whether $\neg(a = a) \equiv a \neq a$ given that $a$ is undefined
- totalize all functions
  - → wraparound semantics for integer arithmetic in old alloy
  - → out-of-bounds applications result in unknown (but determined) value [B, Z]

## differentiating characteristics of our approach

- customized for the bounded setting
- masking quantifier bindings that produce undefinedness