Alin Tomescu | Week 1, Wednesday, February 5th, 2014 | Recitation 1
6.006 Intro to Algorithms | Prof. Srinivas Devadas | Prof. Nancy Lynch | Prof. Vinod Vaikuntanathan

A more convoluted example: $\log \binom{n}{n/2}$

Consider the following function:

$$f(n) = \log \binom{n}{n/2} = \Theta(?)$$

Remember that $\binom{n}{k} = \frac{n!}{k!(n-k)!}$

Let's see how we can simplify $\binom{n}{n/2}$. We assume that n is even.

$$\binom{n}{n/2} = \frac{n!}{\left(\frac{n}{2}\right)! \cdot \left(\frac{n}{2}\right)!} = \frac{1 \cdot 2 \cdot 3 \cdot \ldots \cdot (n-1) \cdot n}{\left(1 \cdot 2 \cdot 3 \cdot \ldots \cdot \left(\frac{n}{2} - 1\right) \cdot \frac{n}{2}\right) \cdot \left(1 \cdot 2 \cdot 3 \cdot \ldots \cdot \left(\frac{n}{2} - 1\right) \cdot \frac{n}{2}\right)}$$
$$= \frac{1 \cdot 2 \cdot 3 \cdot \ldots \cdot (n-1) \cdot n}{\left(\frac{1}{2}\right)^{\frac{n}{2}} (2 \cdot 4 \cdot 6 \cdot \ldots \cdot (n-2) \cdot n) \cdot \left(\frac{1}{2}\right)^{\frac{n}{2}} (2 \cdot 4 \cdot 6 \cdot \ldots \cdot (n-2) \cdot n)} = 2^n \frac{1 \cdot 2 \cdot 3 \cdot \ldots \cdot (n-1) \cdot n}{(2 \cdot 4 \cdot 6 \cdot \ldots \cdot (n-2) \cdot n)^2}$$
$$= 2^n \frac{1 \cdot 3 \cdot 5 \cdot \ldots \cdot (n-3) \cdot (n-1)}{2 \cdot 4 \cdot 6 \cdot \ldots \cdot (n-2) \cdot n}$$

Now, if we take the log of the result we get:

$$f(n) = \log(2^n) + \log\left(\frac{1 \cdot 3 \cdot 5 \cdot \ldots \cdot (n-3) \cdot (n-1)}{2 \cdot 4 \cdot 6 \cdot \ldots \cdot (n-2) \cdot n}\right) = n + \log\left(\frac{1 \cdot 3 \cdot 5 \cdot \ldots \cdot (n-3) \cdot (n-1)}{2 \cdot 4 \cdot 6 \cdot \ldots \cdot (n-2) \cdot n}\right)$$

Now, intuition tells us that $\frac{1 \cdot 3 \cdot 5 \cdot \ldots \cdot (n-3) \cdot (n-1)}{2 \cdot 4 \cdot 6 \cdot \ldots \cdot (n-2) \cdot n}$ will be very small because it can be written as $\frac{1}{2} \cdot \frac{3}{4} \cdot \frac{5}{6} \cdot \ldots \cdot \frac{n-3}{n-2} \cdot \frac{n-1}{n}$ which tends to barely approach 1 as n increases. So informally we can already argue that $f(n) = \Theta(n)$ since $\log(1) = 0$.

More formally, we will prove that $f(n) = \Omega(n)$ and $f(n) = O(n)$ which means $f(n) = \Theta(n)$.

Part 1: Prove that $f(n) = \Omega(n)$

Note that,

$$\frac{1 \cdot 3 \cdot 5 \cdot \ldots \cdot (n-3) \cdot (n-1)}{2 \cdot 4 \cdot 6 \cdot \ldots \cdot (n-2) \cdot n} = \frac{1}{n} \cdot \frac{3 \cdot 5 \cdot \ldots \cdot (n-3) \cdot (n-1)}{2 \cdot 4 \cdot 6 \cdot \ldots \cdot (n-2)} \geq \frac{1}{n}, \forall n > 1$$

This implies that,

$$\log\left(\frac{1 \cdot 3 \cdot 5 \cdot \ldots \cdot (n-3) \cdot (n-1)}{2 \cdot 4 \cdot 6 \cdot \ldots \cdot (n-2) \cdot n}\right) \geq \log\left(\frac{1}{n}\right), \forall n > 1$$

Thus,

$$f(n) = n + \log\left(\frac{1 \cdot 3 \cdot 5 \cdot \ldots \cdot (n-3) \cdot (n-1)}{2 \cdot 4 \cdot 6 \cdot \ldots \cdot (n-2) \cdot n}\right) \geq n + \log\left(\frac{1}{n}\right) = n + \log 1 - \log n = n - \log n, \forall n > 1$$

Since n dominates $\log n$ asymptotically, it follows that $f(n) = \Omega(n)$.

Part 2: Prove that $f(n) = O(n)$

Note that,

$$\frac{1 \cdot 3 \cdot 5 \cdot \ldots \cdot (n-3) \cdot (n-1)}{2 \cdot 4 \cdot 6 \cdot \ldots \cdot (n-2) \cdot n} \le \frac{2 \cdot 4 \cdot 6 \cdot \ldots \cdot (n-2) \cdot n}{2 \cdot 4 \cdot 6 \cdot \ldots \cdot (n-2) \cdot n}, \forall n > 1 \Leftrightarrow \frac{1 \cdot 3 \cdot 5 \cdot \ldots \cdot (n-3) \cdot (n-1)}{2 \cdot 4 \cdot 6 \cdot \ldots \cdot (n-2) \cdot n} \le 1, \forall n > 1$$

This implies that,

$$\log\left(\frac{1 \cdot 3 \cdot 5 \cdot \ldots \cdot (n-3) \cdot (n-1)}{2 \cdot 4 \cdot 6 \cdot \ldots \cdot (n-2) \cdot n}\right) \le \log(1) \le 1, \forall n > 1$$

Thus,

$$f(n) = n + \log\left(\frac{1 \cdot 3 \cdot 5 \cdot \ldots \cdot (n-3) \cdot (n-1)}{2 \cdot 4 \cdot 6 \cdot \ldots \cdot (n-2) \cdot n}\right) \le n + 1, \forall n > 1$$

This is equivalent to saying that $f(n) = O(n)$

Finally, since $f(n) = \Omega(n)$ and $f(n) = O(n)$, this means $f(n) = \Theta(n)$.