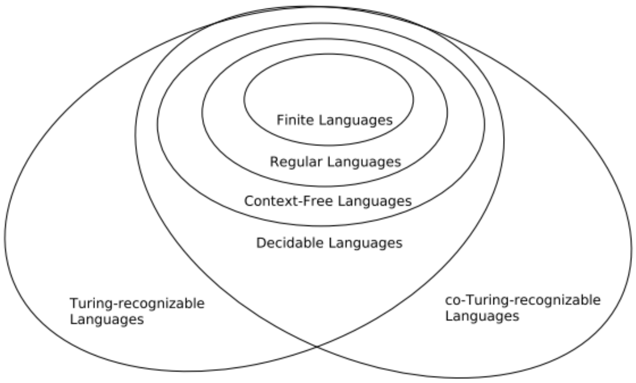
Alin Tomescu, *6.840 Theory of Computation (Fall 2013)*, taught by Prof. Michael Sipser

How to prove Turing decidability of languages

Language hierarchy



Recognizability

Reduce to A_{TM}: R is **T-recog** if it is reducible to A_{TM} ($R \leq_m A_{TM}$)

Reduce to recognizable language: If $R \leq_m R'$ and R' is T-recog, then R is **T-recog**

Give enumerator: R is **T-recog** ⇔ ∃ an enumerator E such that $L(E) = R$

Give recognizer: R is **T-recog** ⇔ ∃ a Turing machine T such that $L(T) = R$ *(by existence of TM recognizer)*

- **WARNING:** Be careful when saying stuff like "I can recognize if this happens by simulating M on all inputs and checking if it accepts". If you do something like this for, let's say, $\overline{E_{TM}}$, then you have to make sure you use **dove-tailing** so as to not get stuck in an infinte loop on a particular input.

R is **T-recog** ⇔ ∃ a language D, such that $R = \{x \mid \exists y(\langle x, y\rangle \in D)\}$ *(by projection of decidable language)*

Decidability

Give lexicographic-order enumerator: A is **T-decidable** ⇔ ∃ an enumerator E such that E prints all of the strings in A in lexicographic order.

Show language and its complement are both recognizable: If A is T-recog and $\bar{A}$ is T-recog then A is **T-decidable**.

- I can take both recognizers and run them in parellel, simulating a step on each one, eventually one will accept, allowing me to decide A. It is important that you run them step-by-step in parallel, as opposed to first running the A recognizer and then running the $\bar{A}$ recognizer. What if the first recognizer never halts?

Reduce to decidable language: If $D \leq_m D'$ and D' is decidable, then D is **T-decidable** *(by mapping-reducibility to decidable language)*

- Because I can map D to D', solve the D' instance, and I will have solved the D instance.

Undecidability

Reduce from A_{TM}: U is **undecidable** if A_{TM} is reducible to U *(by reduction from A_{TM})*

Reduce from undecidable problem: If $U' \leq_m U$ and U' is undecidable, then U is **undecidable** *(by mapping-reducibility from undecidable language)*

Turing-unrecognizability

If $A \leq_m B$ and A is **not T-recognizable**, then B is **not Turing-recognizable** *(by mapping-reducibility to unrecognizable language).*

If A is not decidable, then A or $\bar{A}$ is **not Turing-recognizable**.

If J is undecidable and $J \leq_m \bar{J}$, then both J and $\bar{J}$ are **not Turing-recognizable.**

Examples

Decidable: $A_{DFA}, E_{DFA}, EQ_{DFA}, A_{CFG}, E_{PDA}, A_{LBA}$

Undecidable: $A_{TM}, HALT_{TM}, ALL_{PDA}, EQ_{CFG}, E_{LBA}, PCP$. Also ALL_{TM}.

Unrecognizable: $\overline{A_{TM}}, E_{TM}, EQ_{TM}, \overline{EQ_{TM}}$