Active Learning for Level Set Estimation

Alkis Gotovos[1] Nathalie Casati[1,2] Gregory Hitz[1] Andreas Krause[1]

[1]Department of Computer Science
ETH Zurich

[2]IBM Research – Zurich

IJCAI '13

Swimmers of Lake Zurich, beware!

Steffen Schmidt / EPA

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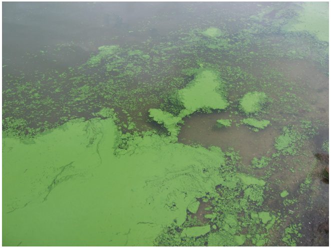
www.limnobotics.ch



Flickr/Dr. Jennifer L. Graham/U.S. Geological Survey

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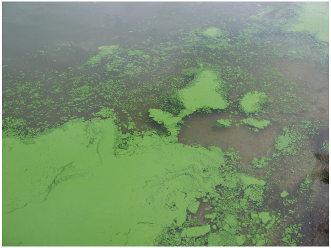

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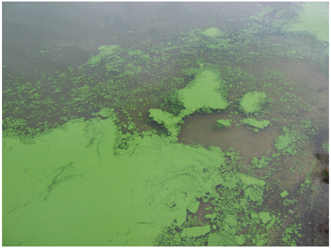

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"Microcystins [...] are cyanotoxins and can be very toxic for plants and animals including humans. Their hepatotoxicity may cause serious damage to the liver."

— *Wikipedia*

Autonomous surface vehicle developed by the Autonomous Systems Lab of ETH

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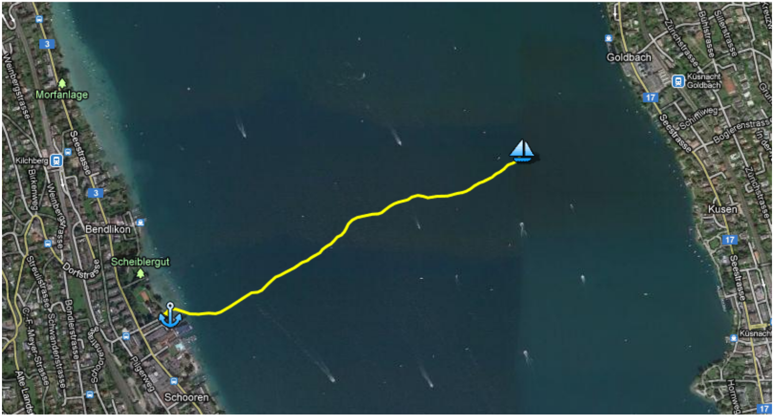
Take measurements on a vertical transect of the lake



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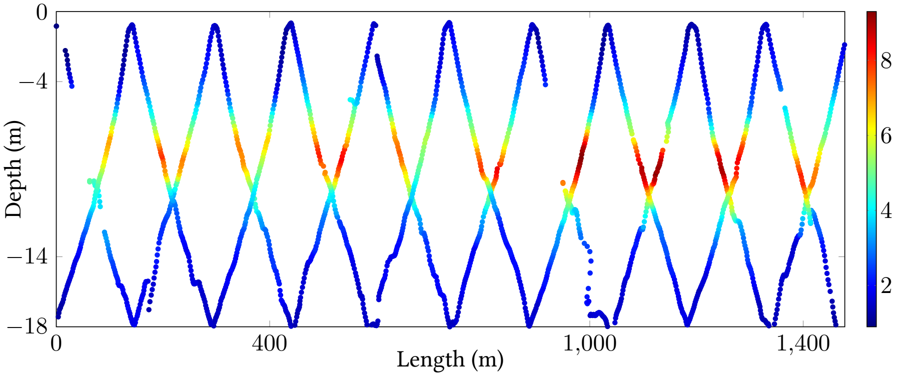
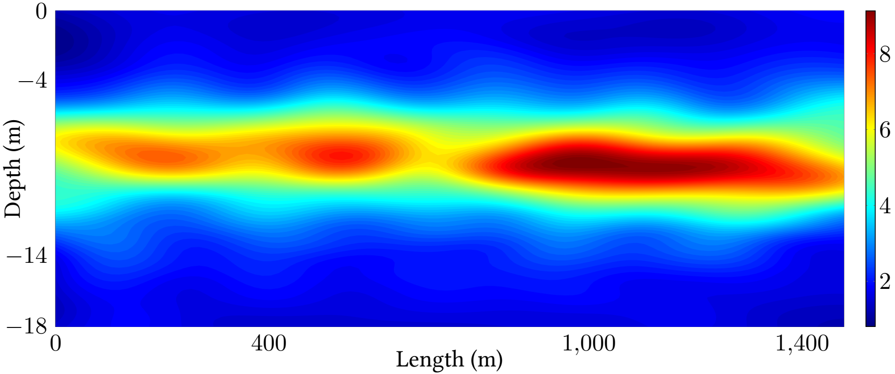
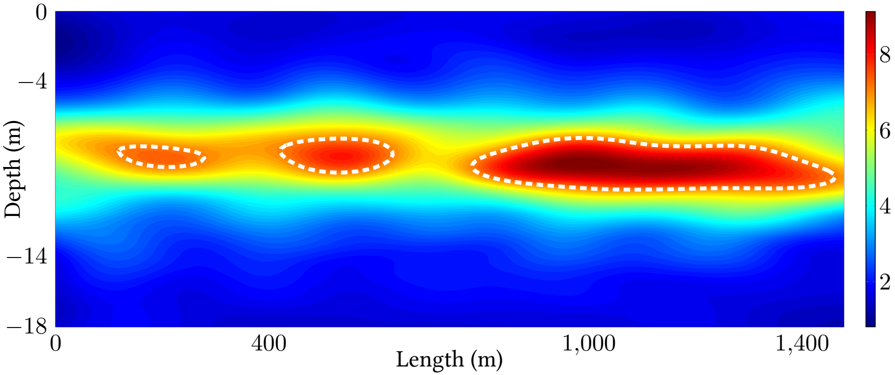
Original algae concentration measurements (~ 2000)



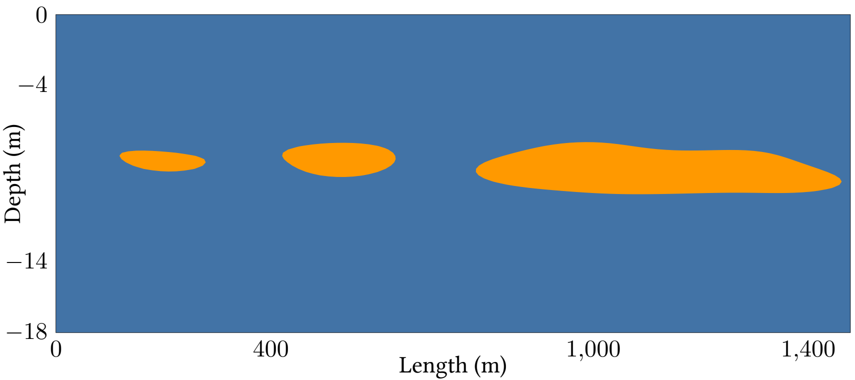
Interpolated algae concentration field



Focus on accurately estimating regions of "high" concentration (e.g. ≥ 7)



Classify transect into a super- and a sublevel set



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- Update our classification estimate

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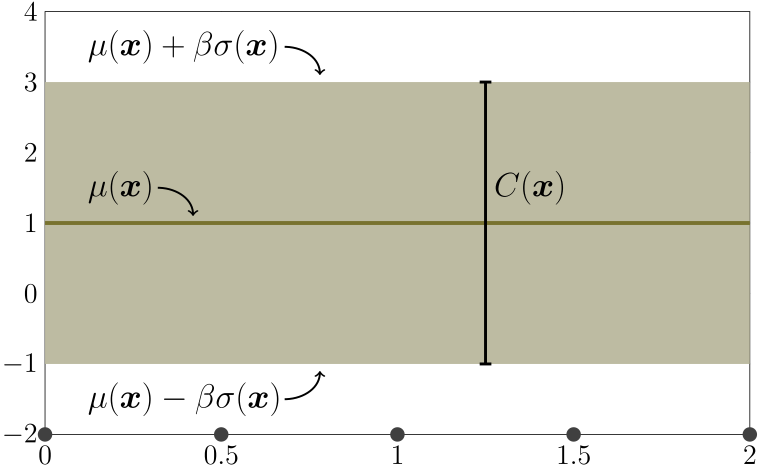
Gaussian processes to the rescue!

Gaussian processes

- Mean **and variance** estimates: construct confidence intervals $C(\boldsymbol{x})$



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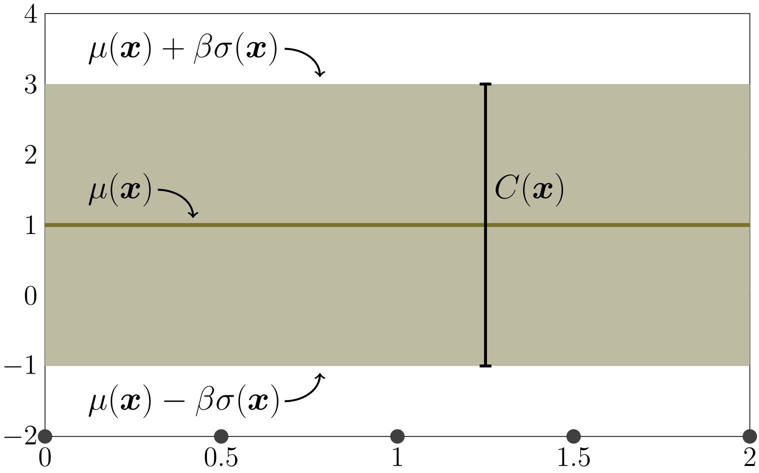

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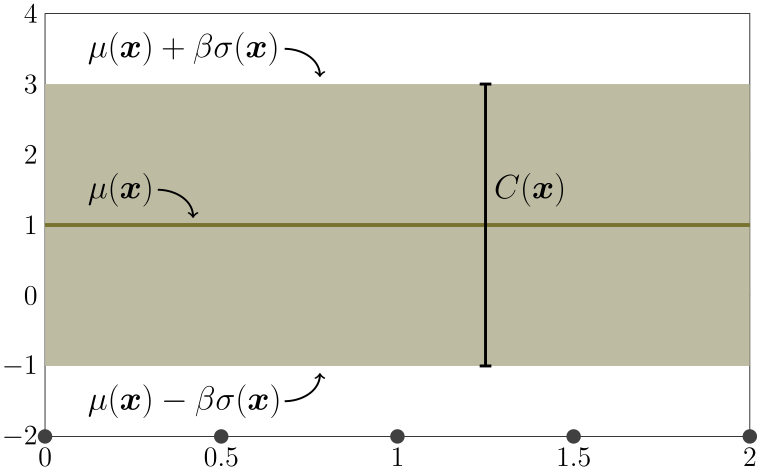

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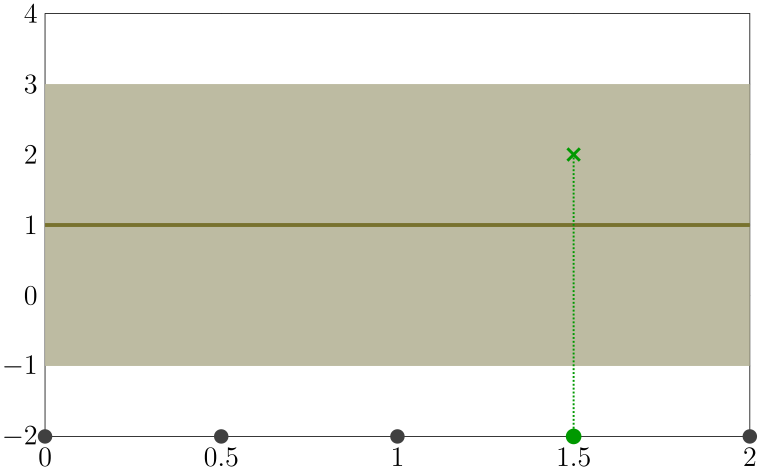

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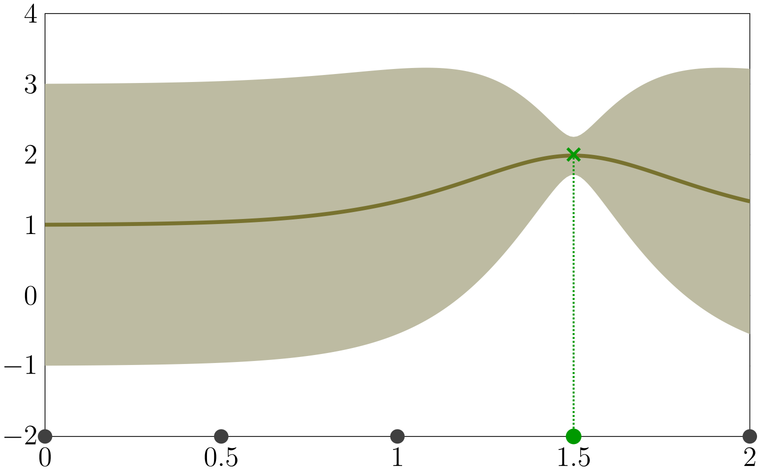

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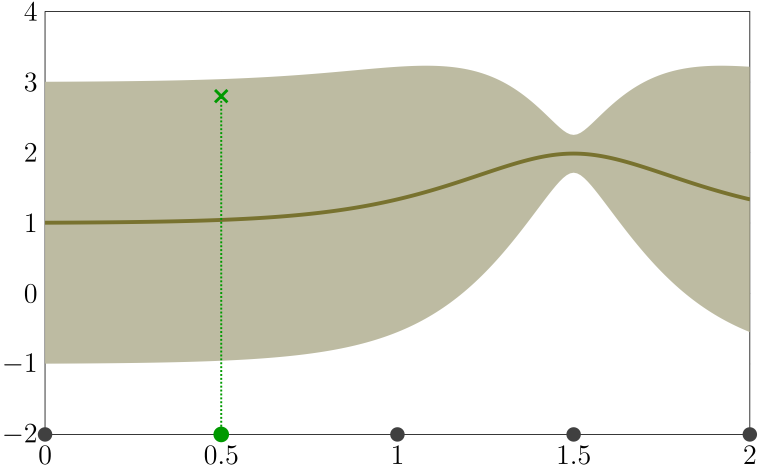

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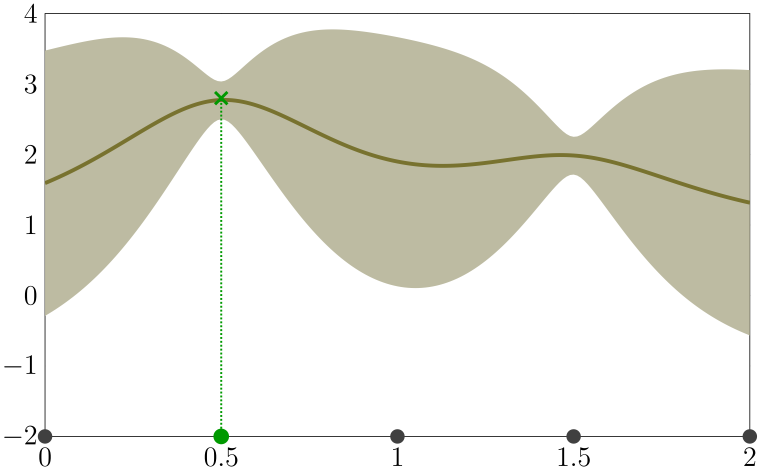

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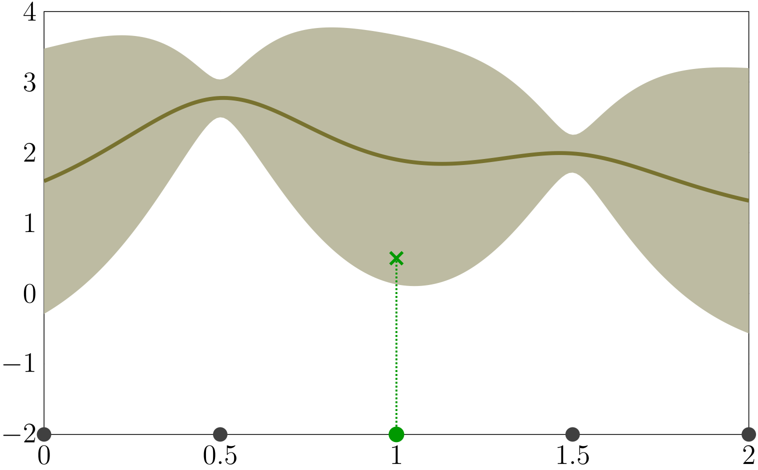

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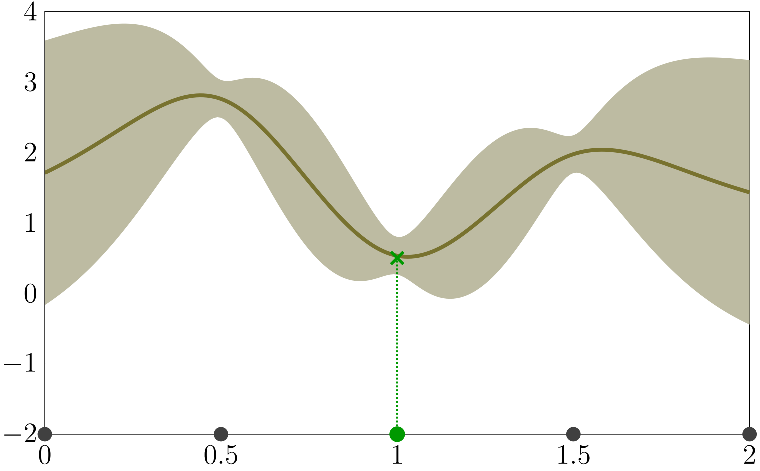

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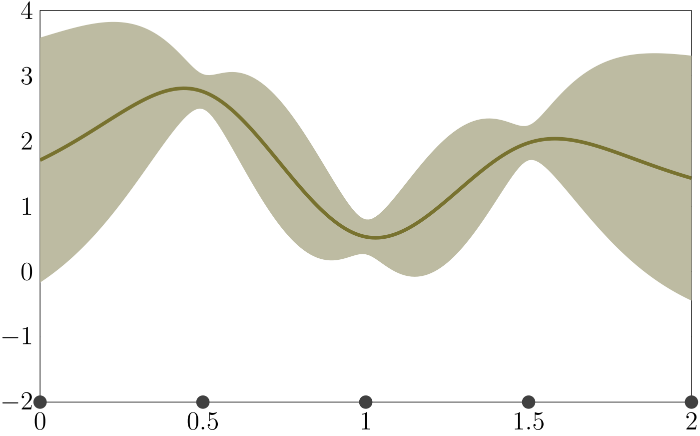

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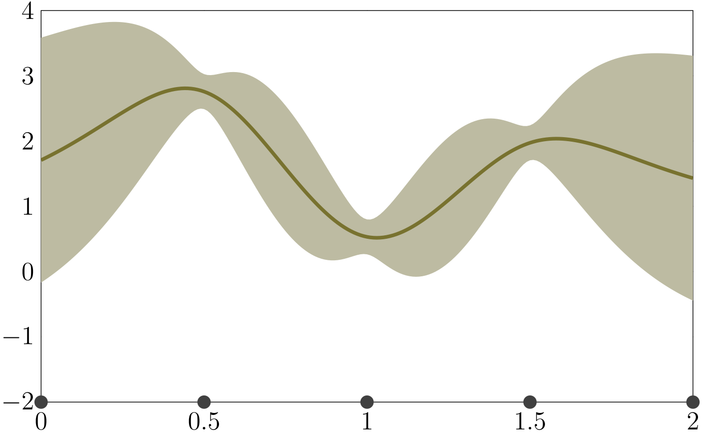

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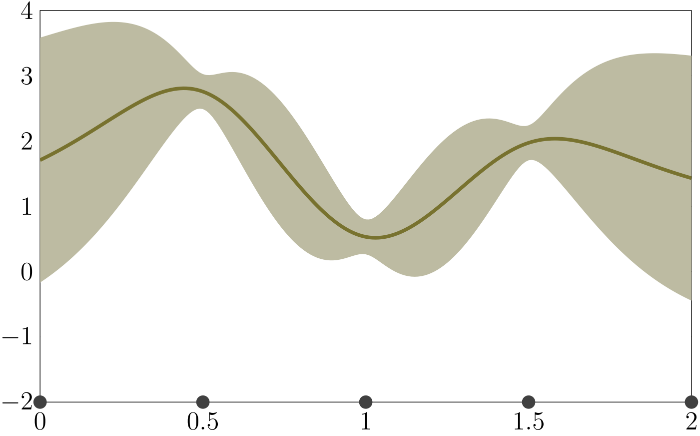

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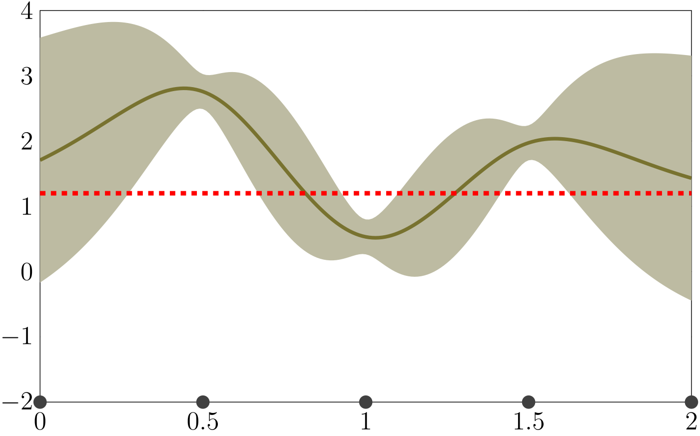

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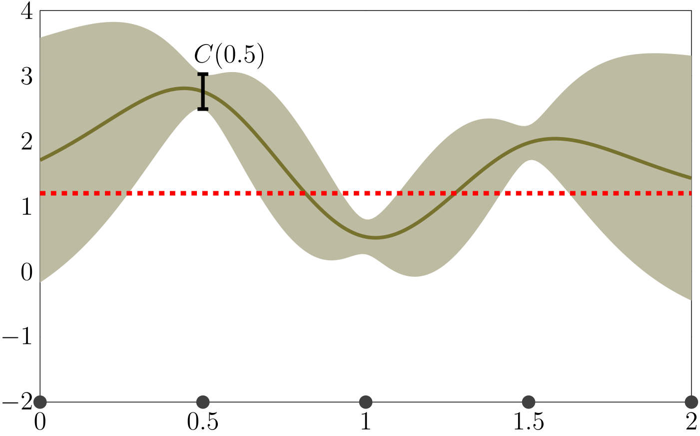

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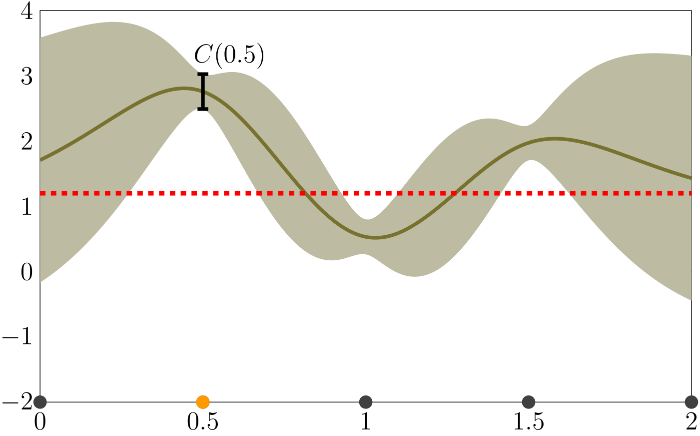

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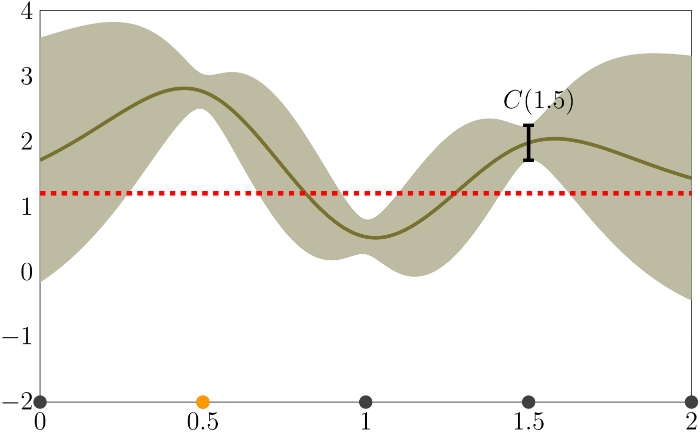

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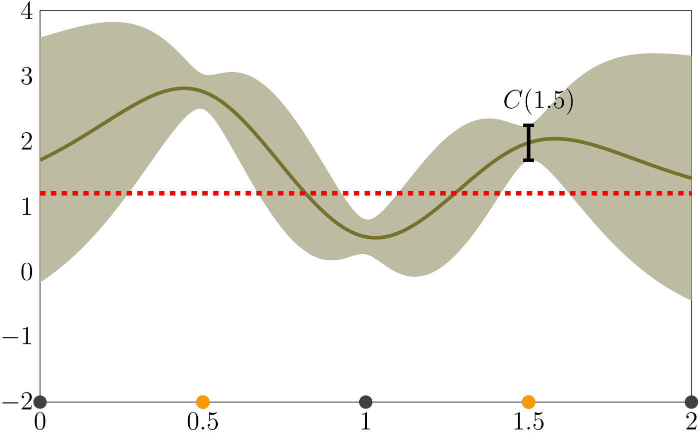

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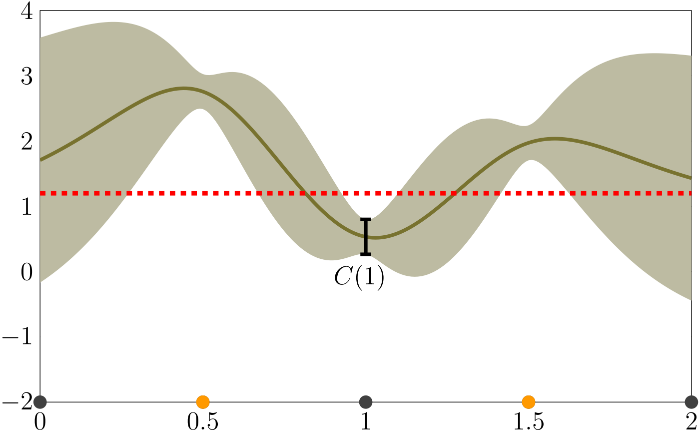

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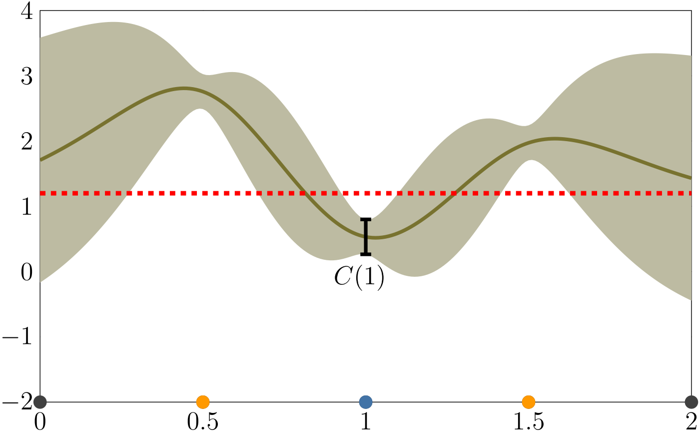

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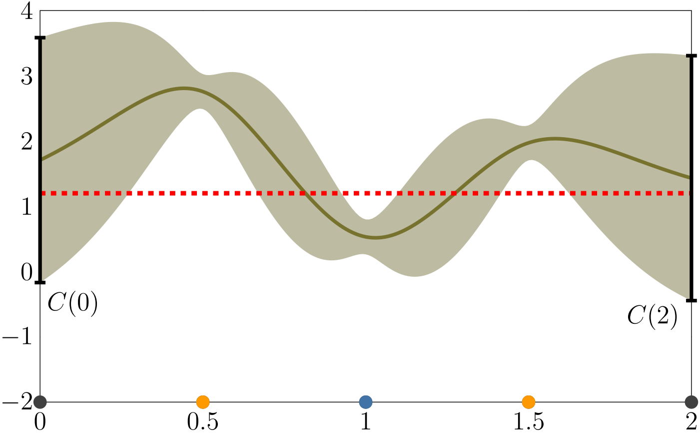

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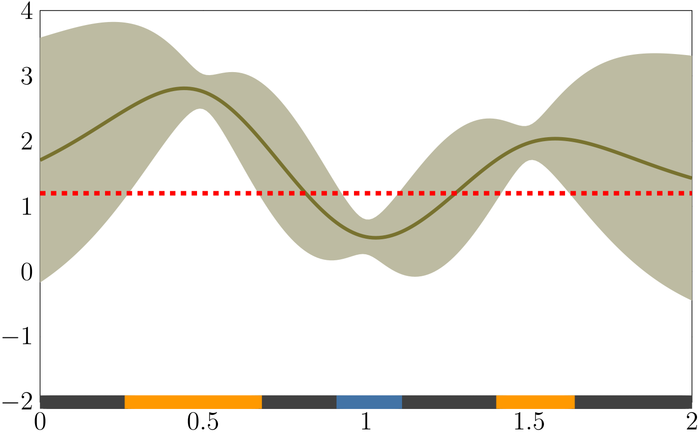

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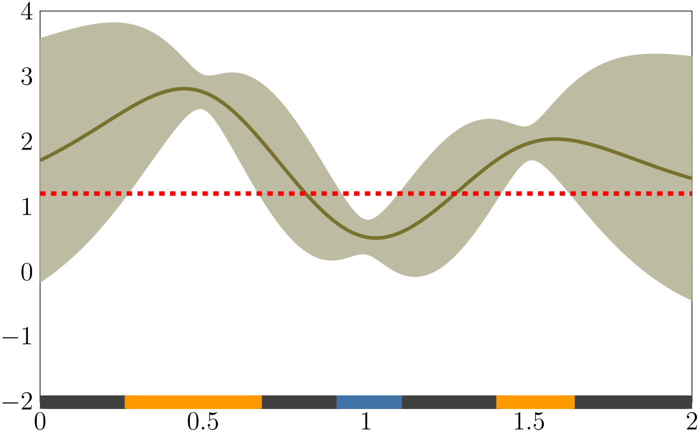

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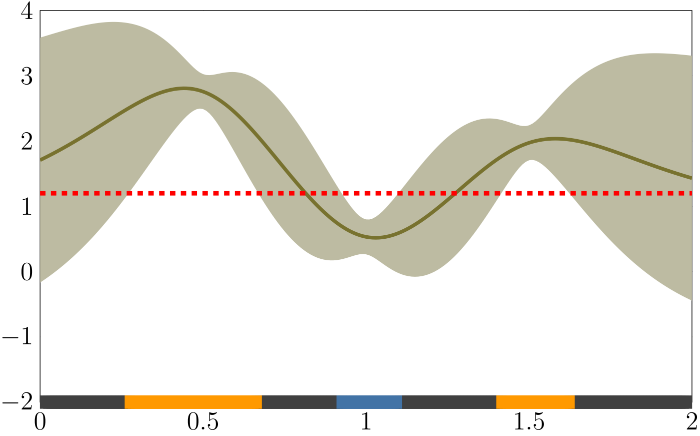

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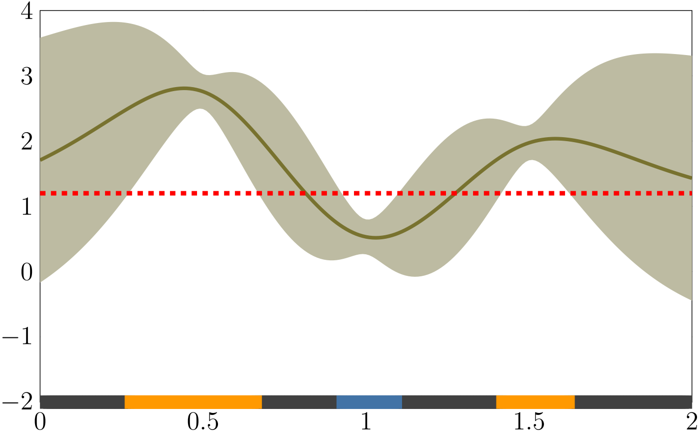

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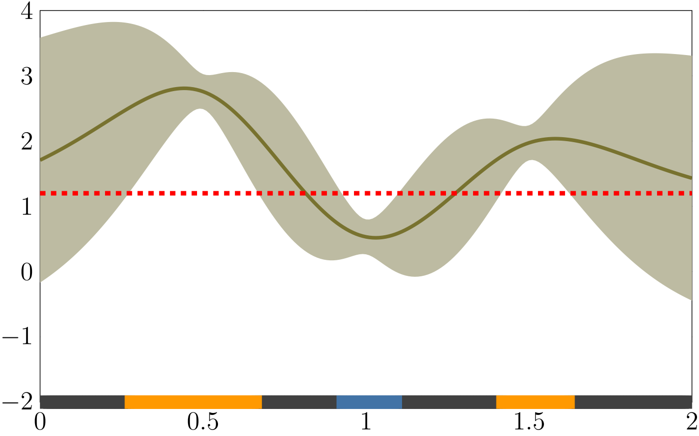

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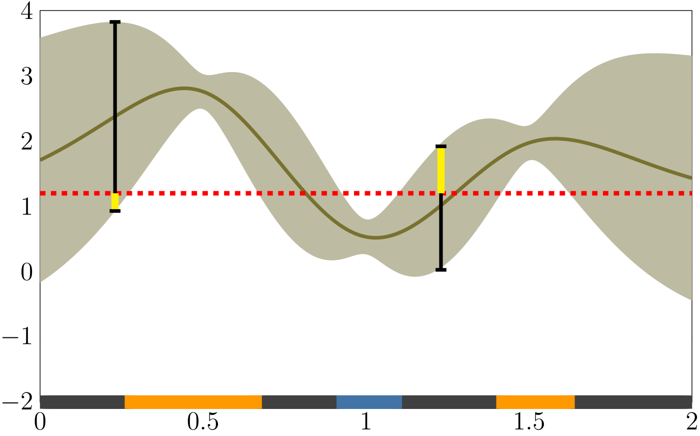

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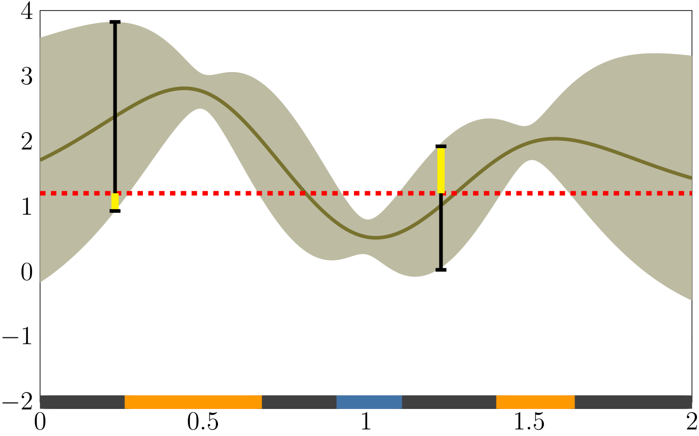

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Output: predicted super- and sublevel sets

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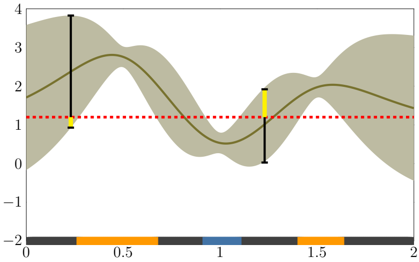
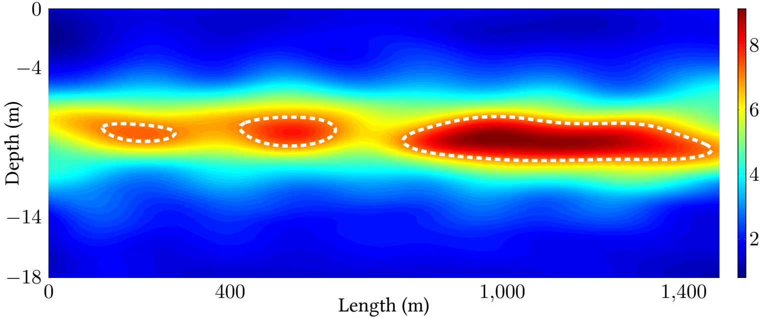
- Relax classification rules by an accuracy parameter ϵ

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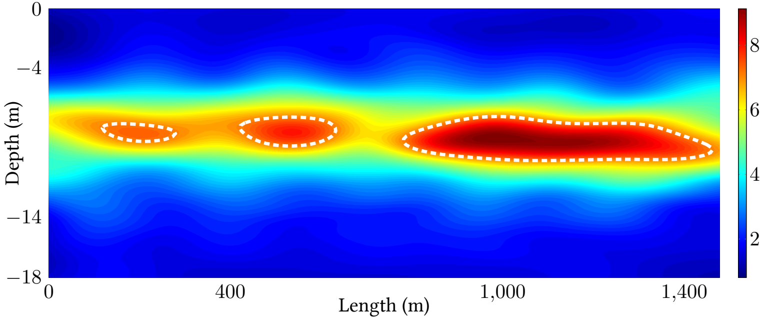
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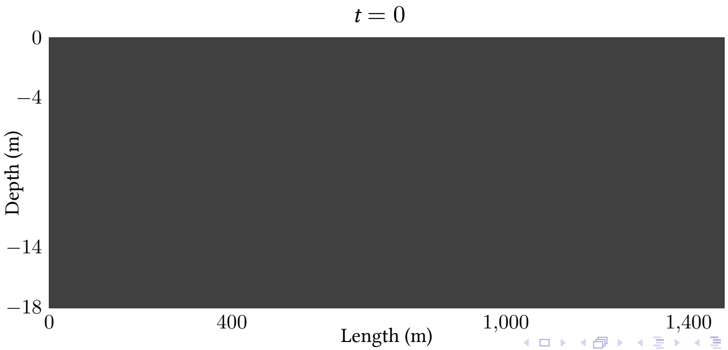
$t = 0$
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Depth (m)
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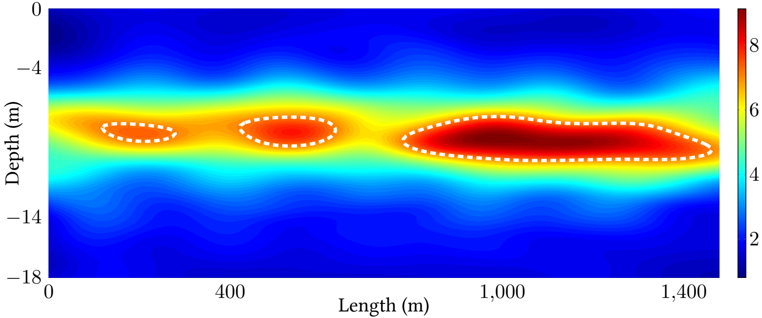
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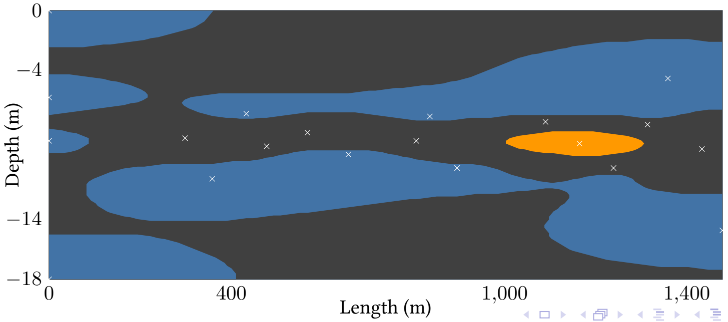
$t = 20$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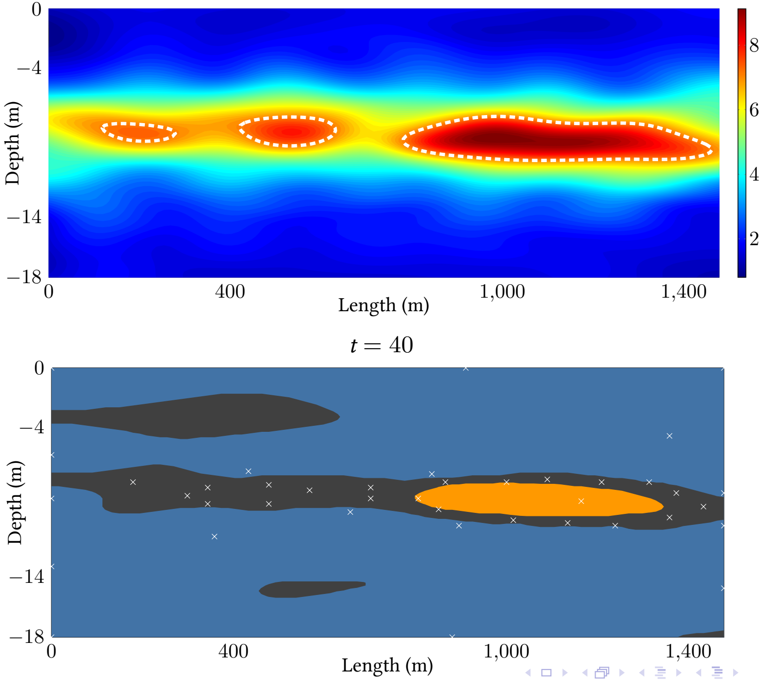
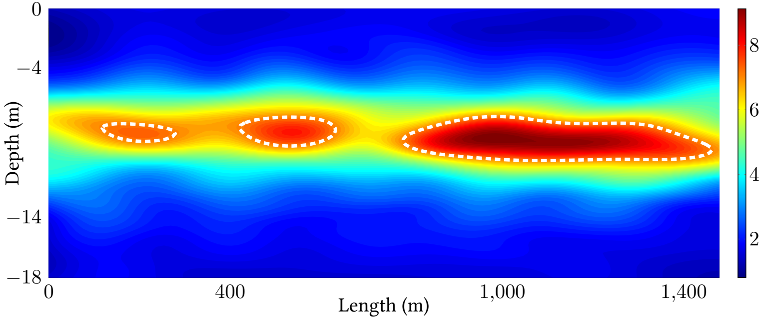
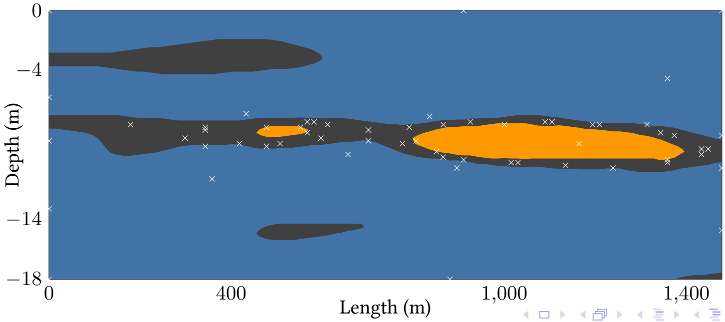
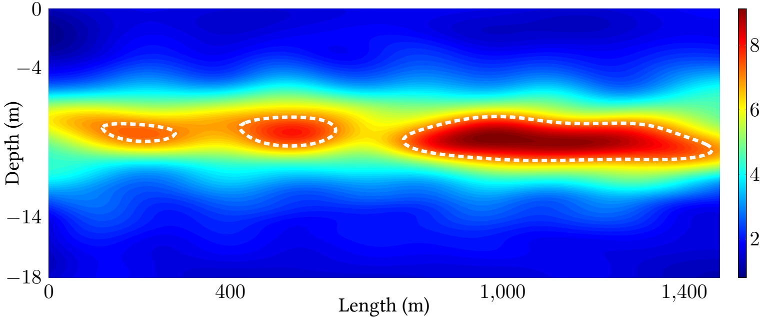
1,400
Length (m)


0
−4
−14
−18
Depth (m)
0
400
1,000
1,400
Length (m)
8
6
4
2
$t = 40$
0
−4
−14
−18
Depth (m)
0
400
1,000
1,400
Length (m)

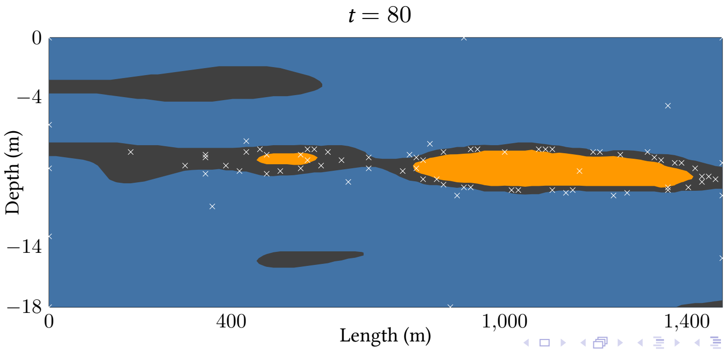

0
−4
−14
−18
Depth (m)
0
400
1,000
1,400
Length (m)
8
6
4
2

$t = 60$

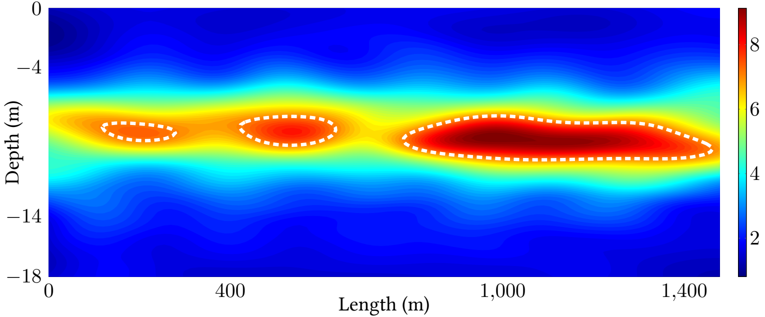

0
−4
−14
−18
Depth (m)
0
400
1,000
1,400
Length (m)



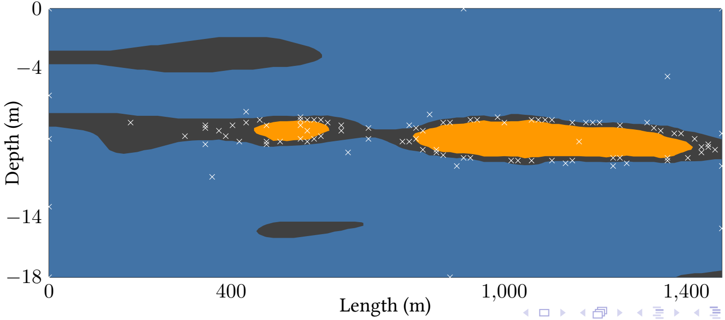
0
−4
−14
−18
Depth (m)
0
400
1,000
1,400
Length (m)
8
6
4
2

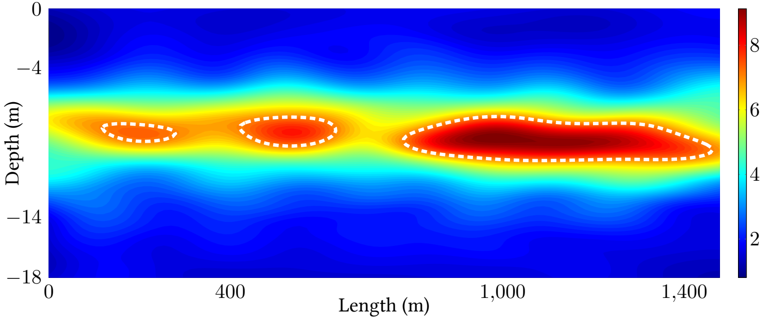


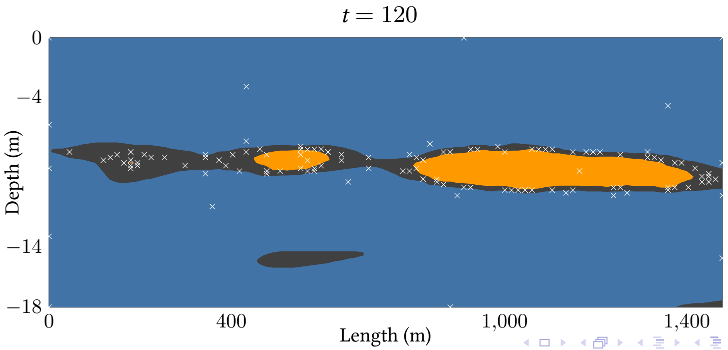
$t = 80$
0
−4
−14
−18
Depth (m)
0
400
1,000
1,400
Length (m)

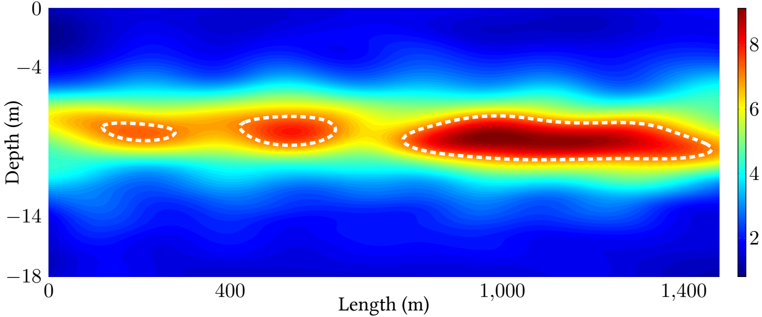

Depth (m)
0
−4
−14
−18
0
400
1,000
1,400
Length (m)
8
6
4
2

$t = 100$

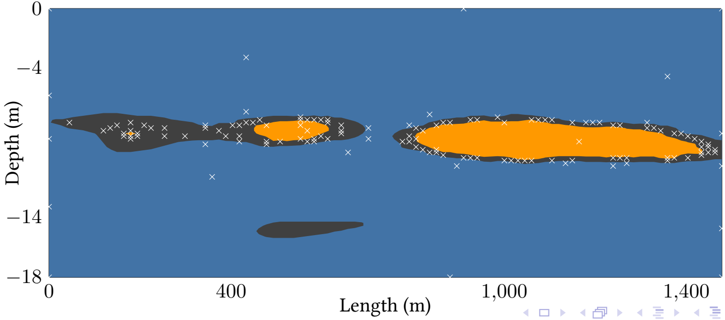

Depth (m)
0
−4
−14
−18
0
400
1,000
1,400
Length (m)

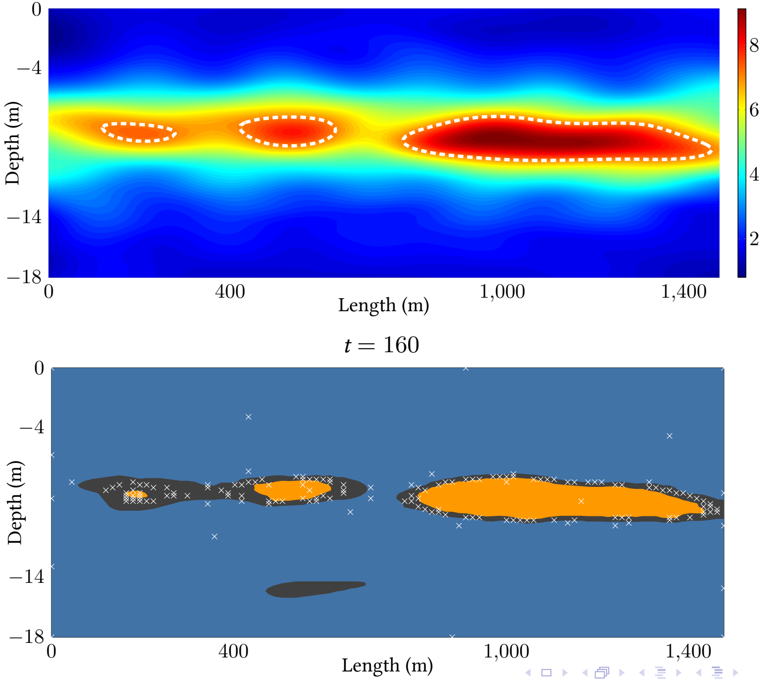

0
−4
−14
−18
Depth (m)
0
400
1,000
1,400
Length (m)
8
6
4
2


$t = 120$
0
−4
−14
−18
Depth (m)
0
400
1,000
1,400
Length (m)

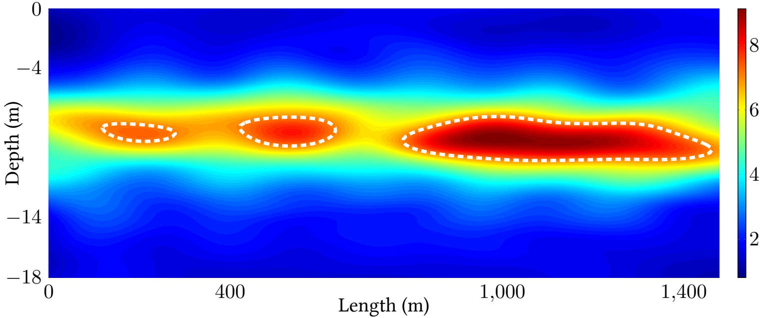

0
−4
−14
−18
Depth (m)
0
400
1,000
1,400
Length (m)
8
6
4
2

$t = 140$

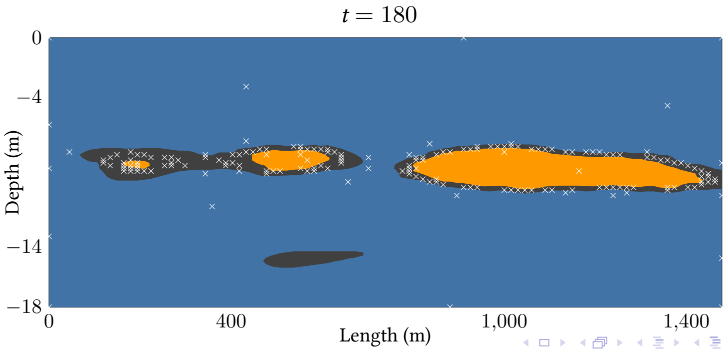

0
−4
−14
−18
Depth (m)
0
400
1,000
1,400
Length (m)



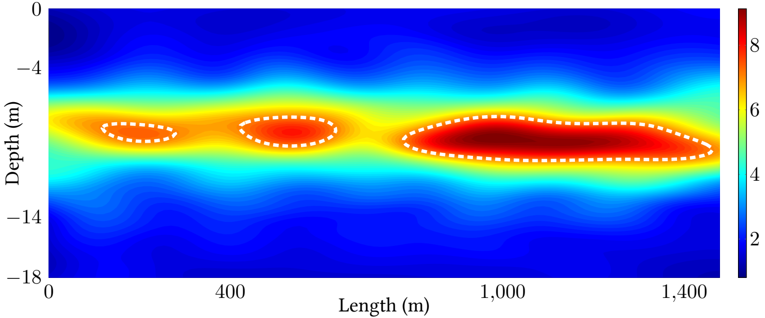
0
−4
−14
−18
Depth (m)
0
400
1,000
1,400
Length (m)
8
6
4
2
t = 160
0
−4
−14
−18
Depth (m)
0
400
1,000
1,400
Length (m)

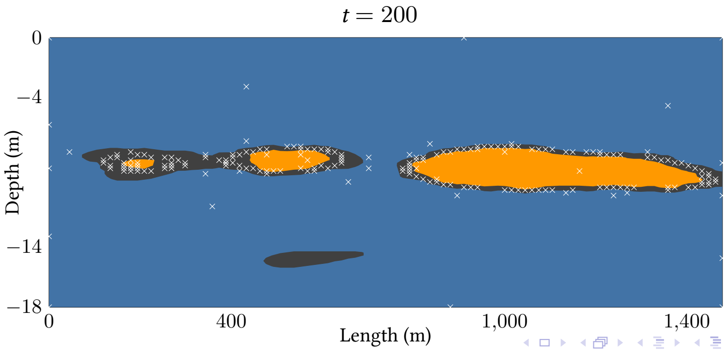


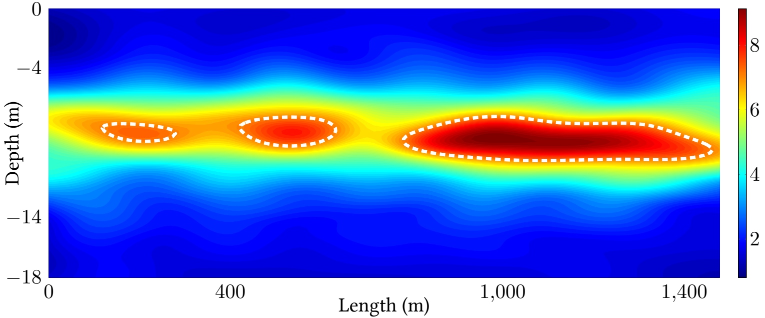
0
−4
−14
−18
Depth (m)
0
400
1,000
1,400
Length (m)
8
6
4
2

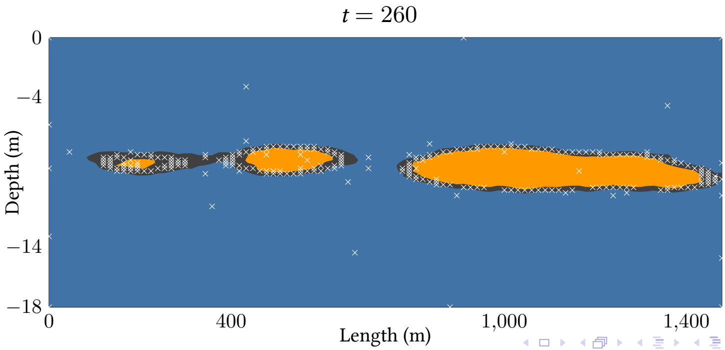


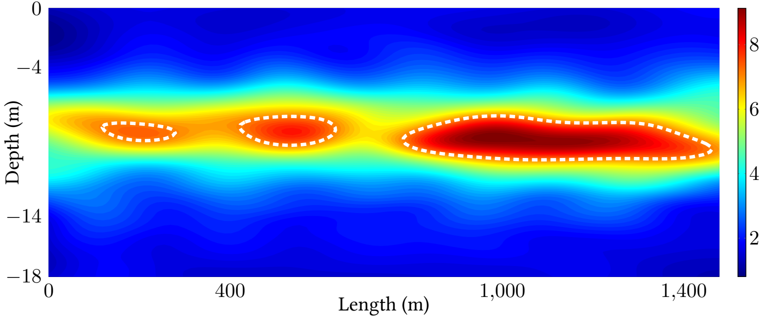
$t = 180$
0
−4
−14
−18
Depth (m)
0
400
1,000
1,400
Length (m)


0
−4
−14
−18
Depth (m)
0
400
1,000
1,400
Length (m)
8
6
4
2

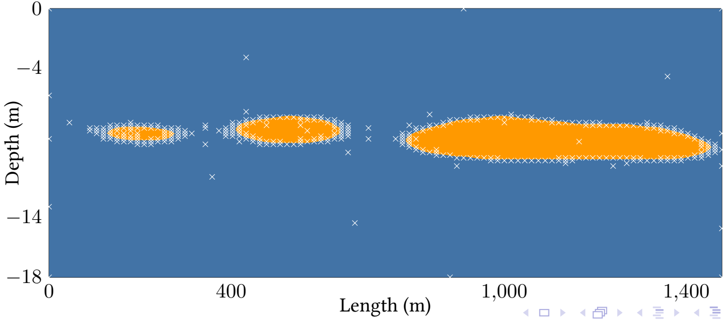

t = 200
0
−4
−14
−18
Depth (m)
0
400
1,000
1,400
Length (m)


0
−4
−14
−18
Depth (m)
0
400
1,000
1,400
Length (m)
8
6
4
2


t = 260
0
−4
−14
−18
Depth (m)
0
400
1,000
1,400
Length (m)


0
−4
−14
−18
Depth (m)
0
400
1,000
1,400
Length (m)
8
6
4
2

$t = 354$


0
−4
−14
−18
Depth (m)
0
400
1,000
1,400
Length (m)

Theorem (Convergence of LSE)

For any $h \in \mathbb{R}$, $\delta \in (0,1)$, *and* $\epsilon > 0$, *if* $\beta_t = 2\log(|D|\pi^2 t^2/(6\delta))$, *LSE terminates after at most* T *iterations, where* T *is the smallest positive integer satisfying*

$$\frac{T}{\beta_T \gamma_T} \geq \frac{C_1}{4\epsilon^2},$$

where $C_1 = 8/\log(1+\sigma^{-2})$.
Furthermore, with probability at least $1-\delta$, *the algorithm returns an* ϵ*-accurate solution, that is*

$$\Pr\left\{\max_{\boldsymbol{x}\in D} \ell_h(\boldsymbol{x}) \leq \epsilon\right\} \geq 1-\delta.$$

Theorem (Simplified)

If we choose β *appropriately (large enough), then:*

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- *LSE terminates after a number of iterations T*

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    1. *smoother kernel $\Rightarrow T\downarrow$*

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    1. *smoother kernel* $\Rightarrow T\downarrow$
    2. $\sigma\uparrow\ \Rightarrow T\uparrow$

Theorem (Simplified)

If we choose β appropriately (large enough), then:

- *LSE terminates after a number of iterations T*
  1. *smoother kernel* $\Rightarrow T\downarrow$
  2. $\sigma\uparrow \Rightarrow T\uparrow$
  3. $\epsilon\uparrow \Rightarrow T\downarrow$

Theorem (Simplified)

If we choose β appropriately (large enough), then:

- *LSE terminates after a number of iterations T*
    1. *smoother kernel* $\Rightarrow T\downarrow$
    2. $\sigma\uparrow \Rightarrow T\uparrow$
    3. $\epsilon\uparrow \Rightarrow T\downarrow$
- *The solution returned is ϵ-accurate with high probability*

Theorem (Simplified)

If we choose β appropriately (large enough), then:

- *LSE terminates after a number of iterations T*
    1. *smoother kernel* $\Rightarrow T\downarrow$
    2. $\sigma\uparrow\Rightarrow T\uparrow$
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Experiments

1. LSE

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2. Maximum variance sampling:

$$\boldsymbol{x}_t = \operatorname*{argmax}_{\boldsymbol{x} \in D} \sigma_{t-1}(\boldsymbol{x})$$

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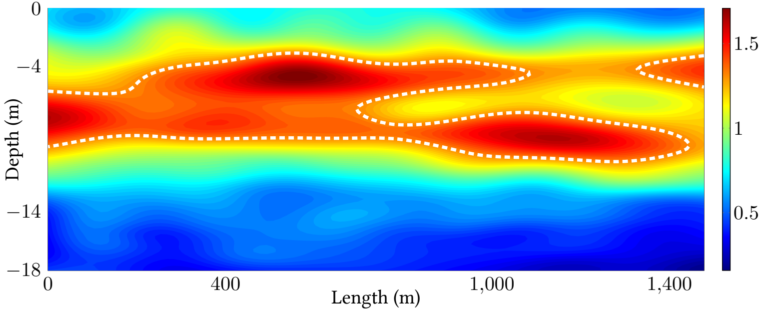
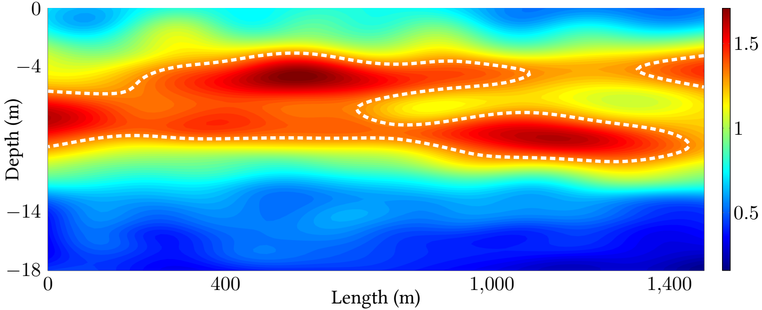
1. LSE
2. Maximum variance sampling:

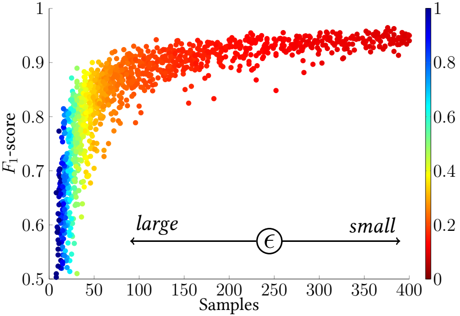
$$\boldsymbol{x}_t = \operatorname*{argmax}_{\boldsymbol{x} \in D} \sigma_{t-1}(\boldsymbol{x})$$

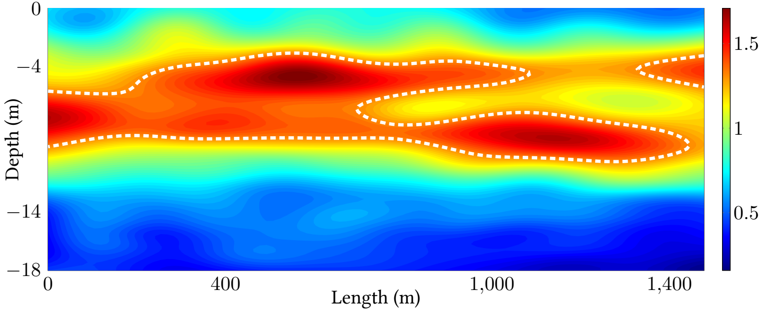
3. State of the art "straddle" heuristic (Bryan *et al.*, 2005):

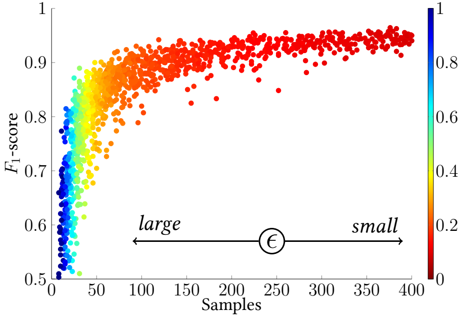
$$\boldsymbol{x}_t \approx \operatorname*{argmax}_{\boldsymbol{x} \in D} a_{t-1}(\boldsymbol{x}) \qquad \text{(for } \beta_t^{1/2} = 1.96\text{)}$$

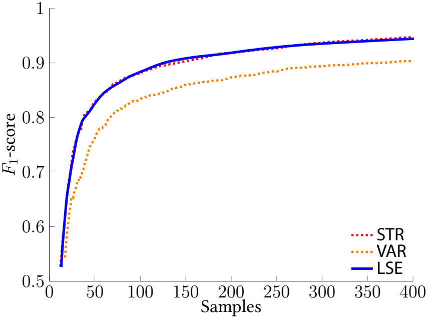

0
−4
−14
−18
Depth (m)
0
400
1,000
1,400
Length (m)
1.5
1
0.5

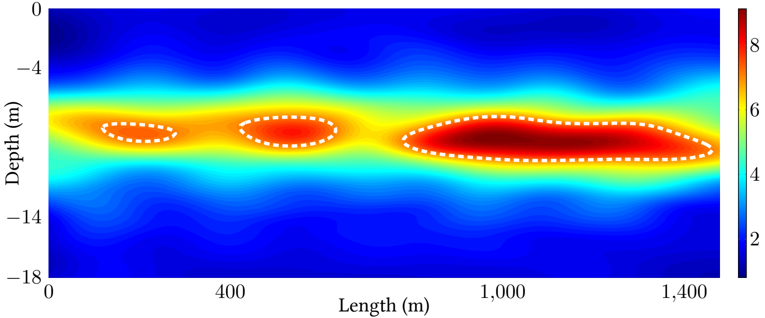

0
−4
−14
−18
Depth (m)
0
400
1,000
1,400
Length (m)
1.5
1
0.5

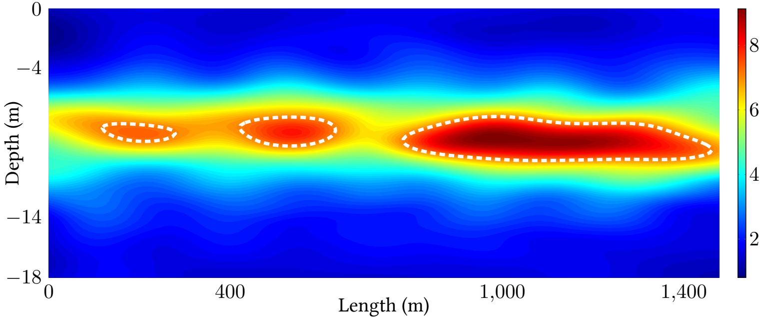

1
0.9
0.8
0.7
0.6
0.5
F_1-score
0
50
100
150
200
250
300
350
400
Samples
large
ϵ
small
1
0.8
0.6
0.4
0.2
0


0
−4
−14
−18
Depth (m)
0
400
1,000
1,400
Length (m)
1.5
1
0.5

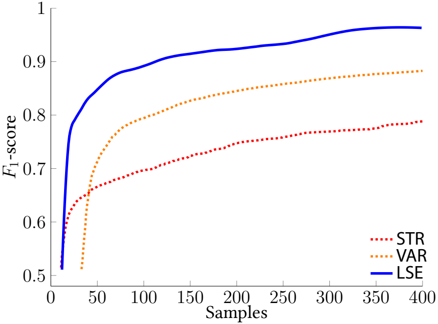

1
0.9
0.8
0.7
0.6
0.5
F_1-score
0
50
100
150
200
250
300
350
400
Samples
large
ϵ
small
1
0.8
0.6
0.4
0.2
0


1
0.9
0.8
0.7
0.6
0.5
F_1-score
0
50
100
150
200
250
300
350
400
Samples
STR
VAR
LSE

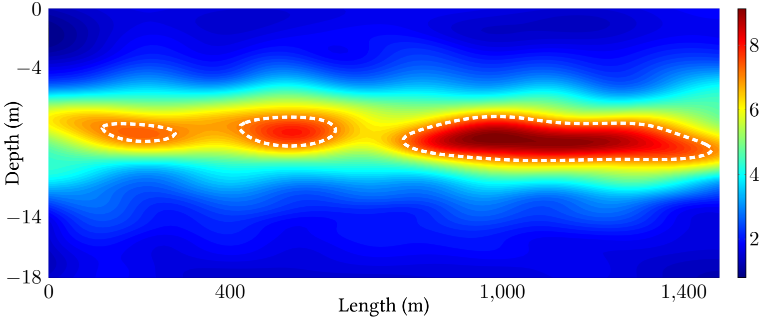

0
−4
−14
−18
Depth (m)
0
400
1,000
1,400
Length (m)
8
6
4
2

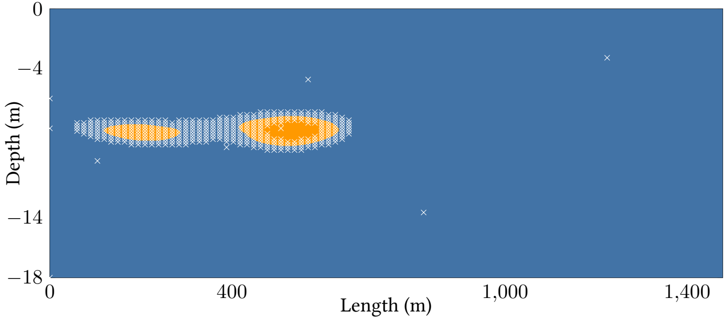


0
−4
−14
−18
Depth (m)
0
400
1,000
1,400
Length (m)
8
6
4
2



1
0.9
0.8
0.7
0.6
0.5
F_1-score
0
50
100
150
200
250
300
350
400
Samples
STR
VAR
LSE


0
−4
−14
−18
Depth (m)
0
400
1,000
1,400
Length (m)
8
6
4
2


0
−4
−14
−18
Depth (m)
0
400
1,000
1,400
Length (m)

Implicit threshold level

- What if we don't know the threshold level h?

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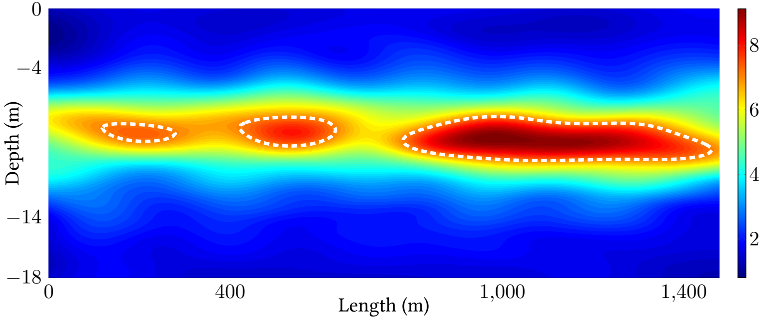
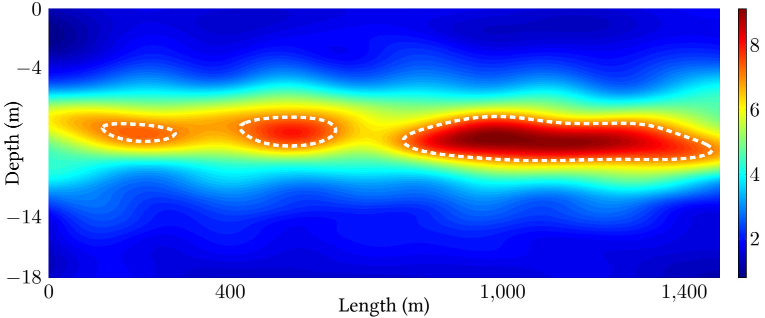
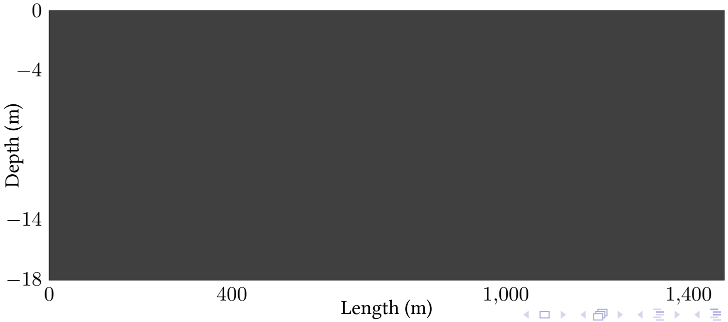
- We extend LSE (and its theory!) to this setting
- **LSE$_{\textbf{imp}}$** algorithm:
  - h is now an estimated quantity $\rightarrow$ modified classification rules
  - Need to accurately estimate the maximum $\rightarrow$ modified selection rule

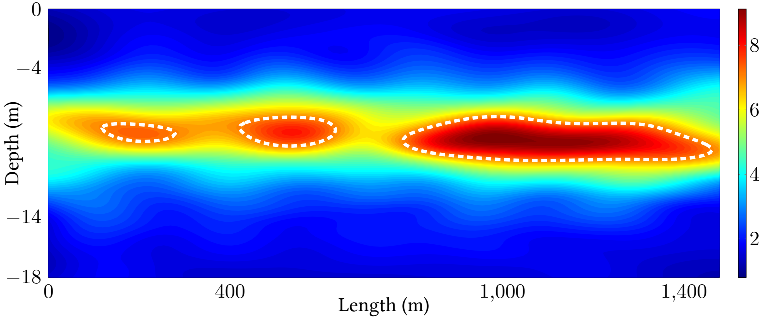


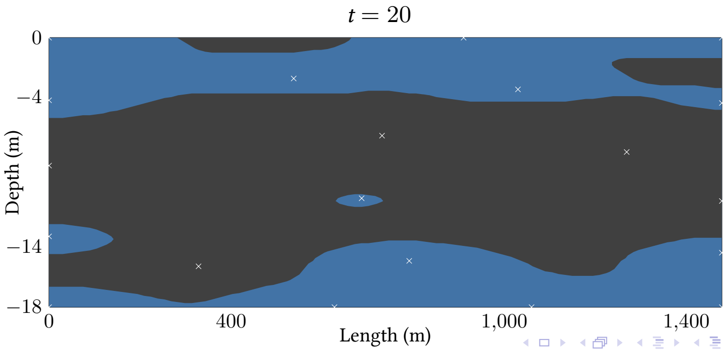
0
−4
−14
−18
Depth (m)
0
400
1,000
1,400
Length (m)
8
6
4
2

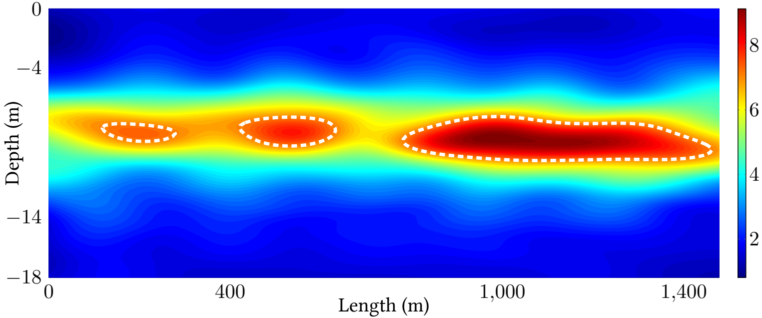

0
−4
−14
−18
Depth (m)
0
400
1,000
1,400
Length (m)
8
6
4
2

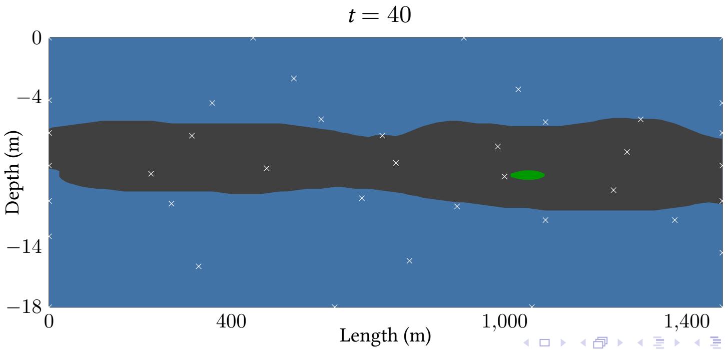
$t = 0$

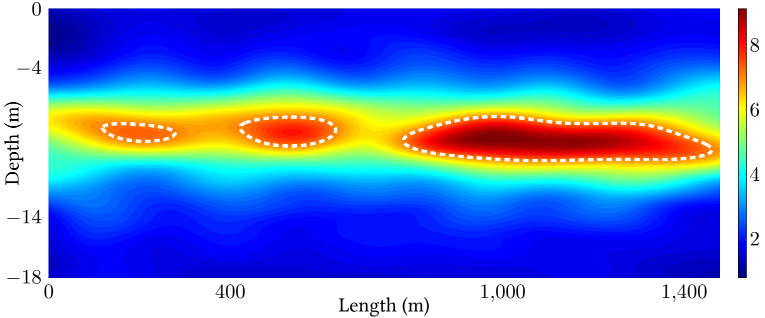

0
−4
−14
−18
Depth (m)
0
400
1,000
1,400
Length (m)


0
−4
−14
−18
Depth (m)
0
400
1,000
1,400
Length (m)
8
6
4
2

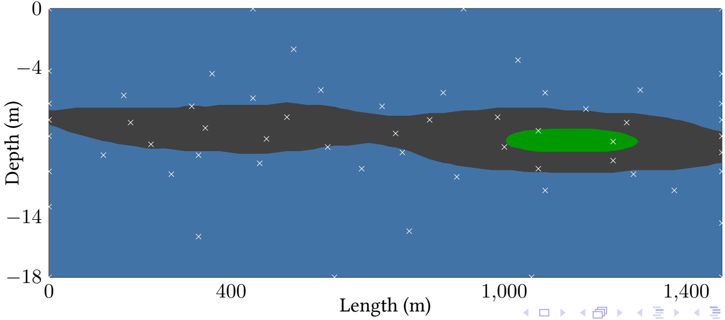

$t = 20$
0
−4
−14
−18
Depth (m)
0
400
1,000
1,400
Length (m)

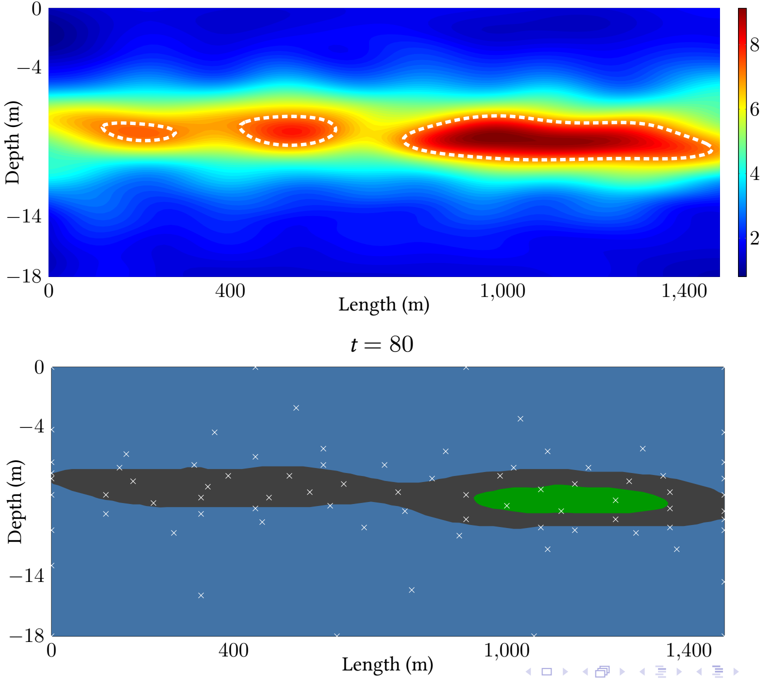

0
−4
−14
−18
Depth (m)
0
400
1,000
1,400
Length (m)
8
6
4
2

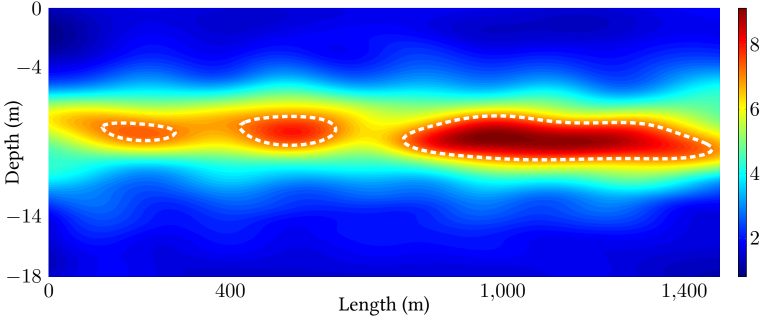

$t = 40$
0
−4
−14
−18
Depth (m)
0
400
1,000
1,400
Length (m)


0
−4
−14
−18
Depth (m)
0
400
1,000
1,400
Length (m)
8
6
4
2

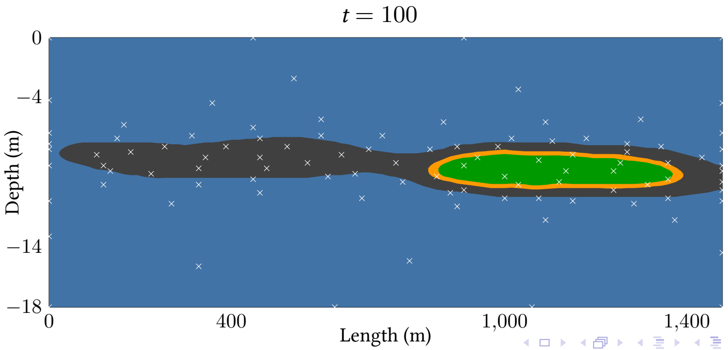
$t = 60$


0
−4
−14
−18
Depth (m)
0
400
1,000
1,400
Length (m)

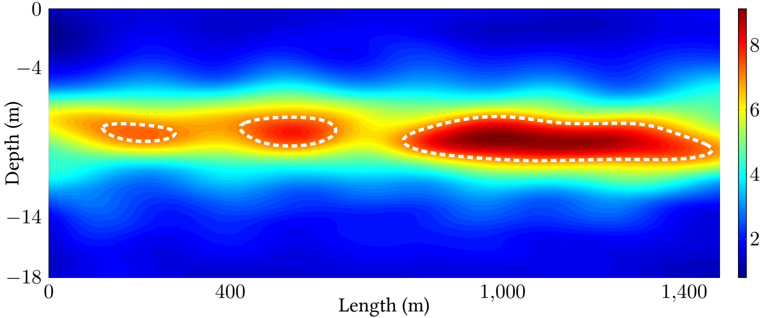

0
−4
−14
−18
Depth (m)
0
400
1,000
1,400
Length (m)
8
6
4
2
$t = 80$
0
−4
−14
−18
Depth (m)
0
400
1,000
1,400
Length (m)



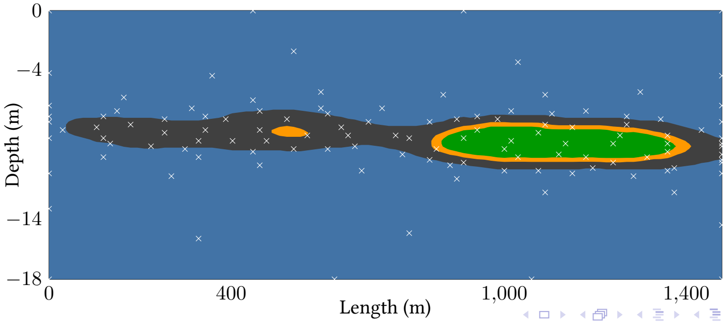
0
−4
−14
−18
Depth (m)
0
400
1,000
1,400
Length (m)
8
6
4
2

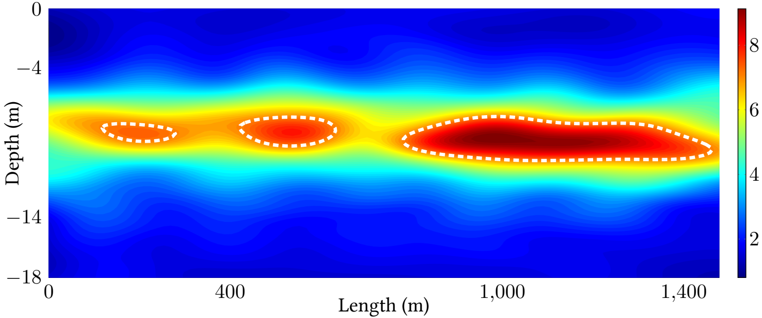


$t = 100$
0
−4
−14
−18
Depth (m)
0
400
1,000
1,400
Length (m)


0
−4
−14
−18
Depth (m)
0
400
1,000
1,400
Length (m)
8
6
4
2

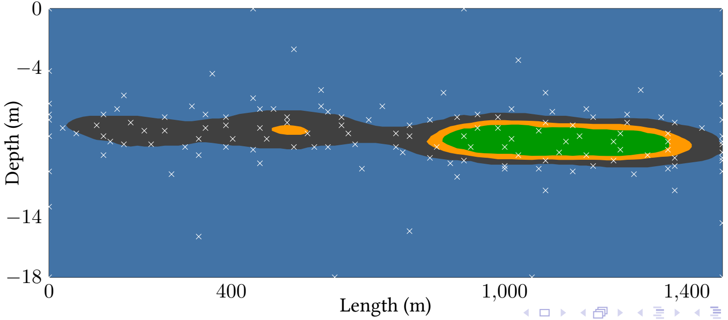
$t = 120$


0
−4
−14
−18
Depth (m)
0
400
1,000
1,400
Length (m)

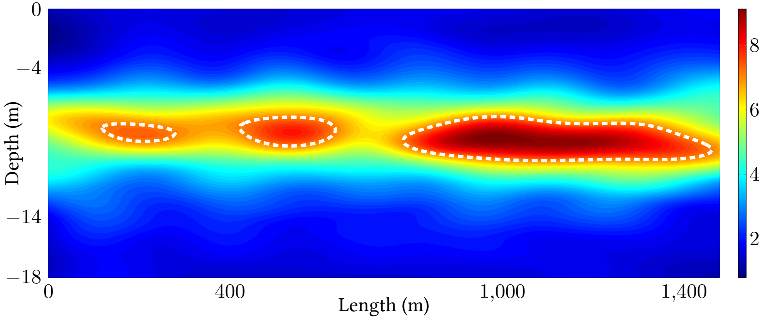


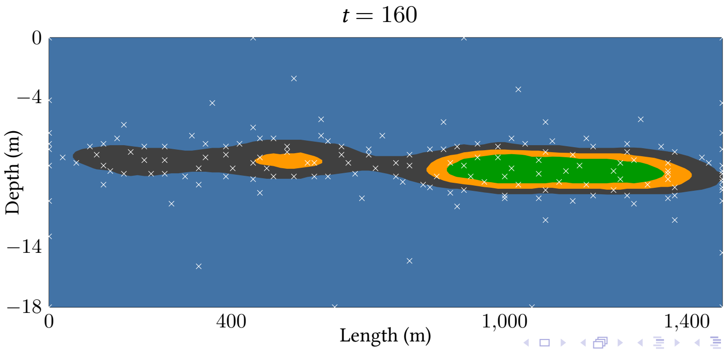
0
−4
−14
−18
Depth (m)
0
400
1,000
1,400
Length (m)
8
6
4
2

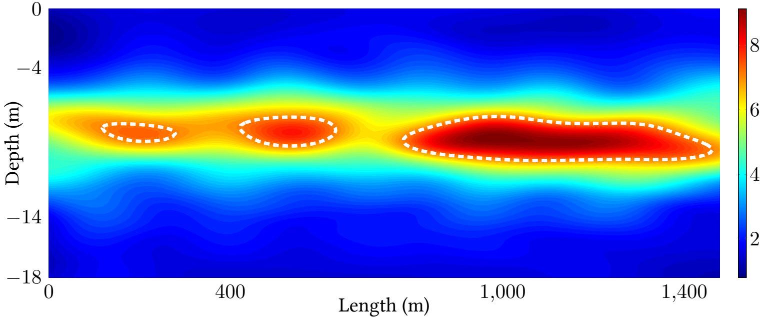
$t = 140$



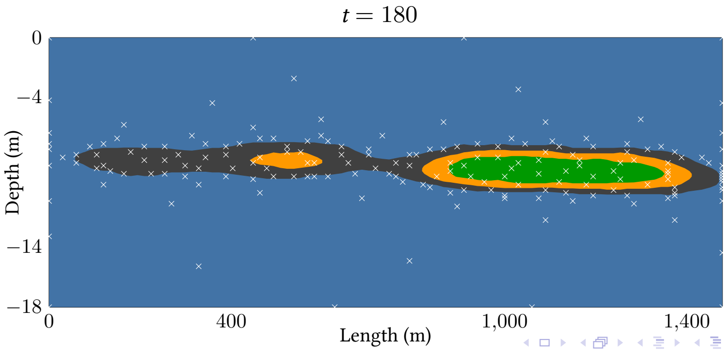
0
−4
−14
−18
Depth (m)
0
400
1,000
1,400
Length (m)


0
−4
−14
−18
Depth (m)
0
400
1,000
1,400
Length (m)
8
6
4
2

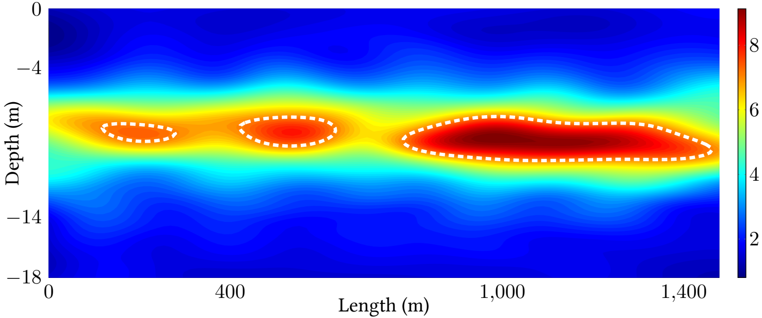

$t = 160$
0
−4
−14
−18
Depth (m)
0
400
1,000
1,400
Length (m)

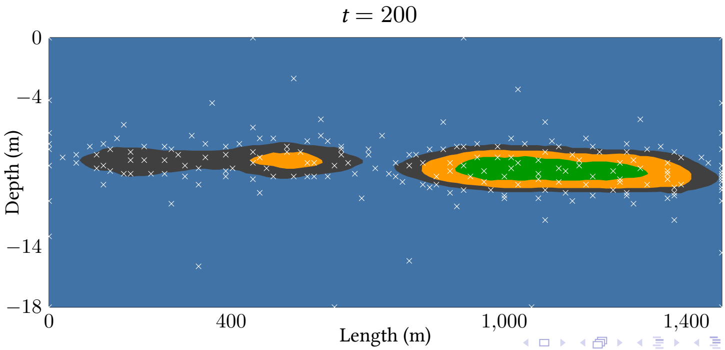


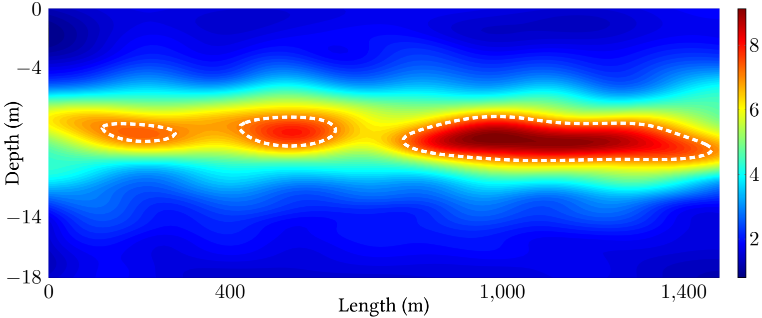
0
−4
−14
−18
Depth (m)
0
400
1,000
1,400
Length (m)
8
6
4
2

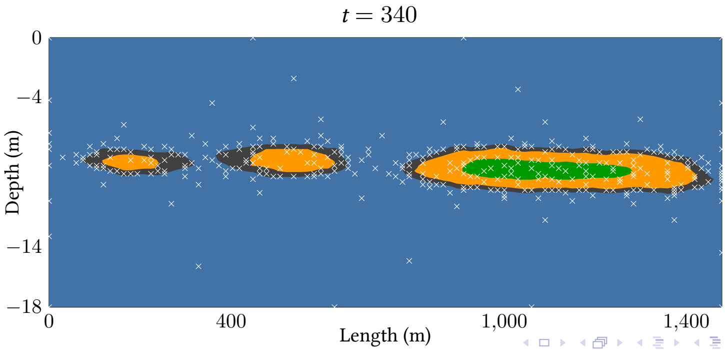


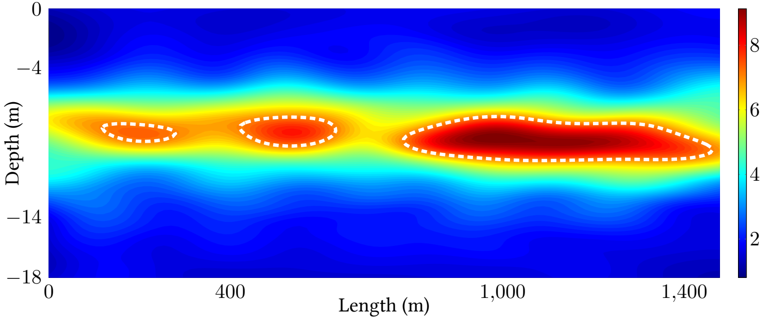
t = 180
0
−4
−14
−18
Depth (m)
0
400
1,000
1,400
Length (m)

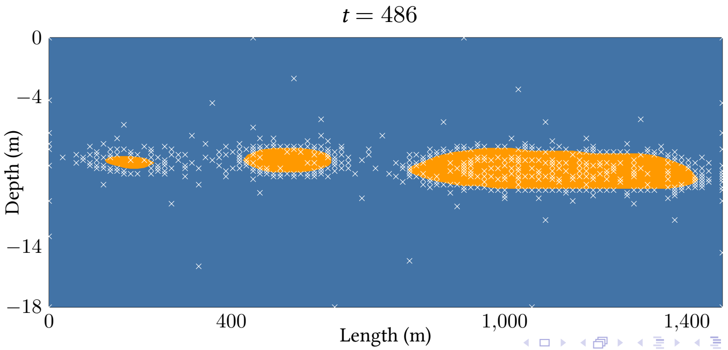

0
−4
−14
−18
Depth (m)
0
400
1,000
1,400
Length (m)
8
6
4
2

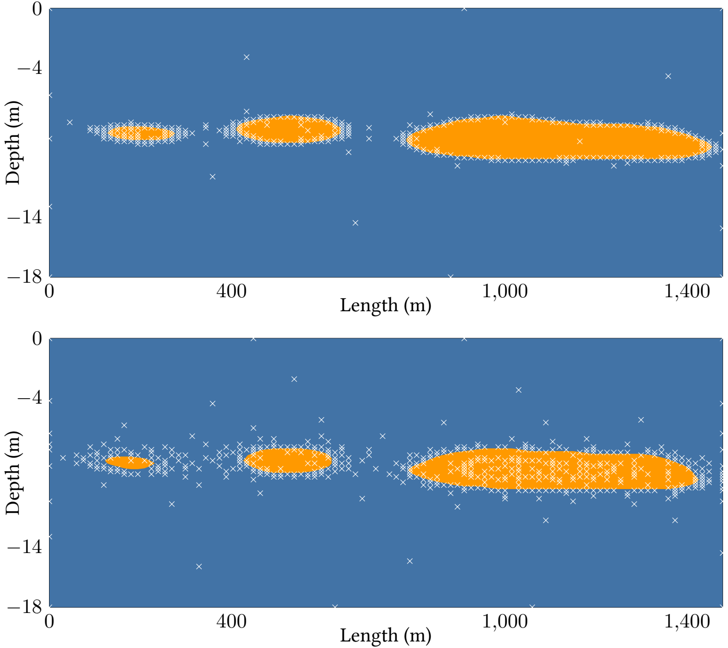

$t = 200$
0
−4
−14
−18
Depth (m)
0
400
1,000
1,400
Length (m)


0
−4
−14
−18
Depth (m)
0
400
1,000
1,400
Length (m)
8
6
4
2


$t = 340$
0
−4
−14
−18
Depth (m)
0
400
1,000
1,400
Length (m)


0
−4
−14
−18
Depth (m)
0
400
1,000
1,400
Length (m)
8
6
4
2


t = 486
0
−4
−14
−18
Depth (m)
0
400
1,000
1,400
Length (m)


0
−4
−14
−18
Depth (m)
0
400
1,000
1,400
Length (m)
0
−4
−14
−18
Depth (m)
0
400
1,000
1,400
Length (m)

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  What about the traveling distance between measurements?

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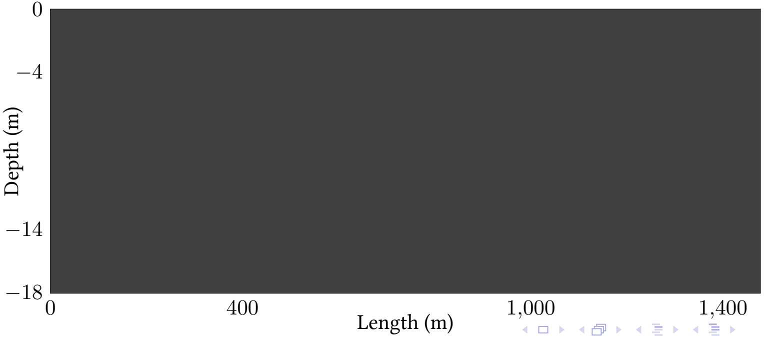

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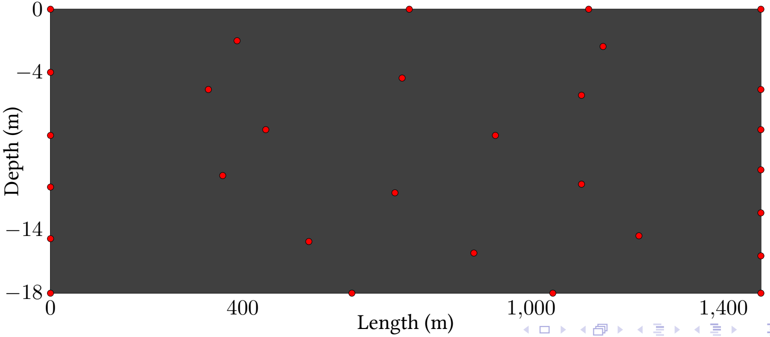

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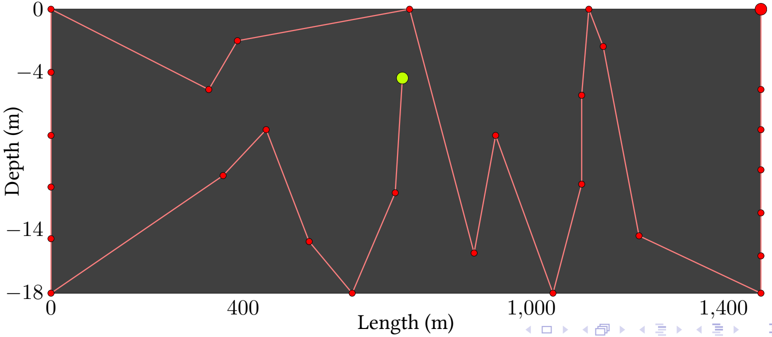

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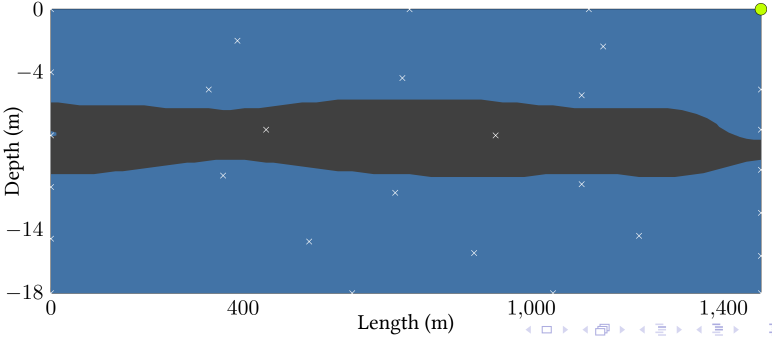

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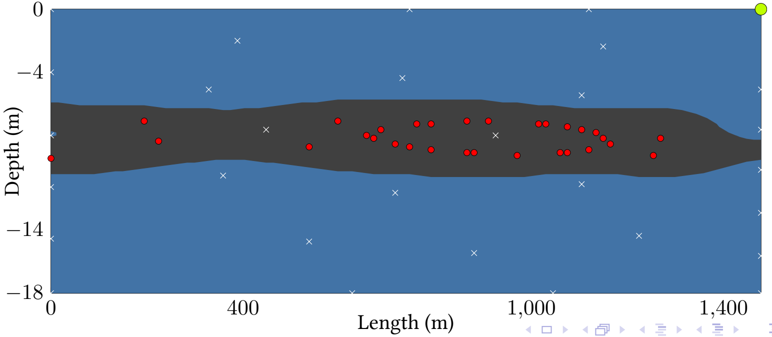

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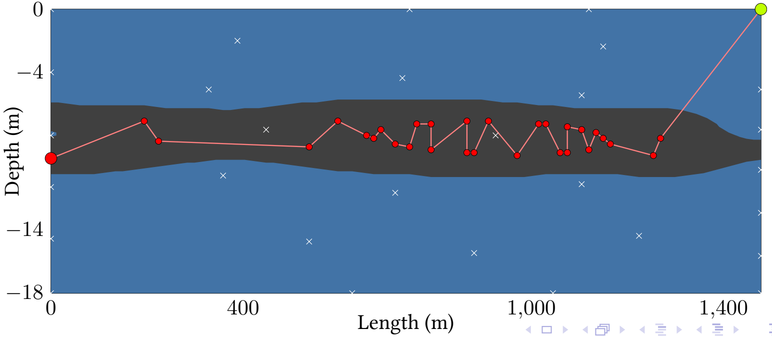

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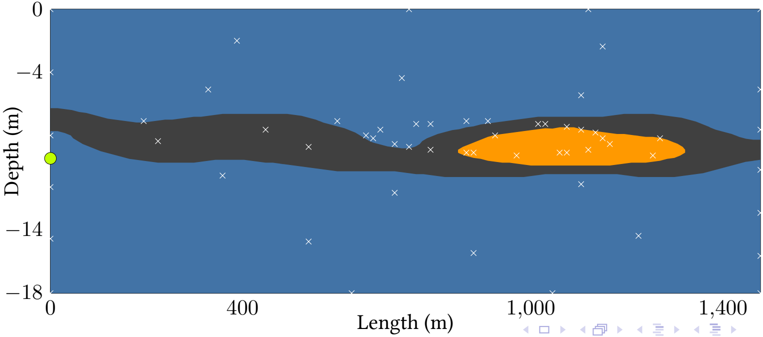

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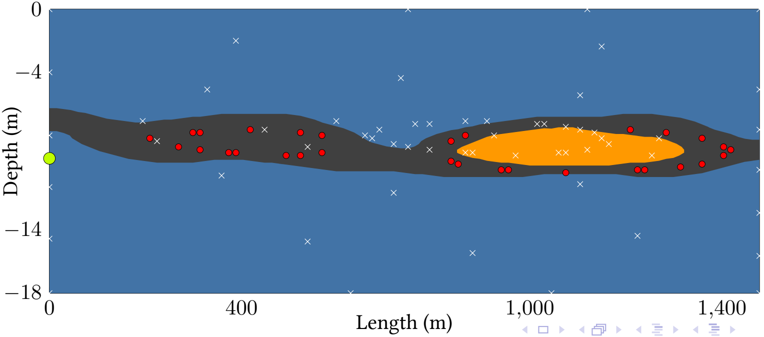

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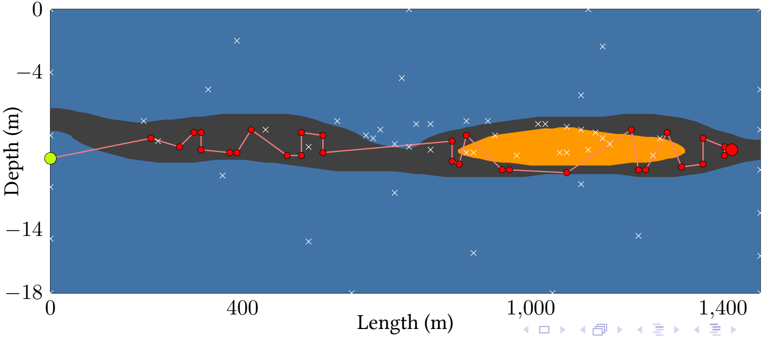

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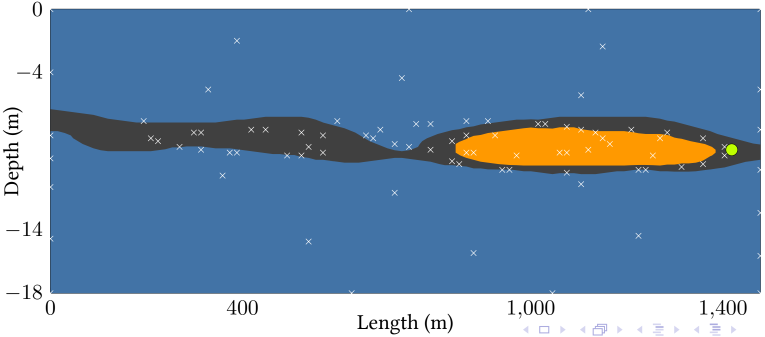

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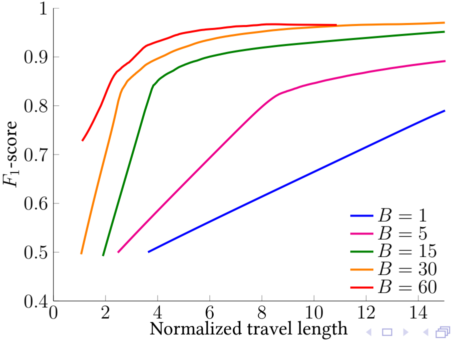

In our paper, more...

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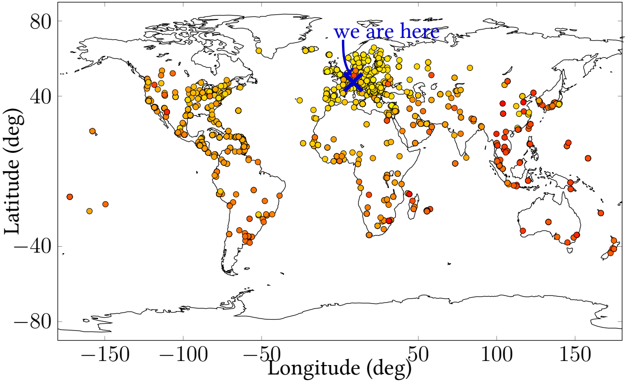
- ...applications: estimate world regions of low internet latency



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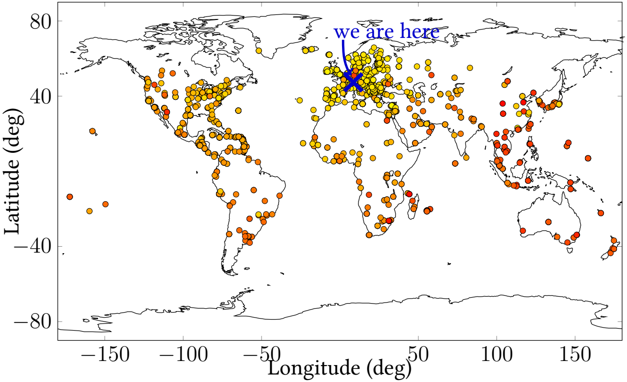

- ...experimental results

Summary

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- LSE algorithm:

Theoretical guarantees

Theorem (Convergence of LSE)

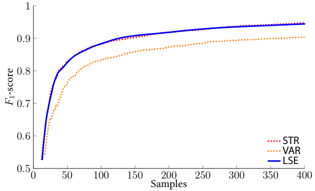
For any $h \in \mathbb{R}$, $\delta \in (0,1)$, and $\epsilon > 0$, if $\beta_t = 2\log(|D|\pi^2 t^2/(6\delta))$, LSE terminates after at most T iterations, where T is the smallest positive integer satisfying

$$\frac{T}{\beta_T \gamma_T} \geq \frac{C_1}{4\epsilon^2},$$

where $C_1 = 8/\log(1+\sigma^{-2})$.
Furthermore, with probability at least $1-\delta$, the algorithm returns an ϵ-accurate solution, that is

$$\Pr\left\{\max_{\boldsymbol{x} \in D} \ell_h(\boldsymbol{x}) \leq \epsilon\right\} \geq 1-\delta.$$

Competitive with the state of the art



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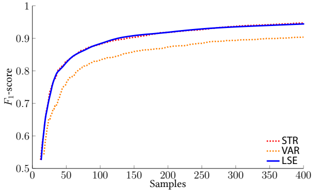
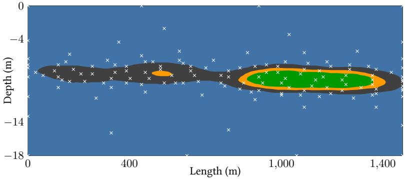


- Two useful extensions:

Implicit threshold level (LSE_{imp})



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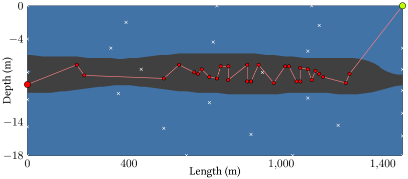

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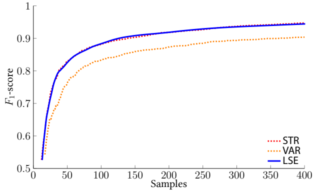

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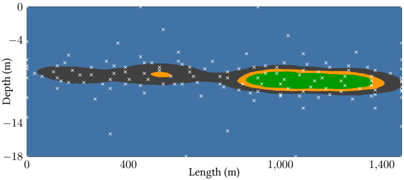

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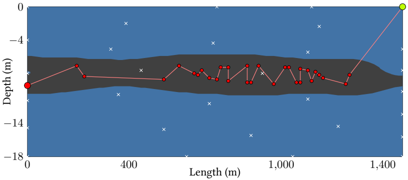

- Look out for algae when swimming in Lake Zurich! :)