... and all that JAZZ...

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Support, Newton Polytope and Other Things

Consider a polynomial $pol(z_1, ..., z_m) = \Sigma\, a_{r_1,...,r_m} \Pi_{1 \leq i \leq m}\, z_i^{r_i}$;
The support is defined as $supp(pol) = \{(r_1, ..., r_m) \in Z_+^m : a_{r_1,...,r_m} \neq 0$

The **Newton Polytope** is defined as
$NP(pol) = CO(supp(pol))$, i.e. the convex hull of the support.

A few examples:

1. $pol(z_1, ..., z_m) = 1 + Sym_1(z_1, ..., z_m) + ... + Sym_m(z_1, ..., z_m)$ then $NP(pol)$ is the box
$Box_m = \{(x_1, ..., x_m) : 0 \leq x_i \leq 1\}$.

2. $pol(z_1, ..., z_m) = \Sigma_{r_1+...+r_m \leq k}\, \Pi_{1 \leq i \leq m}\, z_i^{r_i}$
then $NP(pol) \subset kPyr_m$, where the pyramid
$Pyr_m = \{(x_1, ..., x_m) : \Sigma_{1 \leq i \leq m}\, x_i \leq 1; x_i \geq 0\}$

Degree of a subset

$deg_{pol}(S) = \max_{(r_1,...,r_m) \in supp(pol)} \Sigma_{i \in S} r_i$.

Note that

$(r_1, ..., r_m) \in supp(pol) \Rightarrow \Sigma_{i \in S} r_i \leq deg_{pol}(S), S \subset \{1, ..., m\}$

$(x_1, ..., x_m) \in NP(pol) \Rightarrow \Sigma_{i \in S} x_i \leq deg_{pol}(S), S \subset \{1, ..., m\}$;

Example 0.1: A is $n \times n$ a non-negat. matrix; $Col(j) = \{i : A(i, j) > 0\}$;

$Prod_A(x_1, ..., x_n) =: \Pi_{1 \leq i \leq n} \Sigma_{1 \leq j \leq n} A(i, j) x_j$.

For this polynomial $deg_{Prod_A}(S) = | \cup_{j \in S} Col(j)|$ and $(1, 1, ..., 1) \in supp(Prod_A) \Rightarrow | \cup_{j \in S} Col(j)| \geq |S|$. ∎

Submodular Functions:

$f(S_1 \cup S_2) + f(S_1 \cap S_2) \leq f(S_1) + f(S_2), S_1, S_2 \subset \{1, ..., m\}$

Capacities

Applies to the case of nonnegative coefficients

$$Cap(pol) = \inf_{x_i > 0} \frac{pol(x_1, ..., x_m)}{\Pi_{1 \le i \le m} x_i}$$

Note that $Cap(pol) \ge \frac{\partial^m}{\partial x_1 ... \partial x_n} pol(0)$.

$$C_{pol}(y_1, ..., y_m) =: \inf_{x_i > 0} \frac{pol(x_1, ..., x_m)}{\Pi_{1 \le i \le m} (\frac{x_i}{y_i})^{y_i}}, y_i \ge 0.$$

Note that $C_{pol}(y_1, ..., y_m) > 0$ iff $(y_1, ..., y_m) \in NP(pol)$.

A (discrete) subset $S \subset Z^m$ is called D-convex if

$$CO(S) \cap Z^m = S.$$

...and natural definition of convexity/concavity of functions defined on (non-convex) sets:

$F(a_1 Z_1 + ... + a_k Z_k) \le (\ge) a_1 F(Z_1) + ... + a_k F(Z_k)$:

$Z_1, ..., Z_k, \Sigma_{1 \le i \le k} a_i Z_i \in S; a_i \ge 0, \Sigma_{1 \le i \le k} a_i = 1.$

The Minkowski sum and the Mixed Volume

Minkowski sum: $A+B=\{X+Y : X \in A, Y \in B\}$.

The convexity of A:

$a_1 A + a_2 A + ... + a_k A = (a_1 + ... + a_k)A : a_i > 0$.

$\mathbf{K} = (K_1, ..., K_n)$ is a n-tuple of convex compact subsets in the Euclidean space R^n;

$$V_{\mathbf{K}}(\lambda_1, ..., \lambda_n) =: Vol(\lambda_1 K_1 + \cdots + \lambda_n K_n), \lambda_i \geq 0.$$

Herman Minkowski proved in 1903(?) that $V_{\mathbf{K}}$ is a homogeneous polynomial with non-negative coefficients. The mixed volume:

$$V(K_1, ..., K_n) =: \frac{\partial^n}{\partial \lambda_1 ... \partial \lambda_n} V_{\mathbf{K}}(0, ..., 0).$$

i.e. the mixed volume $V(K_1, ..., K_n)$ is the coefficient of the monomial $\Pi_{1 \leq i \leq n} \lambda_i$ in the Minkowski polynomial $V_{\mathbf{K}}$ (the mixed derivative).

Bernshtein's theorem (1975): number of isolated toric solutions of the system of polynomial equations $p_i(x_1, ..., x_n) = 0; 1 \leq i \leq n$ **is bounded by (and generically equal to) the mixed volume** $V(NP(p_1), ..., NP(p_n))$**.**

Example 0.2 $deg(p_i) \leq D_i$, i.e. $NP(p_i) \subset D_i Pyr_n, 1 \leq i \leq n$. It follows that

$V(NP(p_1), ..., NP(p_n)) \leq V(D_1 Pyr_n, ..., D_n Pyr_n) =$

$= \Pi_{1 \leq i \leq n} D_i n! Vol(Pyr_n) = \Pi_{1 \leq i \leq n} D_i$ ∎

And now there is an "industry" computing this mixed volume...

Examples of the Mixed Volume

Two problems: to evaluate the volume polynomial $V_{\mathbf{K}}(\lambda_1, ..., \lambda_n)$ and to compute its mixed derivative.

1. $K_i = T, 1 \leq i \leq n$; $V(T, ..., T) = n!Vol(T)$; already SharpP-HARD.

2. The convex sets are coordinate boxes:

   $B_i = Diag(A(1,i), ..., A(n,i))Box_n$, the matrix A is nonnegative;

   $Vol(x_1 B_1 + ... x_n B_n) = Prod_A(x_1, ..., x_n)$, where the product polynomial

   $Prod_A(x_1, ..., x_n) =: \Pi_{1 \leq i \leq n} \Sigma_{1 \leq j \leq n} A(i,j)x_j$.

   (Easy to compute). The mixed volume is equal to the permanent:

   $V(B_1, ..., B_n) = Per(A) =: \Sigma_{\sigma \in S_n} \Pi_{1 \leq i \leq n} A(i, \sigma(i))$.

   Is SharpP-Complete if "most" of columns have at least three non-zero entries. In terms of the corresponding Newton Polytopes, that means that the polynomial has at least 8 monomials.

Parallelograms: $K_i = Zon(e_i, Ae_i) = \{xe_i + yAe_i : 0 \leq x, y \leq 1\}$, A is $n \times n$ matrix.

$Vol(x_1 K_1 + ... x_n K_n) =$

$= (\Pi_{1 \leq i \leq n} x_i) Vol(Box_n + Diag(X^{-1}) A Diag(X) Box_n)$.

The evaluation is SharpP-HARD!

Mixed Volume of Parallelograms: $V(K_1, ..., K_n) = MV_A =: \Sigma_{S \subset [1,n]} |\det(A_{S,S})|$

Note that $\Sigma_{S \subset [1,n]} \det(A_{S,S}) = \det(I + A)$. But the sign is a problem: SharpP-Complete, even if A is an unimodular matrix:

$$A = \begin{pmatrix} 0 & I & I \\ Perm_1 & 0 & 0 \\ Perm_2 & 0 & 0 \end{pmatrix},$$

where the three permutation matrices $I, Perm_1, Perm_2$ are not overlaping. (Actually, $MV_A = Per(I + Perm_1 + Perm_2)$.) In terms of polynomials: n polynomials of the form $a + x_i + x_k x_l + x_i x_k x_l$,

$2n$ polynomials of the forms $b + x_j + x_m + x_j x_m$.

What is known in the general case? (I mean polynomial time algorithms.)

We consider the well-presented compact convex sets with weak membership oracles.

1. If the number of **distinct** sets in the tuple $(K_1, ..., K_n)$ is roughly $O(\log(n))$ then there is $FPRAS$ (i.e. $(1+\epsilon)$-approximation, complexity $poly(n, \frac{1}{\epsilon})$) for the mixed volume $V(K_1, ..., K_n)$. (Dyer,Gritzman,Hufnagel;1998).

2. The general case: (Barvinok,1998) - randomized algorithm with $n^{O(n)}$-approximation. (Gurvits,Samorodnitsky;2000,2002) - deterministic algorithm with $n^{O(n)}$-approximation.

3. The general case ; (Gurvits, 2007, 2009) - **randomized** algorithm with e^n-approximation and the better exponents if most of the sets have small dim..

One of the main results

My result is based on the following theorem: Recall the notion of capacity:

$$Cap(pol) = \inf_{x_i > 0} \frac{pol(x_1, ..., x_m)}{\Pi_{1 \le i \le m} x_i}$$

Theorem 0.3: *Let* $\mathbf{K} = (K_1 ... K_n)$ *be a* n*-tuple convex compact subsets in the Euclidean space* R^n*. Then the following inequality holds:*

$$V(K_1, ..., K_n) \le Cap(V_{\mathbf{K}}) \le \frac{n^n}{n!} V(K_1, ..., K_n). \quad (1)$$

The right ineq. in (1) is attained iff $Cap(V_{\mathbf{K}}) = 0$ *or* $K_i = a_i K_1 + \{b_i\}, i \ge 2$.

If affine dimensions are "small", say $aff(K_i) \le d$, then

$$Cap(V_{\mathbf{K}}) \le (\alpha_d)^n V(K_1, ..., K_n), (\alpha_d)^{-1} = \min_{x > 0} \frac{\Sigma_{0 \le i \le d} \frac{x^i}{i!}}{x}.$$

Note that $\alpha_2 = \sqrt{2} + 1 < e$.

Log-Convexity, Log-Concavity and all that JAZZ

$F : K \rightarrow R_+$, $\log(F)$ is convex, i.e $(F(\frac{X_1+X_2}{2}))^2 \leq F(X_1)F(X_2)$ or

$$\begin{pmatrix} F(X_1) & F(\frac{X_1+X_2}{2}) \\ F(\frac{X_1+X_2}{2}) & F(X_2) \end{pmatrix} \succeq 0; X_1, X_2 \in K.$$

This proves that the set of log-convex functions is a convex cone. One interesting sub-set:

$\log(pol(exp(y_1), ..., exp(y_n)))$ is convex on R^n provided the coefficients of the polynomial (entire function) pol are all non-negative. This observation gives the following inequality for such functions:

$$\frac{p(x_1, ..., x_m)}{p(y_1, ..., y_m)} \geq \prod_{1 \leq i \leq m} (\frac{x_i}{y_i})^{g_i}, g_i = \frac{\frac{\partial}{\partial y_i} p(y_1, ..., y_m) y_i}{p(y_1, ..., y_m)}; \tag{2}$$

Log-convexity is a very usefull thing: it allows poly-time algorthms for many things, including **Capacity**: $\log(Cap(pol)) = \inf_{y_1,...,y_m}(\log(pol(exp(y_1), ..., exp(y_n))) - \Sigma_{1 \leq i \leq m} y_i)$.
It is the heart of such seemingly unrelated results as Bregman's upper bound on the permanent of boolean matrices and monotonicity of Baum-Welsh algorithm for HMM. Here is one surprising application:

Keith Ball's Inequality

Let $X_1, ..., X_l \in R^n; ||X_i||^2 =: tr(X_i X_i^T) = 1, 1 \leq i \leq l$ and $\Sigma_{1 \leq i \leq l} a_i X_i X_i^T = I$. Then the following inequality holds:

$$Vol(b_1[X_i] + ... + b_l[X_l]) \geq \prod_{1 \leq i \leq l} (\frac{b_i}{a_i})^{a_i},$$

here the interval $[X_i] = \{aX_i, 0 \leq a \leq 1\}$.

Proof: $Vol(b_1[X_i] + ... + b_l[X_l]) =$

$= \Sigma_{1 \leq j_1 < ... < j_n \leq l} |Det([X_{j_1}|...|X_{j_n}])| \Pi_{1 \leq i \leq n} b_{j_i} \geq$

(the Hadamard's inequality: $|Det([X_{j_1}|...|X_{j_n}])| \leq 1$)

$\geq \Sigma_{1 \leq j_1 < ... < j_n \leq l} |Det([X_{j_1}|...|X_{j_n}|^2 \Pi_{1 \leq i \leq n} b_{j_i} =$

$= Det(b_1 X_1 X_1^T + ... + b_l X_l X_l^T)$.

$$Vol(b_1[X_i]+....+b_l[X_l]) \geq Det(b_1 X_1 X_1^T+...+b_l X_l X_l^T).$$

Define $p(b_1, ..., b_l) =: Det(b_1 X_1 X_1^T + ... + b_l X_l X_l^T)$.

Then

$$\frac{\partial}{\partial a_i} p(a_1, ..., a_l) = 1; p(a_1, ..., a_l) = 1.$$

Using the LOG-CONVEXITY of $p(exp(x_1), ..., exp(x_l))$ (ineq. (2)), we get that:

$$\frac{p(b_1, ..., b_l)}{p(a_1, ..., a_l)} = p(b_1, ..., b_l) \geq \prod_{1 \leq i \leq l} (\frac{b_i}{a_i})^{a_i}.$$

∎

Log-Concavity is not as nice, sum of Log-Concave function is not nec. Log-Concave. Yet, Log-Concavity is supremely powerful tool, especially in proving lower bounds: Brunn-Minkowski, Isoperemetric Theorems, Concentration results in probability theory ....
I will introduce a "slightly" more general, yet completelly natural generalization, which has even more magical proof power.
Next few pages give a brief historical motivaton(or survey).

Newton's inequalities

$x_1, x_2, ..., x_n$ are real (non-negative) numbers;

$P(t) = \Pi_{1 \le i \le n}(t + x_i) = \Sigma_{0 \le i \le n} t^i a_i$. Then

$$\left(\frac{a_i}{\binom{n}{i}}\right)^2 \ge \frac{a_{i-1}}{\binom{n}{i-1}} \frac{a_{i+1}}{\binom{n}{i+1}} : 1 \le i \le n-1$$

In other words the sequence

$(n!P(0), (n-1)!P^{(1)}(0), ..., (n-i)!P^{(i)}(0), ..., P^{(n)}(0))$

is **Log-Concave**

Define the set of **Log-Concave** sequences:

$LC(n+1) = \{(t_0, ..., t_n) \in R_+^{n+1} : t_i^2 \ge t_{i-1}t_{i+1} : 1 \le i \le n-1\}$

Necessary condition for real rootedness, but what does it really mean?

For polynomials with nonnegative coefficients **Newton's inequalities** can be restated as

$$\left(\sqrt[n-i]{p^{(i)}} \right)^{(2)} (0) \leq 0, 0 \leq i \leq n-1. \tag{3}$$

If the roots are real, i.e.
$P(t) = C(t+c_1)...(t+c_n); c_i, C \geq 0$, then $\sqrt[n]{P(t)}$ is concave on R_+
(As simple as $x^2 + y^2 \geq 2xy$).
Note that I just gave another proof of **Newton's inequalities**, albeit for the nonnegative coefficients case.

Theorem 0.4: *The inequalities (3) propagate: i.e. they imply that*

$$\left(\sqrt[n-i]{p^{(i)}} \right)^{(2)} (t) \leq 0, 0 \leq i \leq n-1, t \geq 0.$$

Call a positive sequence $(b_0, ..., b_n)$ good if $(b_0 P(0), b_1 P^{(1)}(0), ..., b_i P^{(i)}(0), ..., b_n P^{(n)}(0)) \in LC(n+1)$ implies the inclusion for all $t \geq 0$.

Theorem 0.5 *Let $(b_0, ..., b_k)$ be a positive sequence. Define $c_i = \frac{b_i}{b_{i+1}}, 0 \leq i \leq k-1$. The sequence $(b_0, ..., b_k)$ is good iff the infinite sequence $(c_0, ..., c_{k-1}, 0, ...)$ is concave,i.e*

$$2c_i \geq c_{i+1} + c_{i-1}, 1 \leq i \leq n-1; 2c_{n-1} \geq c_{n-2}.$$

Homogeneous Polynomials in 2 variables

Let $p(x) = \Sigma_{0 \le i \le d}\, a_i x^i; a_i \ge 0, 0 \le i \le d;$

Homogenation: $H(x, y) = y^d p(\frac{x}{y})$.

Fact 0.6:

1. The roots of p are real iff the polynomial H is **H-Stable**, i.e. $H(z_1, z_2) \neq 0$ if $Re(Z_1), Re(Z_2) > 0$.

2. Polynomial p satisfies **Newton's inequalities** iff the polynomial H is **Strongly Log-Concave**, i.e. the derivatives $(\partial x)^{c_1}(\partial y)^{c_m} H$ are either zero or $\log\left((\partial x)^{c_1}(\partial y)^{c_m} H\right)$ is concave on R_+^m

3. Homogeneous polynomial $H \in Hom_+(2, d)$ is **Strongly Log-Concave** iff the map

   $$Der_H(c_1, c_2) : \{(k, l) : k, l \in Z_+, k + l = d\} \to R_+$$

   is Log-Concave.

How to generalize to many variables?

1. A homogeneous polynomial $p(z_1, ..., z_m)$ is called **H-Stable** if

$$Re(z_i) > 0, 1 \leq i \leq m \rightarrow p(z_1, ..., z_m) \neq 0.$$

2. An entire function $f(z_1, ..., z_n)$ with non-negative coefficients is called **Strongly Log-Concave** if $(\partial x_1)^{c_1} ... (\partial x_m)^{c_m} f$ is either zero or $\log((\partial x_1)^{c_1} ... (\partial x_m)^{c_m} p)$ is concave on R^m_+.

The set of **Strongly Log-Concave** function is invariant respect to partial differentiations(by definition); the same holds for (**H-Stable** polynomials + the zero polynomial)([Gauss-Lukas]).
Therefore **H-Stable** polynomials are **Strongly Log-Concave**.
The set of **H-Stable** polynomials is also invariant respect to positive changes of variables $p(AX)$, where matrices A are positive entry-wise.

Brunn-Minkowski(1903?):

$(V_{\mathbf{K}}(\lambda_1, ..., \lambda_n))^{\frac{1}{n}}$ is concave on R_+^n:

$(Vol(K + S))^{\frac{1}{n}} \geq (Vol(K))^{\frac{1}{n}} + (Vol(S))^{\frac{1}{n}}$.

Why it was such a big deal?

Consider just the univariate case

$P(t) = Vol(K + tBall(1)) = t^n Vol(Ball(1)) + ... + a_1 t + Vol(K)$. Now, $a_1 = P^{(1)}(0)$ is the surface area of the convex body K. The log-concavity gives that:

$$P(t)^{\frac{1}{n}} \leq Vol(K)^{\frac{1}{n}} + \frac{t}{n} \frac{a_1}{(Vol(K))^{1-\frac{1}{n}}}.$$

Dividing left and right sides by t and taking the limit $t \rightarrow \infty$ we get

$$(Vol(K))^{\frac{n-1}{n}} \leq \frac{a_1}{n} (Vol(Ball(1)))^{-\frac{1}{n}}.$$

Which proves that the **Balls** have maximum volume for the fixed surface area. Does it remind you of Newton Inequlities?

Simple, yet crucial, differential inequality

We need the following elementary result, its proof is very similar to the isoperemetric proof above:

Lemma 0.7: *Consider a function $f : R_+ \to R_+$ such that the derivative $f'(0)$ exists.*

1. *If $f^{\frac{1}{k}}$ is concave on R_+ for $k > 1$ then* $f'(0) \geq (\frac{k-1}{k})^{k-1} \inf_{t>0} \frac{f(t)}{t}$.

2. *If f is log-concave on R_+ then* $f'(0) \geq \frac{1}{e} \inf_{t>0} \frac{f(t)}{t}$.
   If, additionally, the function f is analytic and $f'(0) = \frac{1}{e} \inf_{t>0} \frac{f(t)}{t}$ *then $f(t) = exp(at), a > 0$.*

Alexandrov(1937), Fenchel(?):

Brunn-Minkowski theory - Log-Concavity of the volume polynomial $V_{\mathbf{K}}$ on R^n_+:
the backbone of convex geometry and its numerous applications ...
Its generalization, **Alexandrov-Fenchel theory**, is based on the very deep fact that the functionals

$$\left(\frac{\partial^k}{\partial\lambda_1 ... \partial\lambda_k} V_{\mathbf{K}}(0, .., 0, \lambda_{k+1}, ..., \lambda_n)\right)^{\frac{1}{n-k}}$$

are concave on R^{n-k}_+ for all $1 \leq k \leq n-1$. In other words the volume polynomials $V_{\mathbf{K}}$ are **Strongly Log-Concave**.

Theorem 0.8: *[Shephard, 1960] A homogeneous polynomial* $H \in Hom_+(2, n)$ *is* **Strongly Log-Concave** *(i.e. the univariate polynomial satisfies* **Newton Inequalities***)*

iff there exist two convex compact sets $K_1, K_2 \in R^n$ *such that*

$$H(x, y) = Vol_n(xK_1 + yK_2).$$

Corollary 0.9: *[L.G. 08] If the polynomials* $H_1 \in Hom_+(2, k), H_2 \in Hom_+(2, l)$ *are* **Strongly Log-Concave** *then the product* $H_1 H_2 \in Hom_+(2, k + l)$ *is also* **Strongly Log-Concave***.*

(Previous Proofs are (boringly) long,..., very usefull in geometric funct. analysis, exact Khintchine Constants...)

Remark 0.10:

1. **Alexandrov-Fenchel Inequalities** :

   $$V(K_1, K_2, K_3, ...)^2 \geq V(K_1, K_1, K_3, ...)V(K_2, K_2, K_3, ...) \tag{4}$$

   Equivalent to the **Strongly Log-Concavity** of the volume polynomial.

2. Alexandrov also proved that the determinantal polynomials $\det(x_1 Q_1 + ... + x_n Q_n)$ where $Q_i \succeq 0$ are **Strongly Log-Concave**, i.e. the determinantal analogue of (4). He did not realize that such determinantal polynomials are **H-Stable**.

3. (Petrovsky, 1937; Garding 1950s): *A homogeneous polynomial* $p(x_1, ..., x_m)$ *is called hyperbolic in direction* $e \in R^m$ *if the roots of* $p(V - te) = 0$ *are real for all real vectors* $V \in R^n$. *The hyperbolic cone is the (convex) set of the vectors with non-*

negative roots.

A homogeneous polynomial $p \in Hom_C(m, n)$ is **H-Stable** iff it is hyperbolic in direction $e = (1, ..., 1)$, and its hyperbolic cone contains the positive orthant R^m_{++}, i.e. the roots of $p(X - te) = 0$ are positive real numbers for all positive real vectors $X \in R^m_{++}$.

Moreover $\frac{p}{p(X)} \in Hom_+(m, n)$ for all $X \in R^m_{++}$ and

$|p(z_1, ..., z_m)| \geq p(Re(z_1), ..., Re(z_m)) : Re(z_i) \geq 0$.

4. Important for this talk: volume polynomials are **Strongly Log-Concave**; the determinantal polynomials as above are **H-Stable**.

5. A **H-Stable** polynomial $p(x,y,z), p(1,1,1) > 0$ has a "positive" determinantal representation: $p(x,y,z) = \det(xQ_1 + yQ_2 + yQ_3) : Q_i \succeq 0, Q_1 + Q_2 + Q_3 \succ 0$.
   Hermitian case [B. Dubrovin, 1983], Real symmetric case[V.Vinnikov, 1993].

▮

Strong Log-Concavity and Lower Bounds: Easy Induction

Let $p(x_1, ..., x_{n-1}, x_n)$ be **Strongly Log-Concave** function. Fix positive numbers $(x_1, ..., x_{n-1})$ and define univariate function $f(t) = p(x_1, ..., x_{n-1}, t)$. Note that $f(t)$ is Log-Concave on R_+ and
$\frac{\partial}{\partial x_n} p(x_1, ..., x_{n-1}, 0) = f^{(1)}(0)$.
Define $q_{n-1}(x_1, ..., x_{n-1}) = \frac{\partial}{\partial x_n} p(x_1, ..., x_n, 0)$
Recall the definition of capacity:
$Cap(p) = \inf_{x_i > 0} \frac{p(x_1,...,x_n)}{\Pi_{1 \leq i \leq n} x_i}$.
So, $f(t) \geq tCap(p)x_1....x_{n-1}$. The elementary differential inequality above gives

1. In the general case
   $Cap(q_{n-1}) \geq e^{-1} Cap(p)$.

2. If $p \in Hom_+(n, n)$ then $Cap(q_{n-1}) \geq G(n)Cap(p)$, where $G(i) = \left(\frac{i-1}{i}\right)^{i-1}, i > 1; G(1) = 1$.

3. If $p \in Hom_+(n,n)$ is **H-Stable** then

   $Cap(q_{n-1}) \geq G(deg_p(\{n\}))Cap(p)$;

   (just reminding) where $G(i) = \left(\frac{i-1}{i}\right)^{i-1}, i > 1; G(1) = 1$.

And now the induction is easy!:

$$q_n = p; q_i(x_1, ..., x_i) = \frac{\partial^{n-i}}{\partial x_n ... \partial x_{i+1}} p(x_1, ..., x_i, 0, ..., 0);$$

Note that $Cap(p) \geq Cap(g_{n-1}) \geq ... \geq Cap(q_0)$ and, most importantly,

$Cap(q_0) = \frac{\partial^n}{\partial x_n ... \partial x_1} p(0, ..., 0).$

For instance, in the general **Strongly Log-Concave** case we get that

$Cap(q_{k-1}) \geq e^{-1} Cap((q_k).$

In the homogeneous **Strongly Log-Concave** case we get that

$Cap(q_{k-1}) \geq G(k) Cap((q_k).$

In the (homogeneous) **H-Stable** case we get that

$Cap(q_{k-1}) \geq G(deg_{q_k}(\{k\})) Cap((q_k).$

And $G(2)...G(n) = vdw(n) =: \frac{n!}{n^n}.$

Theorem 0.11:

1. *Let* $f \in Ent_+(n)$ *be* **Strongly Log-Concave** *entire function in n variables. Then the follow. inequality holds:*

$$Cap(f) \geq \frac{\partial^n}{\partial x_1 ... \partial x_n} f(0) \geq \frac{1}{e^n} Cap(f) \qquad (5)$$

Note that the right inequality in (5) becomes equality if $f = exp(\Sigma_{1 \leq i \leq n} a_i x_i)$ *where* $a_i > 0, 1 \leq i \leq n$.

2. *Let a homogeneous polynomial* $p \in Hom_+(n, n)$ *be* **Strongly Log-Concave**. *Then the follow. inequality holds:*

$$Cap(f) \geq \frac{\partial^n}{\partial x_1 ... \partial x_n} f(0) \geq vdw(n) Cap(p), vdw(n) = \frac{n!}{n^n} \qquad (6)$$

Moreover, the right inequality in (6) becomes equality iff $Cap(p) = 0$ *or* $p = (\Sigma_{1 \leq i \leq n} a_i x_i)^n$ *where* $a_i > 0, 1 \leq i \leq n$.

In the **H-Stable** we have a stronger result: Recall $G(i) = \left(\frac{i-1}{i}\right)^{i-1}, i > 1; G(1) = 1$. This function G is strictly decreasing and $G(k) = \frac{wdv(k)}{wdv(k-1)}, vdw(k) =: \frac{k!}{k^k}$ for integer k

Theorem 0.12:

$$\frac{\partial^n}{\partial x_1 ... \partial x_n} p(0) \geq Cap(p) \prod_{2 \leq i \leq n} G(deg_{q_i}(\{i\})). \qquad (7)$$

As $deg_{q_i}(\{i\}) \leq \min(i, deg_p(\{i\}))$ we get that

$$\frac{\partial^n}{\partial x_1 ... \partial x_n} p(0) \geq Cap(p) \prod_{2 \leq i \leq n} G(\min(i, deg_p(\{i\}))). \qquad (8)$$

If $deg_p(\{i\}) \leq k, 1 \leq i \leq n$ or $deg_{q_i}(\{i\}) \leq k, 1 \leq i \leq n$ then

$$\frac{\partial^n}{\partial x_1 ... \partial x_n} p(0) \geq Cap(p) wdv(k) G(k)^{n-k}. \qquad (9)$$

The inequality (9) is asymptotically sharp with assumption $deg_p(\{i\}) \leq k, 3 \leq k \leq n-1$ and exactly sharp with assumption $deg_{q_i}(\{i\}) \leq k, 1 \leq i \leq n$.

Multivariate Newton-Like Inequalities

Recall

$$C_f(y_1, ..., y_m) =: \inf_{x_i > 0} \frac{f(x_1, ..., x_m)}{\Pi_{1 \le i \le m} (\frac{x_i}{y_i})^{y_i}}, y_i \ge 0.$$

Corollary 0.13: *Let* $f \in Ent_+(m)$ *be* **Strongly Log-Concave** *entire function in* m *variables. Then for all integer vectors* $R = (r_1, ..., r_m) \in Z_+^m$ *the following inequalities hold:*
$(\Pi_{1 \le i \le m} vdw(r_i)) \, C_f(r_1, ..., r_m) \ge (\partial x_1)^{r_1} ... (\partial x_m)^{r_m} f(0) \ge exp(-|R|_1) C_f(r_1, ..., r_m)$

But if f is just Log-Concave then $C_f(y_1, ..., y_m)$ is also Log-Concave. This observation gives the following Newton-Like Inequalities:

Consider integer vectors $Y_0, Y_1, ..., Y_k \in Z_+^m$ such that

$Y_0 = \Sigma_{1 \le i \le k} a_i Y_i; a_i \ge 0, \Sigma_{1 \le i \le k} a_i = 1.$

For a non-negative integer r we define $vdw(r) = \frac{r!}{r^r}$, and for a non-negative integer vector $Y = (r_1, ..., r_m) \in Z_+^m$ we define $VDW(Y) = \Pi_{1 \le i \le m} vdw(r_i)$.

If the entire function $f \in Ent_+(m)$ is **Strogly Log-Concave** then

$Der_f(Y_0) \ge$

$\ge exp(-|Y_0|_1) \Pi_{1 \le i \le k} (VDW(Y_i))^{-a_i} \Pi_{1 \le i \le k} (Der_f(Y_i))^{a_i}.$

Corollary 0.14: *The supports of* **Strogly Log-Concave** *entire functions $f \in Ent_+(m)$ are D-convex, i.e.*

$$CO(supp(f)) \cap Z^m = supp(f).$$

Log-Concavity alone is not sufficient

Log-concavity of f alone is not sufficient for D-convexity of the support $supp(f)$ even for univariate polynomials with non-negative coefficients.

Indeed, consider $p(t) = t + t^3$.

The fourth root $\sqrt[n-i]{p(t)}$ is concave on R_+:

$$(p^{(1)}(t))^2 - \frac{4}{3}p(t)p^{(2)}(t) = (1+3t^2)^2 - \frac{4}{3}(t+t^3)6t = (t^2-1)^2 \geq 0.$$

This example can be "lifted" to a "bad" log-concave homogeneous polynomial $q \in Hom_+(4,4)$:

$q(x,y,v,w) = (x+y)^3(v+w) + (v+w)^3(x+y)$.

It is easy to see that $Cap(q) = 2^5$

but $\frac{\partial^4}{\partial x \partial y \partial v \partial w} q(0) = 0$.

A few words about the Permanent

Specializing to the permanent

(and the mixed discriminant, which is the mixed derivative of $\det(x_1 Q_+ ... + x_n Q_n)$):

the generating polynomial for the permanent $Per(A)$ is

$$Prod_A(x_1, ..., x_n) = \prod_{1 \le i \le n} \sum_{1 \le j \le n} A(i,j) x_j.$$

I.e. the mixed derivative of $Prod_A$ is equal to $Per(A)$.

If A is non-negative and $Prod_A \neq 0$ then $Prod_A$ is **H-Stable**.

$deg_{Prod_A}(j) = |col(j)| =$ number of non-zero entries in jth column.

If A is doubly-stochastic then $Cap(Prod_A) = 1$.

Theorem 0.15: *If A is a doubly-stochastic $n \times n$ matrix then*

$$Per(A) \ge \prod_{2 \le j \le n} G(\min(|col(j)|, j)) \ge \prod_{2 \le i \le n} G(j) = \frac{n!}{n^n}.$$

$$\textit{If } |col(j)| \leq k < n \textit{ for } k+1 \leq j \geq n \textit{ then}$$

$$Per(A) \geq \left(\frac{k-1}{k}\right)^{(k-1)(n-k)} \frac{k!}{k^k} > \left(\left(\frac{k-1}{k}\right)^{k-1}\right)^n \tag{10}$$

The ineq. (10) is sharp only for $k = 2, n$. But the following lower bound is sharp:

$$\frac{\partial^n}{\partial x_1 \ldots \partial x_n} p(0) \geq Cap(p) \prod_{2 \leq i \leq n} G(deg_{q_i}(\{i\})) \tag{11}$$

Example 0.16: Doubly-stochastic matrices with the pentagon pattern:

$DS(n,k) = \{A \in DS(n) : A(i,j) = 0 : j-i \geq n-k\}$.

Then $\min_{A \in DS(n,k)} Per(A) = \left(\frac{k-1}{k}\right)^{(k-1)(n-k)} \frac{k!}{k^k}$,

and $deg_{q_i}(\{i\}) \leq k, 1 \leq i \leq n$.

∎

A. Schrijever (1998): $A = \{\frac{d(i,j)}{n} : 1 \leq i, j \leq n\}$, All rows and columns of the **integer** matrix D sum to $k \leq n$ (i.e. k-regular bipartite graph with multiple edges). Then

$$Per(A) \geq \left(\frac{k-1}{k}\right)^{(k-1)n}. \tag{12}$$

The inequality (10) gives a stronger version of the very discrete Schrijvers's inequality (12). Moreover, our inequality works in much more general real valued case. Amazingly, the exponent $\left(\frac{k-1}{k}\right)^{k-1}$ is optimal. This optimality follows from a forgotten H. Wilf's 1966 paper. Was rediscovered by Schrijver and Valiant in 1981.

In the case of the mixed discriminant of doubly-stochastic tuples (i.e. $tr(Q_i) \equiv 1, \Sigma_{1 \leq i \leq n} Q_i = I$):

$$D(A_1, ..., A_n) \geq \prod_{2 \leq j \leq n} G(\min(Rank(A_j), j)).$$

This leads to the deterministic poly-time algorithms to approximate as $\Sigma_{S \subset \{1,...,n\}} |\det(A_{S,S})|$(the mixed volume of parallelograms) as well
$\Sigma_{S \subset \{1,...,n\}} |\det(A_{S,S})|^2$ with the factor $\frac{2^n}{n^m}$. But the permanent is apparently quite special: if A is doubly-stochastic then [L.G, 2011]

$$Per(A) \geq \prod_{1 \leq i,j \leq n} (1 - A(i,j))^{1-A(i,j)} \qquad (13)$$

And it is just a beginning...

A bit of Complexity Theory, Separation of Variables

A polynomial $p(x_1, ..., x_m)$, $p \in Hom_+(n, m)$ with (non-negative) integer coefficients given as evaluation oracle; i.e. we can evaluate it at rational vectors with bounded bit-wise complexity. The following questions seem to be natural and practical:

1. Does integer vector $(r_1, ..., r_m) \in supp(p)$?
2. Does rational vector $(b_1, ..., b_m) \in NP(p)$?
3. What is $deg_{q_i}(\{i\})$?
4. Can we factorize
   $p(x_1, ..., x_m) = P(x_i : i \in S)Q(x_j, j \in T)$, where $S \cup T$ is a nontrivial partition of variables.
5. Can we split monomials, i.e. does there exist a non-trivial partition such that $deg_p(S) + deg_p(T) = n$?

Note that in the homogeneous case splitting of monomials is necessary for the separation of variables.

6. Can we approximate (within relative error) the coefficients?

7. If $p(1, ..., 1) = 1$ then we have a probabilistic distribution on $\{(d_1, ..., d_m) \in Z_+^m : \Sigma_{1 \leq i \leq m} d_i = n\}$. Can we sample (with small error) from that distribution?

Separation of variables is in BPP, using (Schwartz, Zippel):
construct the following undirected graph with m vertices : (i, j) are connected iff

$$((\partial x_i)p)((\partial x_j)p) - p((\partial x_i \partial x_j)p) \neq 0.$$

The variables can be saparated iff the graph is not connected. But the splitting of monomials is NP-HARD.

Theorem 0.17: *let* $p \in Hom_+(n,m)$ *be* **H-Stable**. *Then(all vectors sum to n and non-negative)*

1. *The degree function* $deg_p(S)$ *is submodular.*
2. $(r_1, ..., r_m) \in supp(p)$ *iff* $\Sigma_{i \in S} r_i \leq deg_p(S), S \subset \{1, ..., m\}$.
3. $(b_1, ..., b_m) \in NP(p)$ *iff* $\Sigma_{i \in S} b_i \leq deg_p(S), S \subset \{1, ..., m\}$ *and* $\Sigma_{1 \leq i \leq m} b_i = n$.
4. *The separation of variables is equivalent to the the splitting of monomials (just the hyperbolicity in direction (1,...,1) would do).*

Using submodular minimization, this result allows for **H-Stable** polynomials deterministic strongly poly-time algorithms for memberships, separation of variables, splitting of monomials.

My proof is based on Dubrovin's hermitian determinantal representation of **H-Stable** polynomials.

Theorem 0.18:

1. *Let $p \in Hom(n,m)$ be non-zero homogeneous polynomial which log-concave on some open set. Consider a non-trivial partition $S \cup T$ of variables. If $deg_p(S) = 1$ and $deg_p(S) + deg_p(T) = n$ then variables are separated.*
   Note that this result gives log-concavity characterization of rank-one tensors.

2. *Let $p \in Hom(n,m)$ be non-zero homogeneous polynomial. Assume that there is an open subset $U \subset R^m$ and open subset of matrices $M \in R^{m^2}$ such that the polynomials $p(AX), A \in M$ are* **Strogly Log-Concave** *on U. Then the separation of variables is equivalent to the the splitting of monomials.*