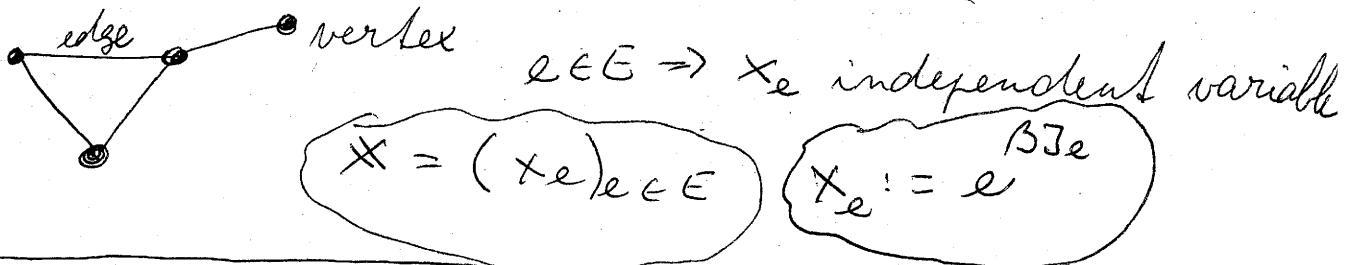


Complexity of Graph Polynomials

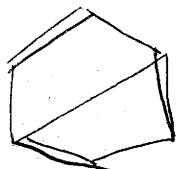
Martin LOEBL, Charles University, Prague, Czech R.

$G = (V, E)$ graph



Dimer partition function

$$P_G(x) = \sum_{\text{M perf. matching}} \prod_{e \in M} x_e$$



even set of edges:

(V, E') each degree even



Ising partition function

$$Z_G(x) = \sum_{\rho: V \rightarrow \{-1, 1\}} \prod_{\{u, v\} \in E} x_e^{s(u)s(v)}$$

||| van der Waerden

$$E_G(x) = \sum_{E' \text{ even}} \prod_{e \in E'} x_e$$

Generating function of even sets

A Concrete calculations (algorithms, general weights)

B formulas: very special weights

Theory of Pfaffian orientations

$$\mathbb{X}_A = \prod_{e \in A} x_e$$

signed dimer p.f. $P(D, M_0, *) = \sum_{\substack{\text{Perfect} \\ \text{Matching}}} \text{sign}(D, M_0, P) \mathbb{X}_P$

Pfaffian [determinant-type expression; Gaussian-type elimination]

Theorem (Kasteleyn 67, Galluccio-Bell 99, Tesler 2000, Limasini-Reshetikhin 2007)

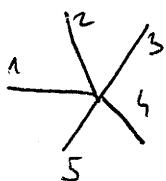
G embeds into orientable surface of genus $\sum g$ then

$$P(G, *) = 2^g \sum_{S \in \{0, 1\}^{2g}} (-1)^{\text{Arf}(S)} P(D_S, M_0, *); \quad \text{sign}(D_S, M_0, P) = (-1)^{q_p(M_0 \Delta P)}$$

Arf-invariant formula

q_p : quadratic form on first homology group $H_1(\sum_g; \mathbb{F}_2)$ of \sum_g (coef. in \mathbb{F}_2).

$$* E(G, *) = P(G_1, *)'$$



Is the number of the needed Pfaffians exponential

$$\# \{ \text{only 3 terms, one is zero!!} \}$$

?

$$x = (x_e)_{e \in E}$$

Norine (2000's) $c_{\text{match}}(G) = \min \# \text{orientations } D_i \text{ of } G \text{ so that}$
 $P_G(x)$ is a linear combination of the Pfaffian polynomials
 $P(D_i, M_0, x)$.

Conjecture (Norine) $c_{\text{match}}(G)$ is always a power of 4.

Thm (Norine) $c_{\text{match}}(G)$ cannot be 2, 3 or 5.

Thm. (Miranda, Lucchesi).

Norine's conjecture not true. There is G with $c_{\text{match}}(G) = 6$.

Loeb, Masbaum (2011) Norine's conjecture true for $E_G(x)$ (Ising f.f.)

$$\mathcal{E}(G, \mathbf{x}) = P(G_\Delta, \mathbf{x}) \quad \left\{ \begin{array}{l} \tilde{\Delta} = (\tilde{\Delta}_v)_{v \in V(G)} \text{ choice of orientations} \\ \text{of the gadgets} \end{array} \right.$$

Oriented drawing G_Δ is $(\tilde{\Delta}$ -admissible) if \mathcal{D} restricts to $\tilde{\Delta}_v$ on every gadget.

$C_{\tilde{\Delta}}(G)$: min cardinality of a set of $\tilde{\Delta}$ -admissible orientations \mathcal{D}_i of G_Δ s.t. $\mathcal{E}(G, \mathbf{x})$ is a linear combination of the Pfaffians $P(\mathcal{D}_i, M_0, \mathbf{x})$.

$$C_{\text{ring}}(G) = \min_{\tilde{\Delta}} C_{\tilde{\Delta}}(G).$$

Theorem (Bebl, Masbaum, Adv. in Math 2011)

$$\textcircled{1} \quad C_{\tilde{\Delta}}(G) \text{ is always a power of 4. } \textcircled{2} \quad C_{\text{ring}}(G) = 4^g \left[g : \text{genus } G \right]$$

Corollary: Even drawing : each even set has an even # self-intersections

Min genus of an orientable surface which supports an even drawing of G
 = embedding genus of G .

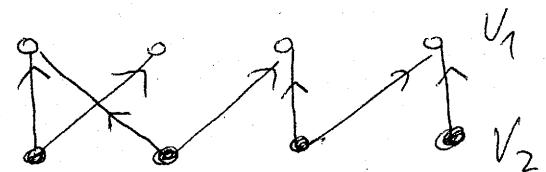
Is there a complex-weights generalisation of Kasteleyn orientations?

Motivation Alvarez-Gaume, Bost, Moore, Vafa: Partition function

of free fermion on a closed Riemann surface of genus g is linear combination of 2^g Pfaffians of Dirac operators of Σ .

Kuperberg (98): Kasteleyn curvature

$$w: E(G) \rightarrow \mathbb{C}$$

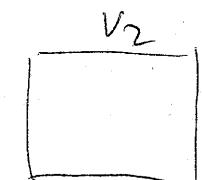


C : cycle of G with a way of traversal (\curvearrowleft or \curvearrowright)

$$c(C) = (-1)^{|C|/2 + 1} \frac{\prod_{e \in C^+} w(e)}{\prod_{e \in C^-} w(e)}$$

Kuperberg: G plane,

$w: E(G) \rightarrow \mathbb{C}$ is [Kasteleyn flat] w Kasteleyn flat then each if $c(F) = 1$ for each face F . perfect matching contributes the same to $\det(N(G))$.



Cimasoni (2010). Kasteleyn flatness of ω is not sufficient for existence of Arf-invariant fla for $P_G(\omega)$, G toroidal.

Way out

Ising partition function $E_G(\omega)$ replacing $P_G(x)$ and discrete Thara-Selberg function replacing Pfaffian.

[Cimasoni 2010, Bebl-Somborg 2011]

$M = M(G)$: $2|E| \times 2|E|$ matrix of transitions between orientations of edges

$$I(M) = \prod_{\tau} (1 - M(\tau))$$

$\xrightarrow{\text{prime reduced closed walk}}$

$$M(\tau) = \prod_t M_t$$

discussions
 with
 Misha Chertkov
 of τ
 Vladimir Chernyak

Background results

Feynman, Bass:

G planar $\Rightarrow E_G^2(\omega) = I(\tilde{M})$ includes rotation

$I(M)$ is a determinant

Cimasoni, Masbaum, Bebl, Somborg:
 $(2010-11)$

Arf-invariant fla true for discrete Thara-Selberg fctn and graphs embedded in \mathbb{S}_g , $g > 0$.

Higher dimensional systems

(Leibl, Ryzin) (discussions with Misha Cherkov,
Vladimir Chernyak)

G graph ; E_G : set of even sets of G .

E_G is binary linear code , $E_G \subseteq \{0,1\}^{|E|}$ (cycle space of G)

G : graph in \mathbb{R}^3 : 1-dimensional simplicial complex (system of edges)

G is geometric representation of E_G ; useful to get
(Arf-invariant) fla for $E(G, \star)$.

② Do general (binary) linear codes have a geometric representation which carries the weight enumerator?

$S = \{s_i ; i \in I\}$ set of words over $\{0, 1, \dots, k\}$ then weight
enumerator $W_S(x) = \sum_i x^{\# \text{non-zero entries}}$ [or different variables
for different entries]

Theorem (Ryškovič 2011).

T system of Δ (2d simplicial complex); p prime

$\text{Ker}_p(T) = \text{Kernel of incidence matrix of } T \text{ over } \mathbb{F}_p$

1	0	1	1
0	1	0	
1	0	1	
3			1

① If linear code over \mathbb{F}_p , p prime. Then one can construct system of Δ T so that $W_C(\mathbf{x})$ can be simply derived from $W_{\text{Ker}_p(T)}(\mathbf{x})$.

② $W_{\text{Ker}_2(T)}$ is a 3 permanent;

$$A = (a_{ijk})_{i,j,k=1}^n \quad 3\text{ matrix}$$

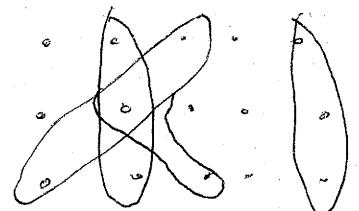
$$\text{3per } A = \sum_{\substack{\text{perm. of } n \\ \Pi_1, \Pi_2}} \prod_{i=1}^n a_{i \Pi_1(i) \Pi_2(i)}$$

$$\text{3det } A = \sum_{\substack{\text{perm. of } n \\ \Pi_1, \Pi_2}} \bigcirc \prod_{i=1}^n a_{i \Pi_1(i) \Pi_2(i)}$$

$$\bigcirc = \text{sign } \Pi_1 \cdot \text{sign } \Pi_2$$

3per is generating function of perfect matchings in

$P_T(\mathbf{x})$

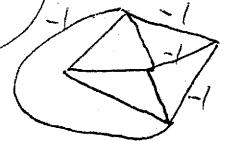


② Does $P_T(\mathbf{x})$, T system of Δ , have some physics meaning?

① L, R : $P_T(\mathbf{x})$ is 3det for large class of T 's. \square FLAS

Non-degenerated groundstates in antiferromagnetic Ising model on triangulations

Timmer, Kivi, Coebl



Theorem (Ésperet, Kádár, King, Král, Norine; longstanding Conjecture by Lovasz, Plummer) 2011

Cubic bridgeless graphs have exponential # perfect matchings.

~~G cubic bridgeless~~ ^{loopless} embedded in a Riemann surface Σ , G^* geometric dual; G^* is a triangulation [dir. cycle double cover conj. needed]

G planar \Rightarrow perfect matchings of G are exactly groundstates of antiferromagnetic Ising model on G^* . Frustrated edges of

$\hookrightarrow G^*$ planar triangulation, then antiferromagnetic degeneracy is exponential. [planar: Chudnovsky, Seymour] Contradicts strong

Andrea Timmer: ~~Given~~ There are arbitrarily large toroidal ~~triangulations~~ with ~~antiferromagnetic~~ EXACTLY 2^{\downarrow} GROUNDSTATES (2011) !

belief that geometrically fully frustrated systems have large degeneracy