Should We Believe in Numbers Computed by Loopy Belief Propagation?

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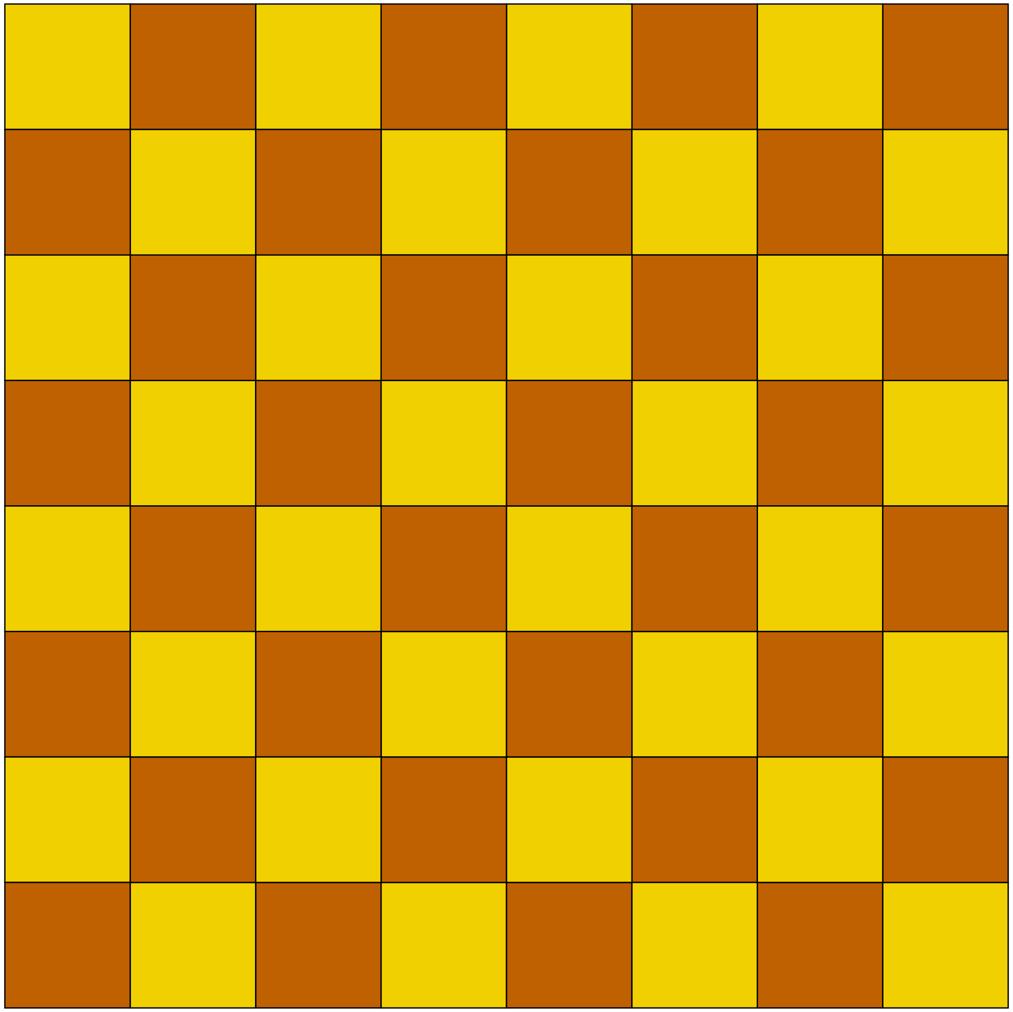
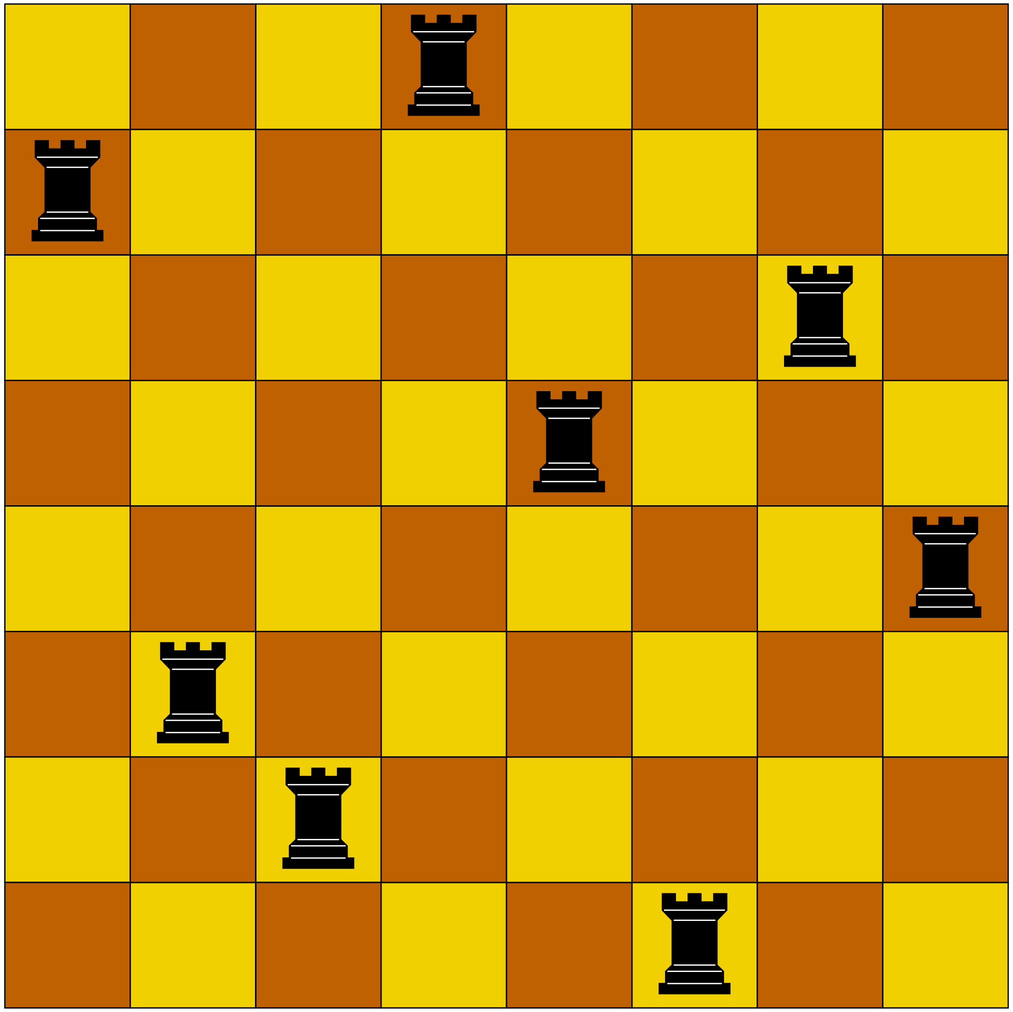
Information Theory Research Group
Hewlett-Packard Laboratories Palo Alto

CIOG Workshop, Princeton University, November 3, 2011

Chess Board



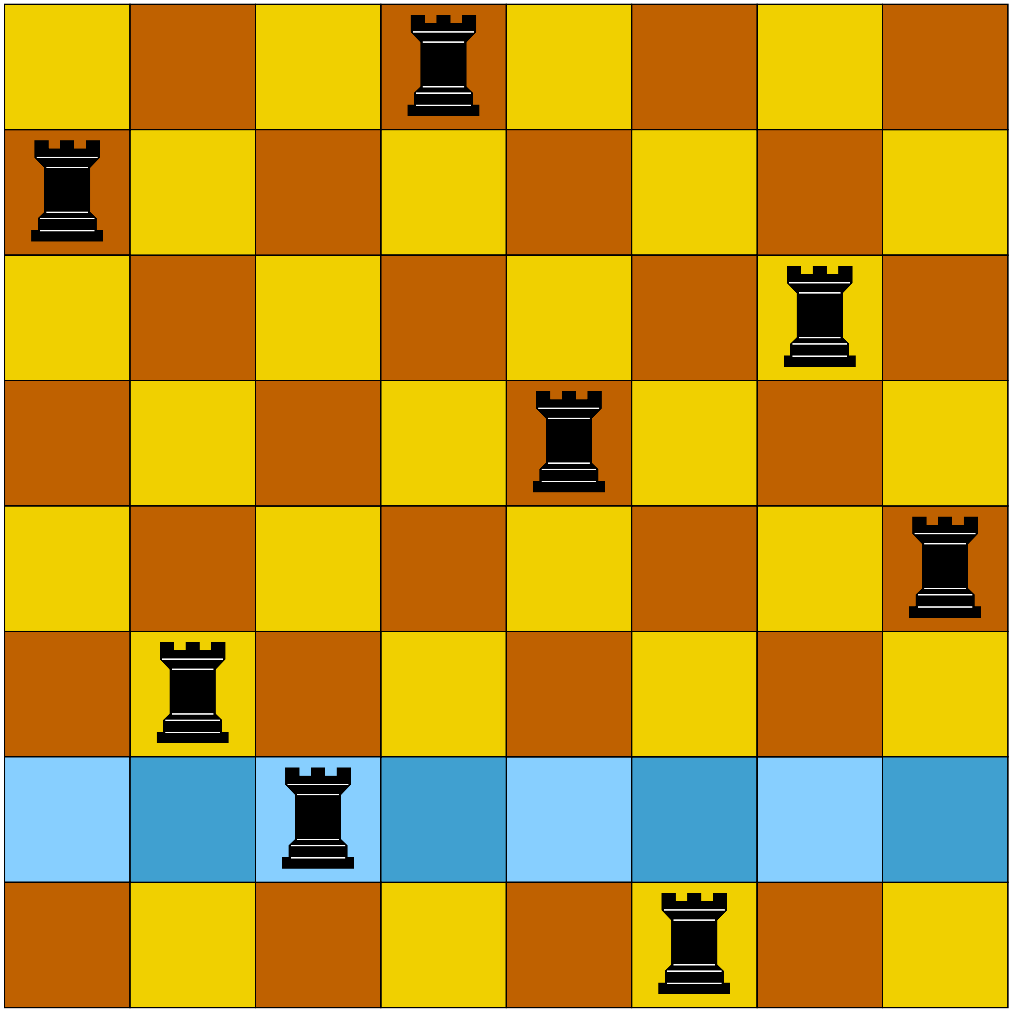
Chess Board



Question: in how many ways can we place 8 non-attacking rooks on a chess board?

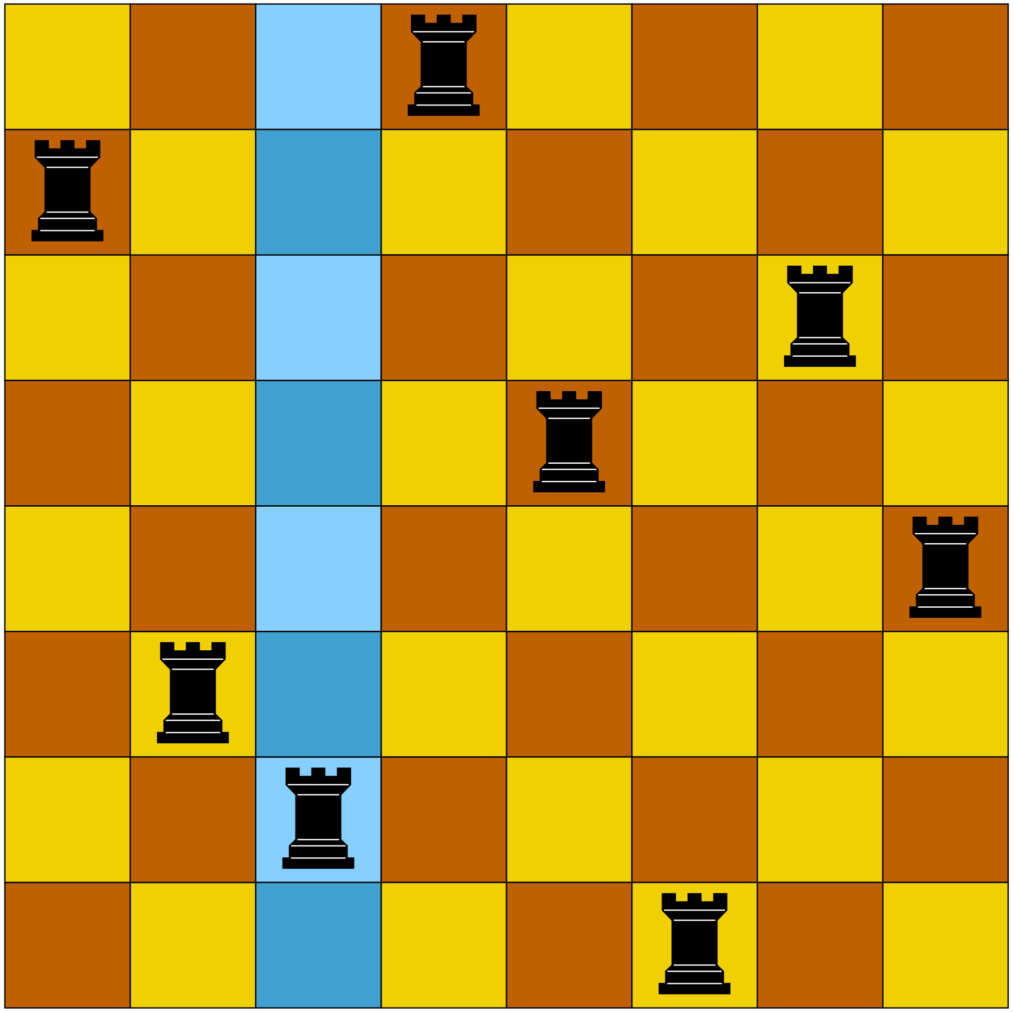
LABS hp

Chess Board



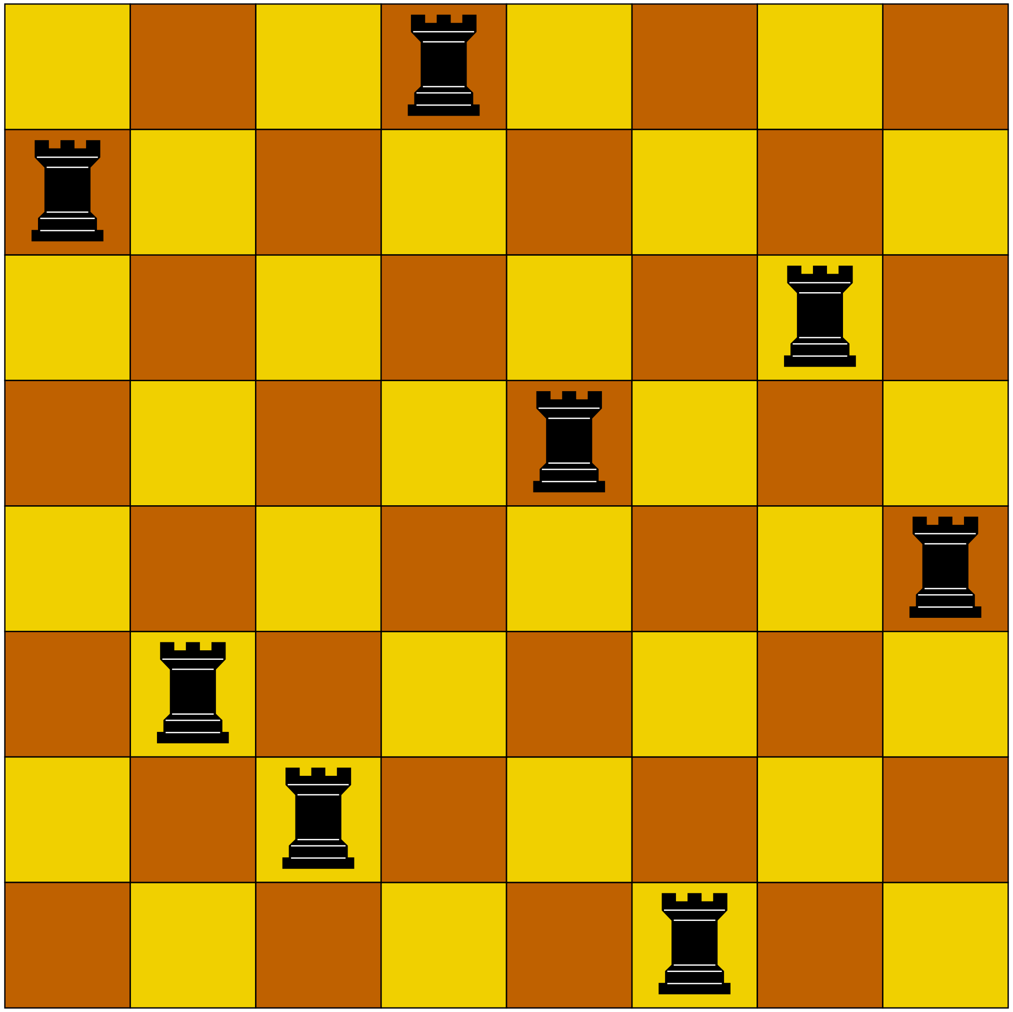
Row condition: exactly one rook per row.

Chess Board



Column condition: exactly one rook per column.

Chess Board



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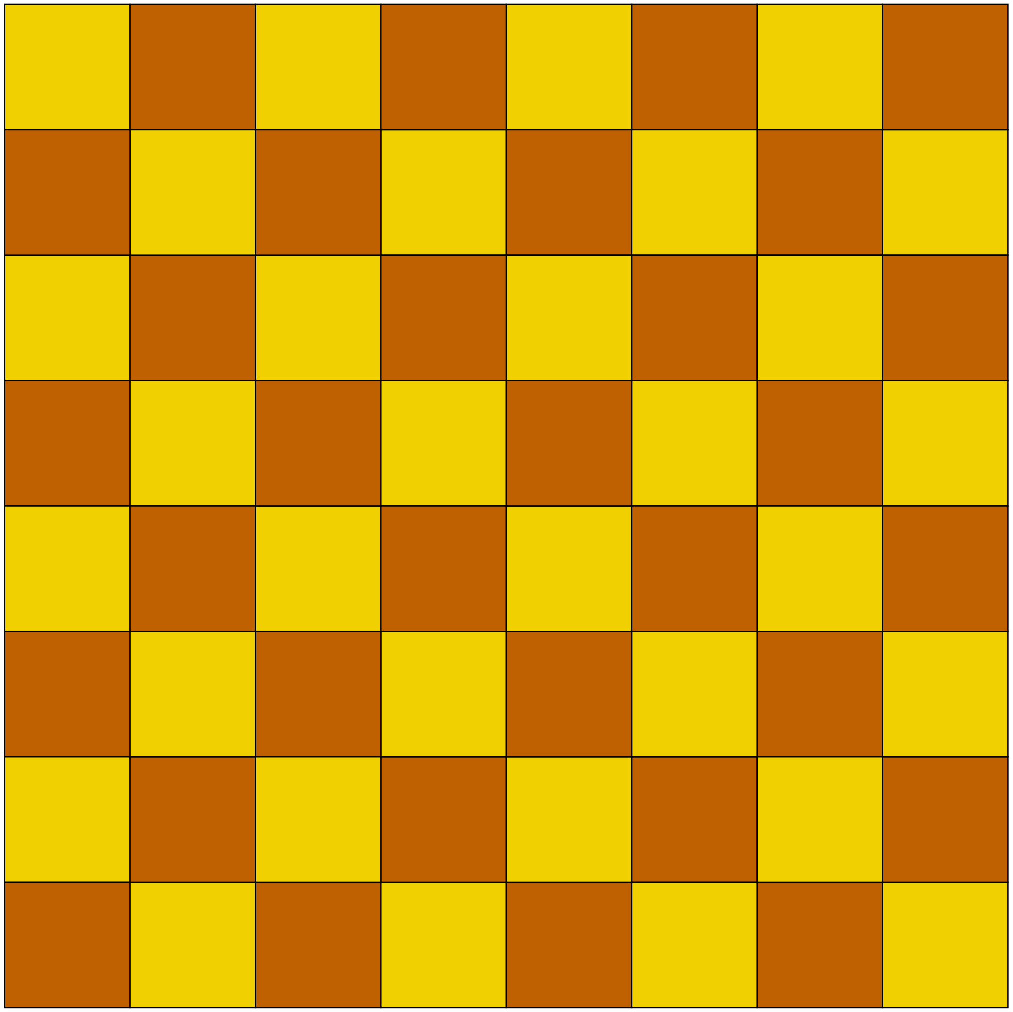

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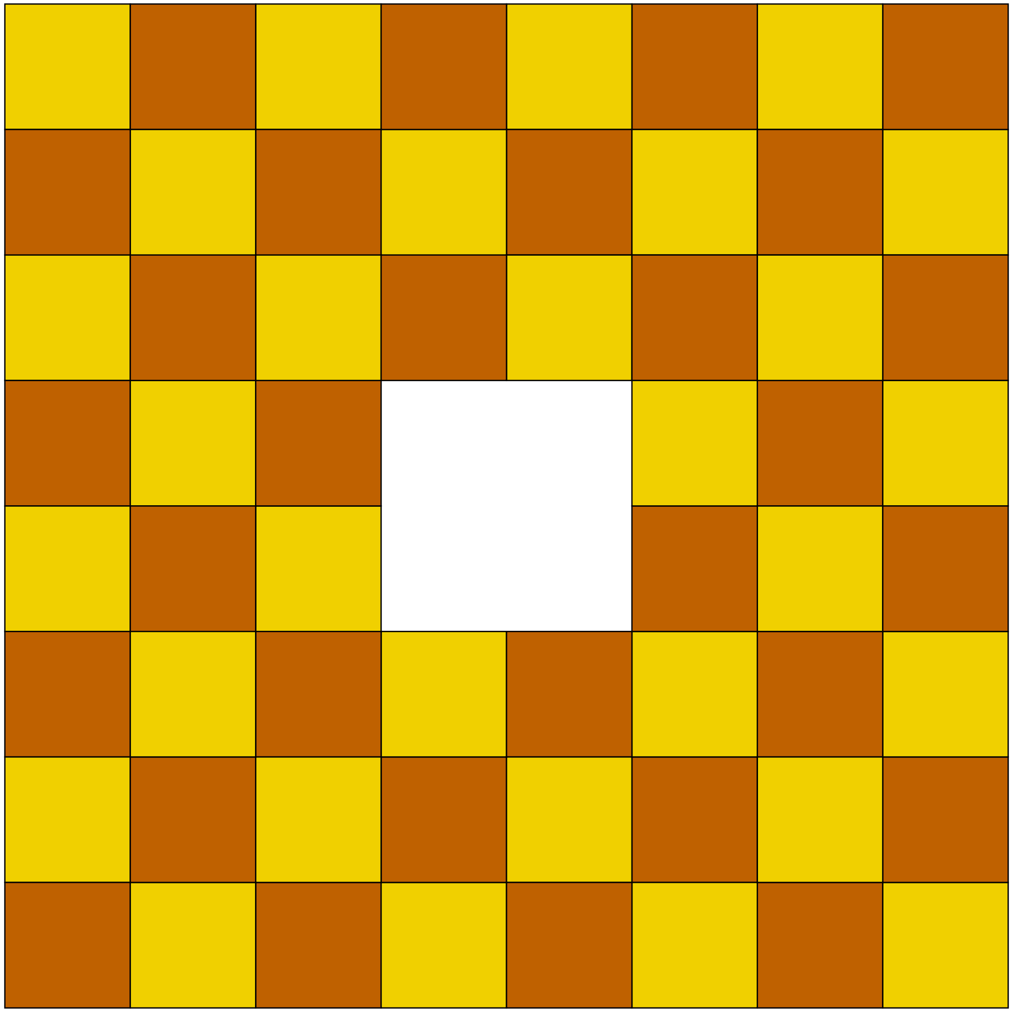
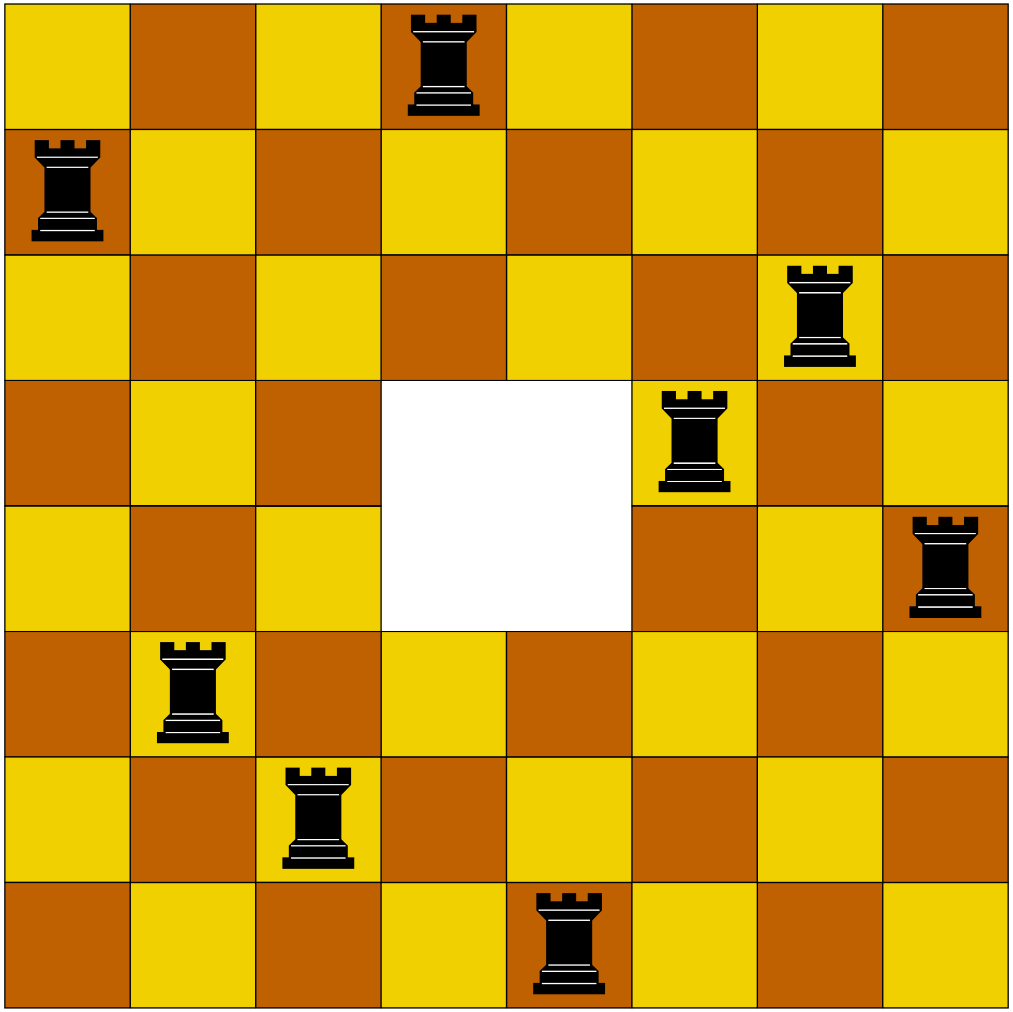
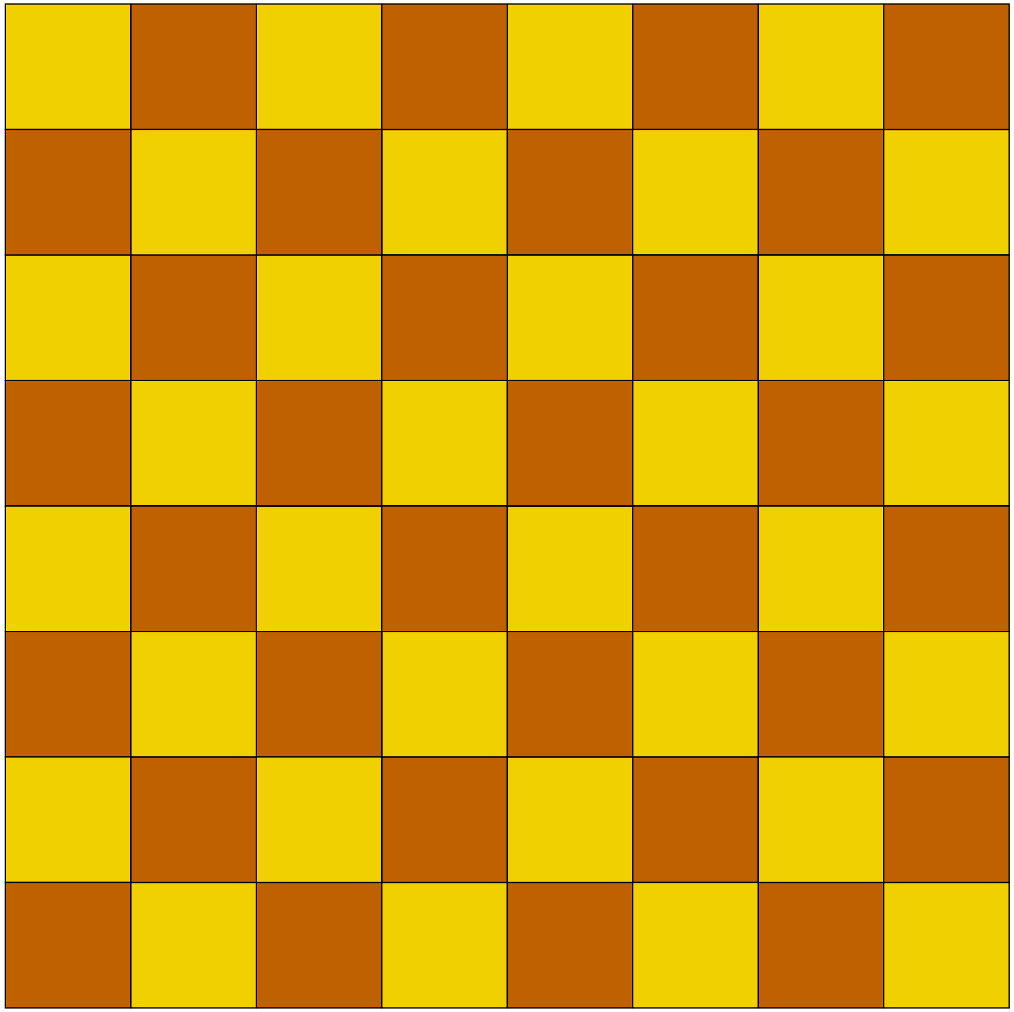


Chess Board

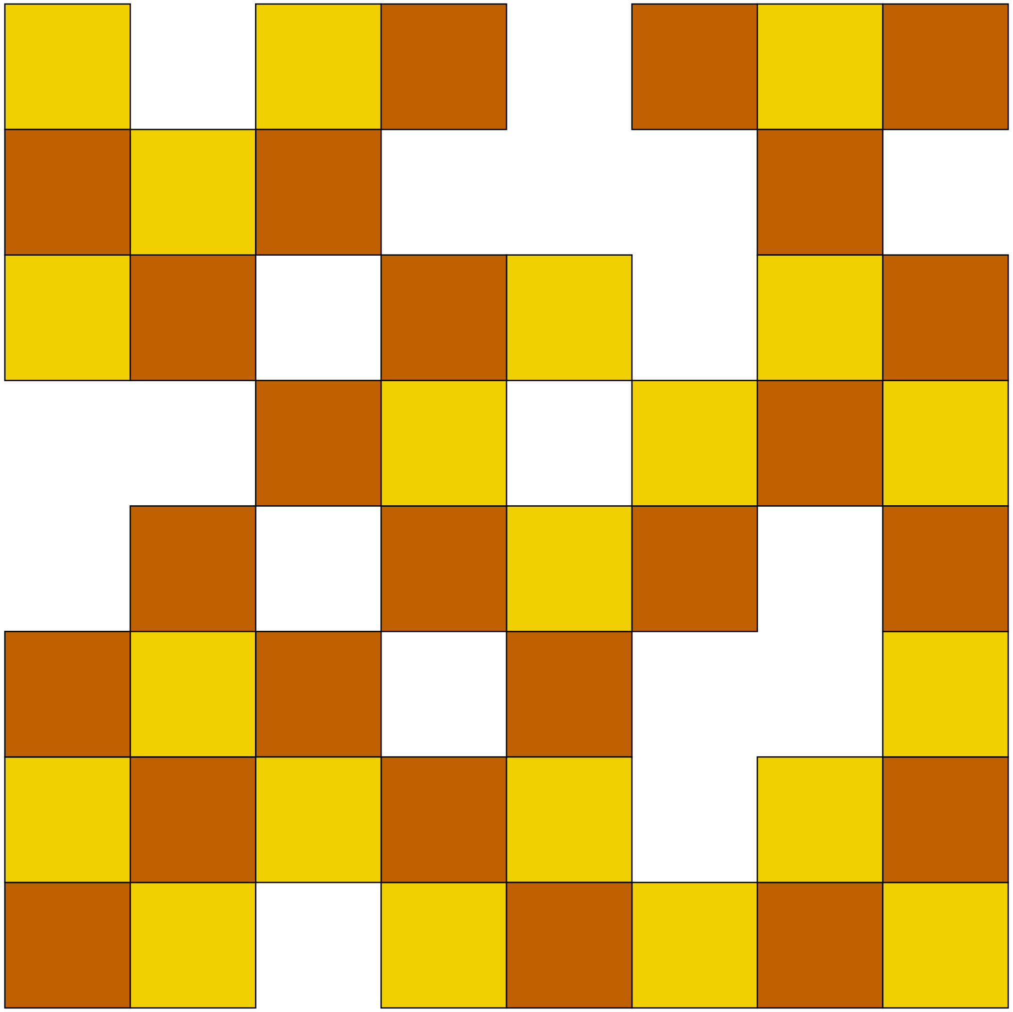


Question: in how many ways can we place 8 non-attacking rooks on this modified chess board?

Chess Board



Chess Board



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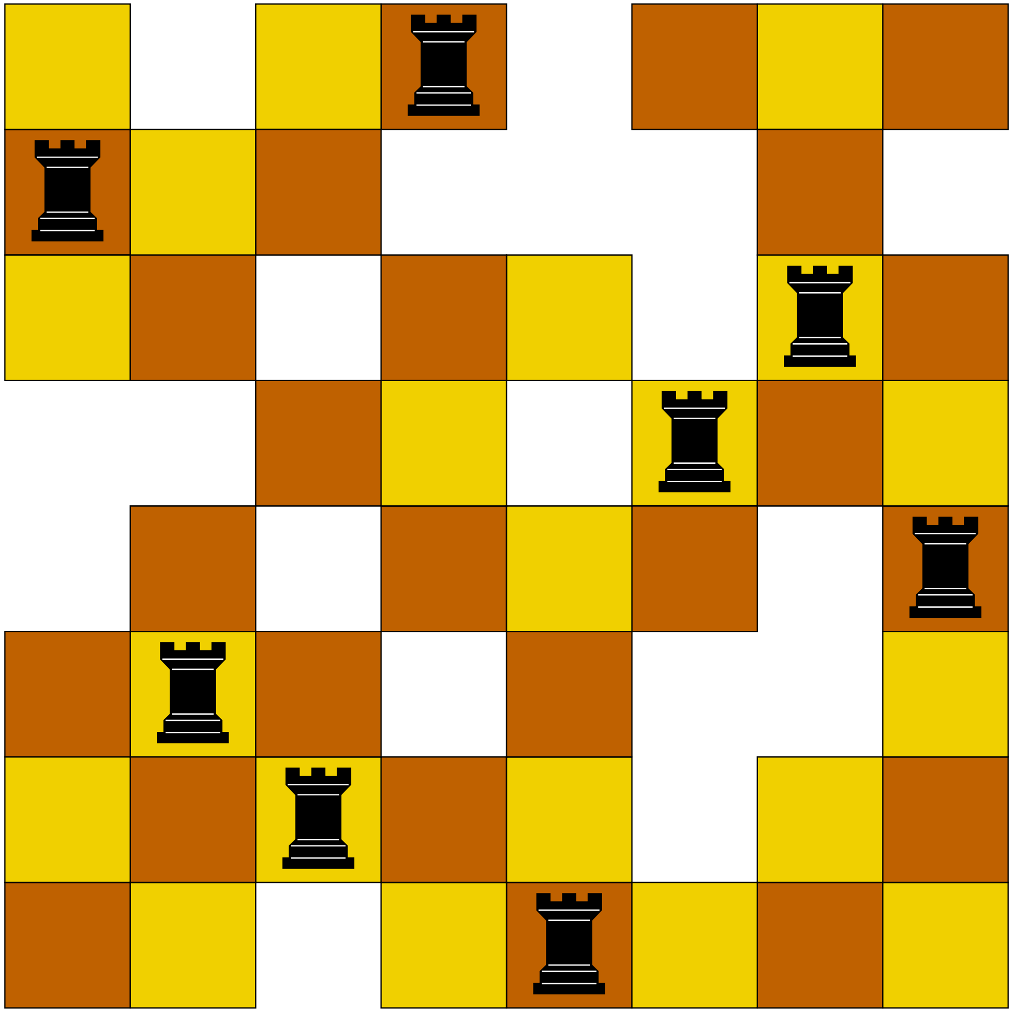

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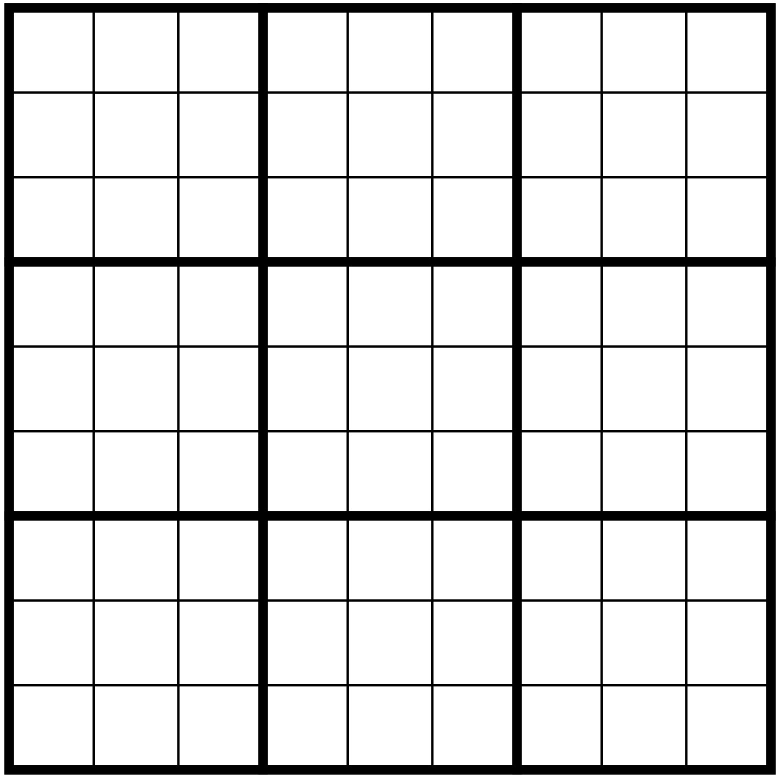
Sudoku



Sudoku

1	2	5	3	9	6	8	7	4
4	6	3	8	7	5	2	9	1
7	9	8	2	4	1	5	3	6
5	4	7	6	1	2	9	8	3
2	3	9	5	8	4	1	6	7
8	1	6	9	3	7	4	2	5
6	8	1	7	5	9	3	4	2
3	7	4	1	2	8	6	5	9
9	5	2	4	6	3	7	1	8

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8	1	6	9	3	7	4	2	5
6	8	1	7	5	9	3	4	2
3	7	4	1	2	8	6	5	9
9	5	2	4	6	3	7	1	8

Question: how many Sudoku arrays are there?

(More technically: how many valid configurations are there?)

Sudoku

1	2	5	3	9	6	8	7	4
4	6	3	8	7	5	2	9	1
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8	1	6	9	3	7	4	2	5
6	8	1	7	5	9	3	4	2
3	7	4	1	2	8	6	5	9
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Row condition: numbers $1, \ldots, 9$ appear exactly once.

Sudoku

1	2	5	3	9	6	8	7	4
4	6	3	8	7	5	2	9	1
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2	3	9	5	8	4	1	6	7
8	1	6	9	3	7	4	2	5
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Column condition: numbers $1, \ldots, 9$ appear exactly once.

Sudoku

1	2	5	3	9	6	8	7	4
4	6	3	8	7	5	2	9	1
7	9	8	2	4	1	5	3	6
5	4	7	6	1	2	9	8	3
2	3	9	5	8	4	1	6	7
8	1	6	9	3	7	4	2	5
6	8	1	7	5	9	3	4	2
3	7	4	1	2	8	6	5	9
9	5	2	4	6	3	7	1	8

Sub-block condition: numbers $1, \ldots, 9$ appear exactly once.

Sudoku

1	2	5	3	9	6	8	7	4
4	6	3	8	7	5	2	9	1
7	9	8	2	4	1	5	3	6
5	4	7	6	1	2	9	8	3
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Other Sudoku Setups

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3	5	8	1	9	6	2	7	4
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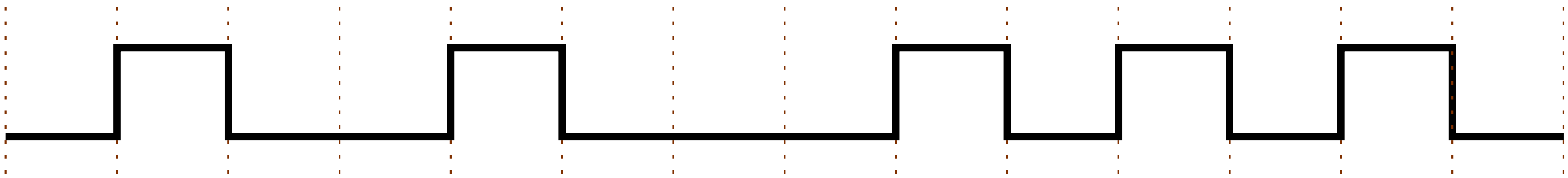
7	3	4	6	1	5	9	8	2
5	6	1	2	8	9	3	4	7

1D constraints in communications

RLL Constraints

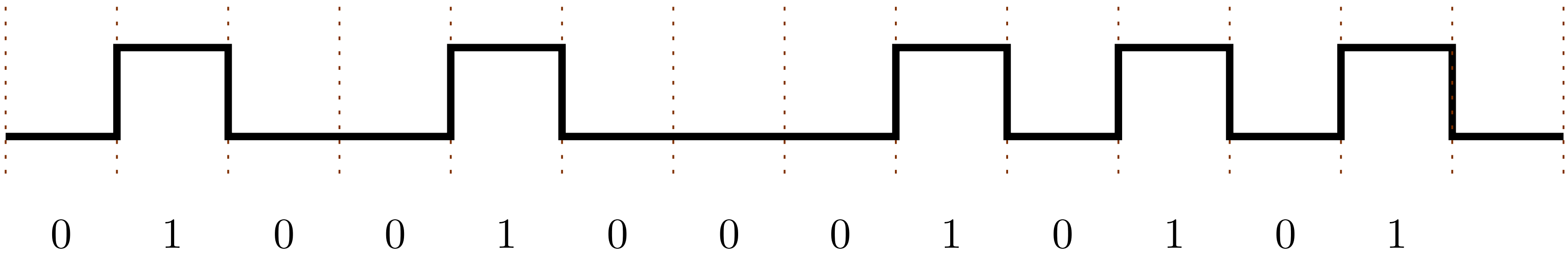


0 1 0 0 1 0 0 0 1 0 1 0 1

A (d, k) RLL constraint imposes:

- At least d zero symbols between two ones.
- At most k zero symbols between two ones.

RLL Constraints



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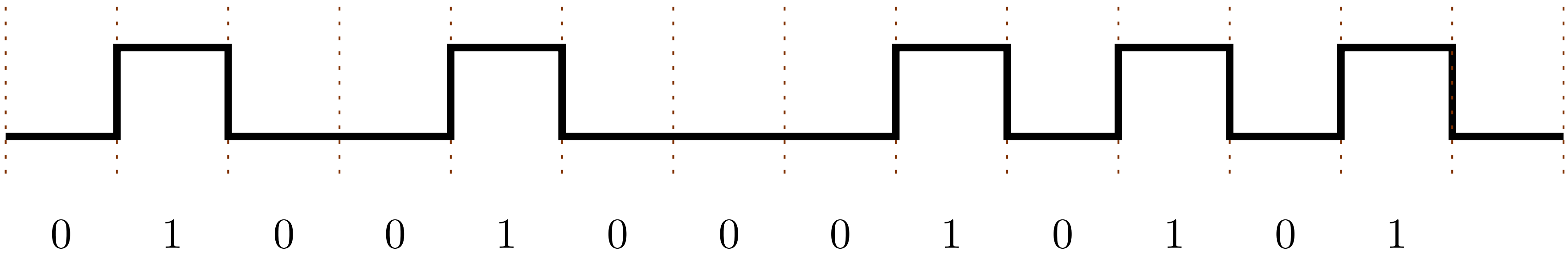

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$$N(T) = \exp\big(C \cdot T + o(T)\big).$$

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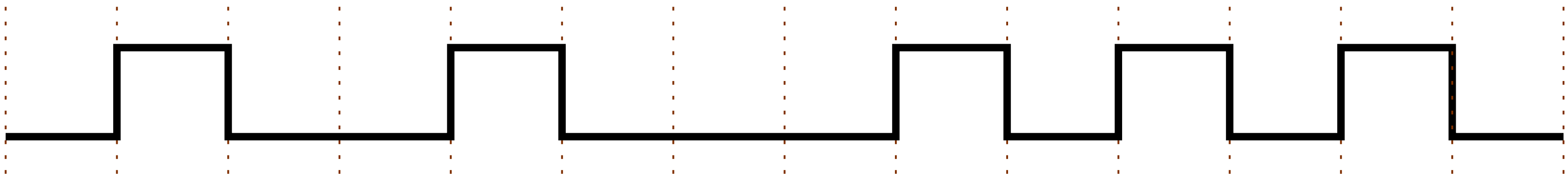

0 1 0 0 1 0 0 0 1 0 1 0 1

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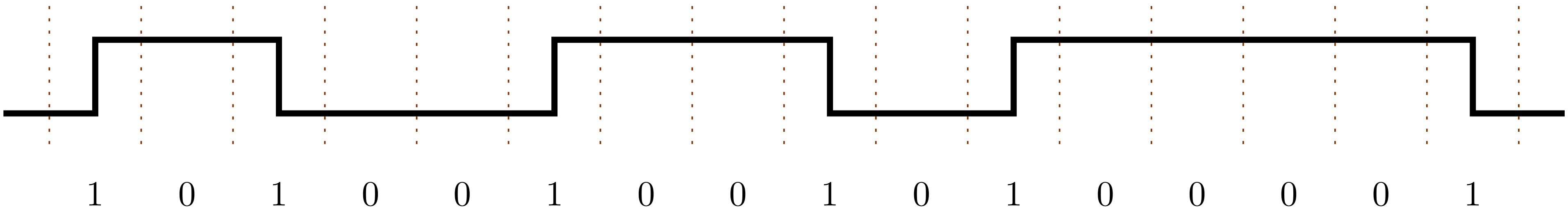

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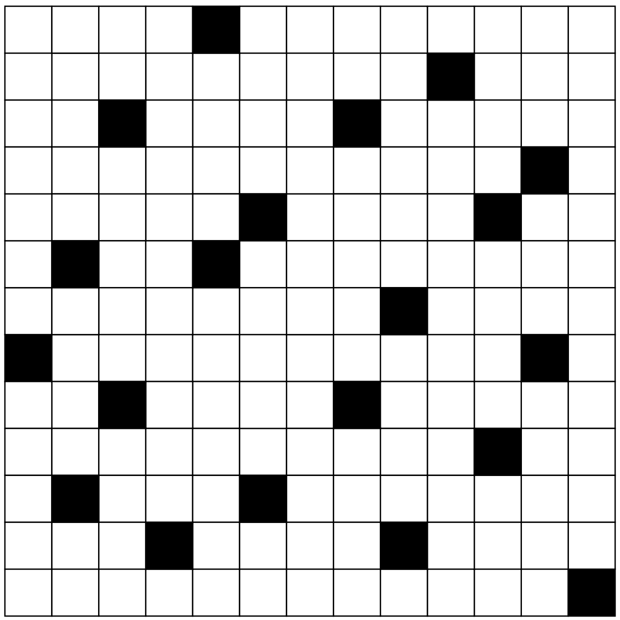
2D constraints in communications

Two-Dimensional RLL Constraints



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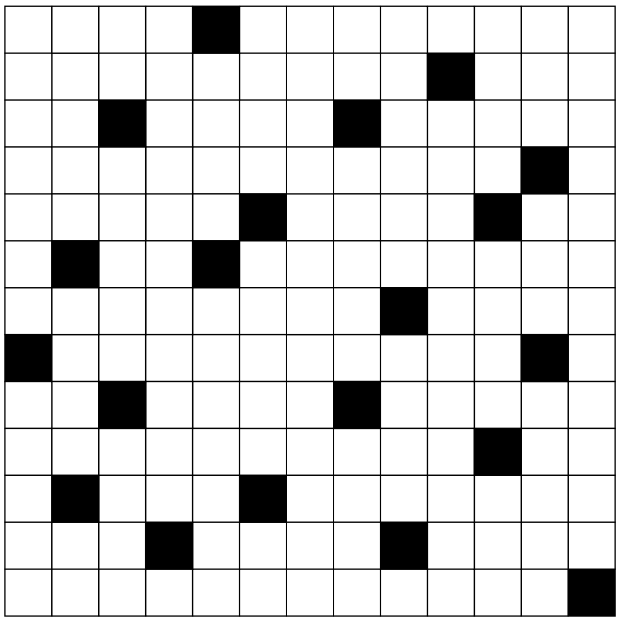

A $(d_1, k; d_2, k_2)$ RLL constraint imposes:

- ...
- ...

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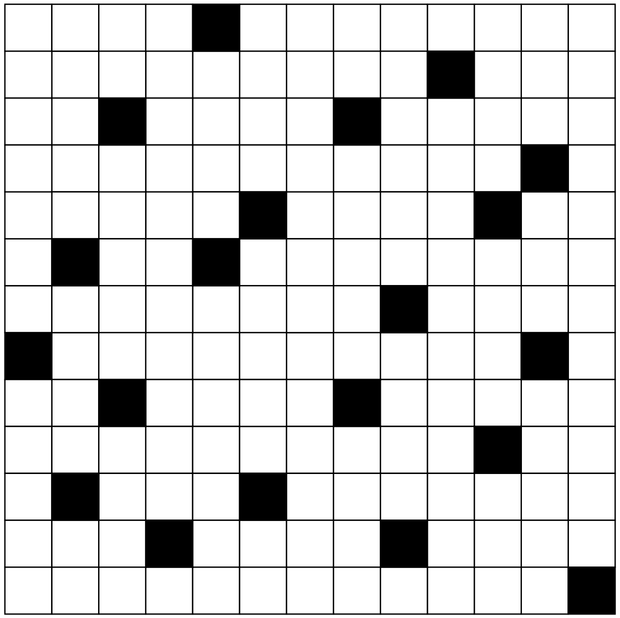

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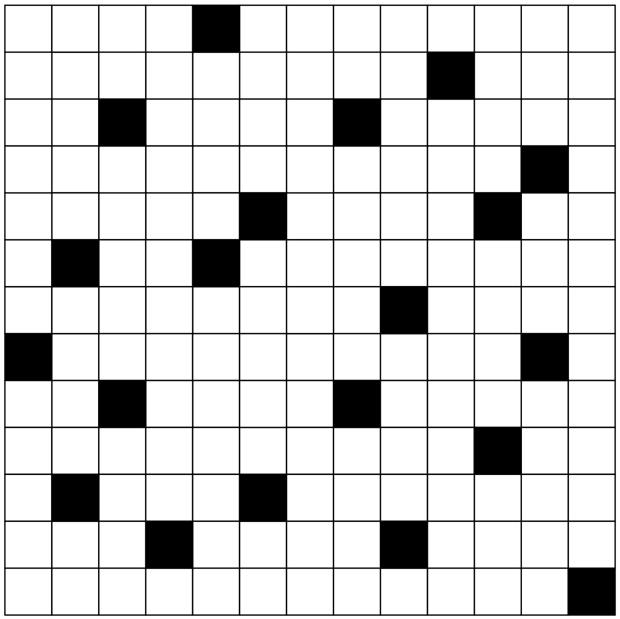

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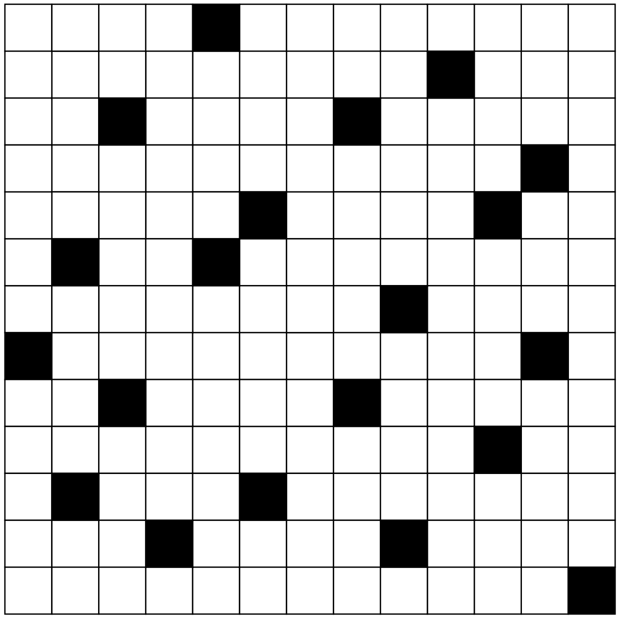

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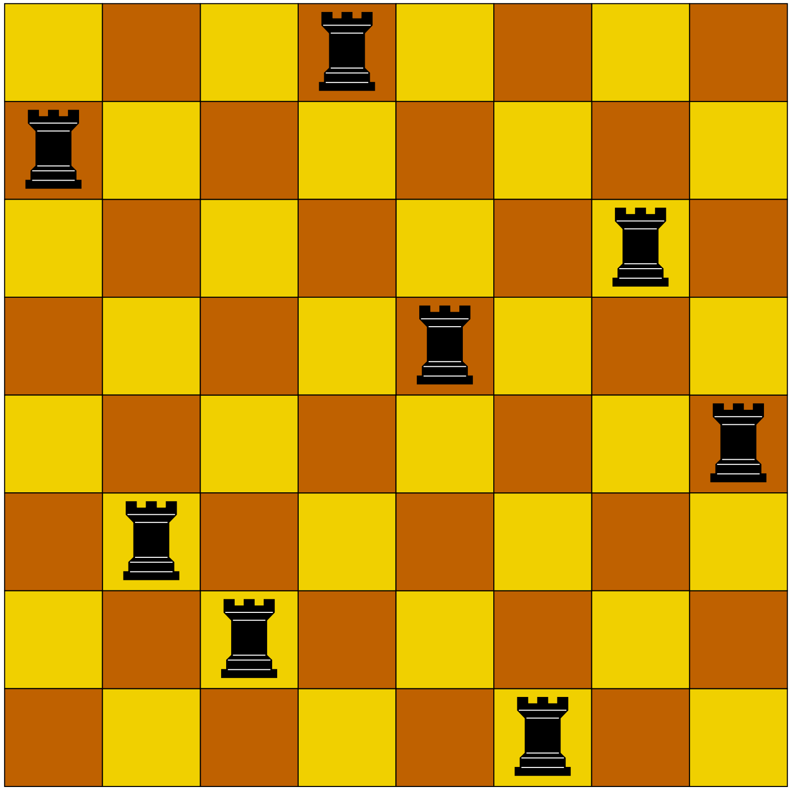
Towards a graphical model

Towards a Graphical Model



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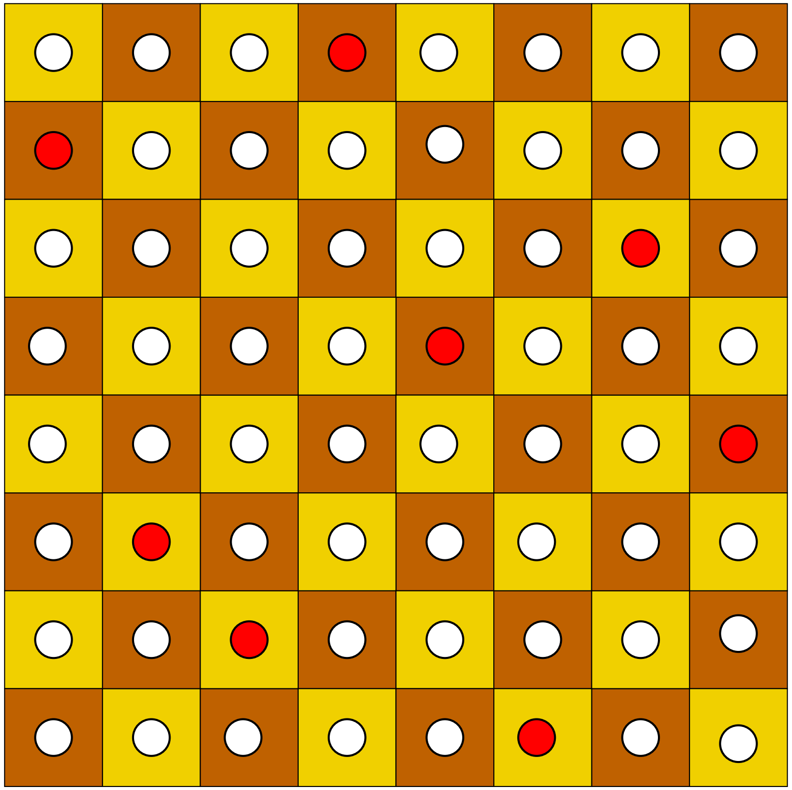

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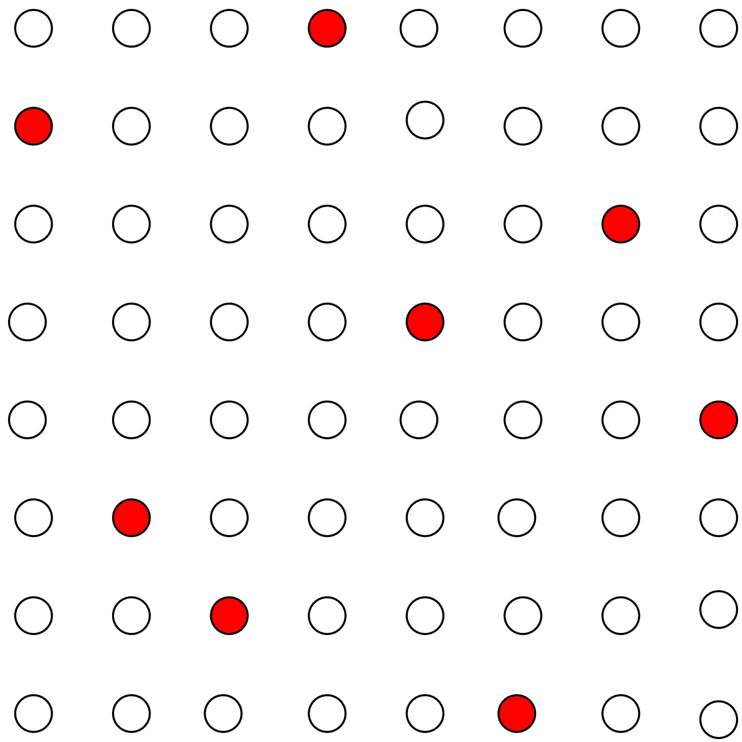

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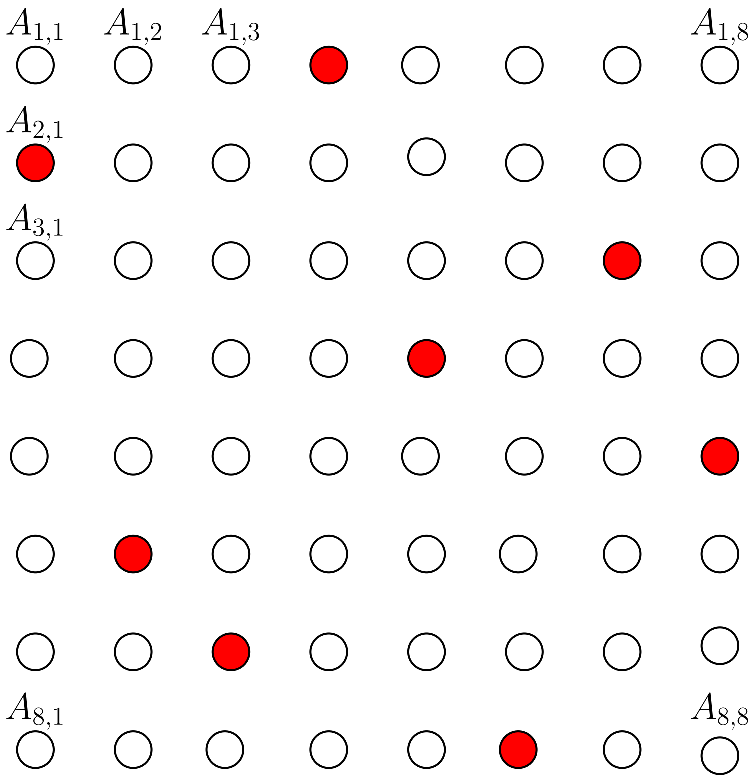

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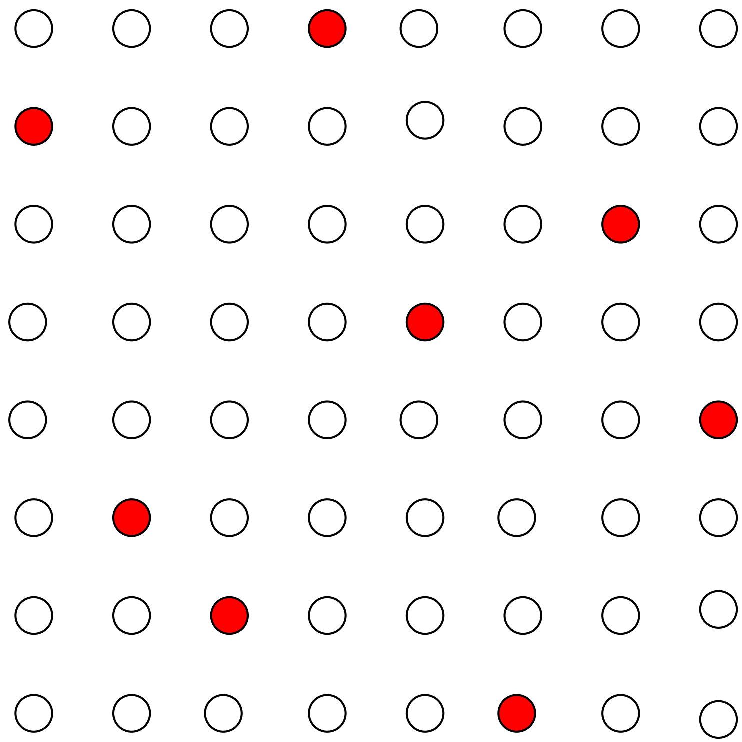

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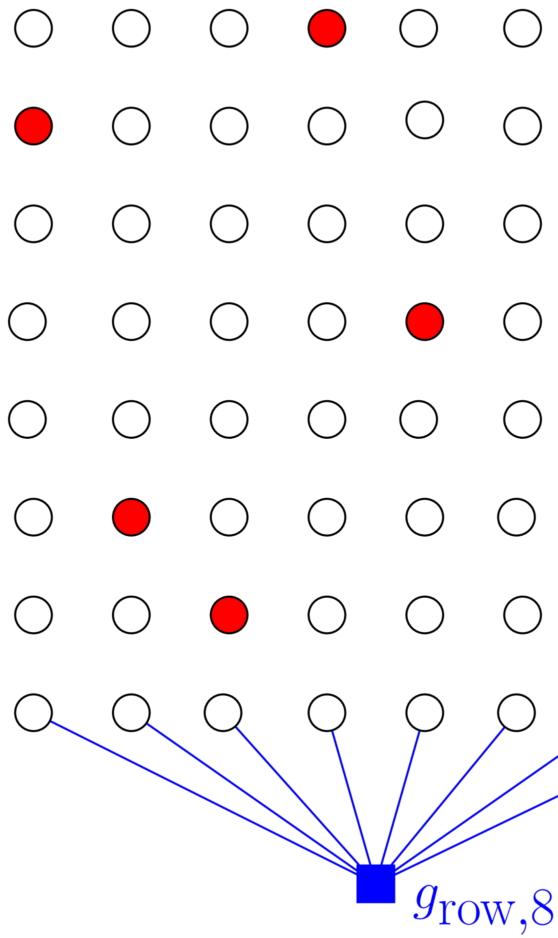
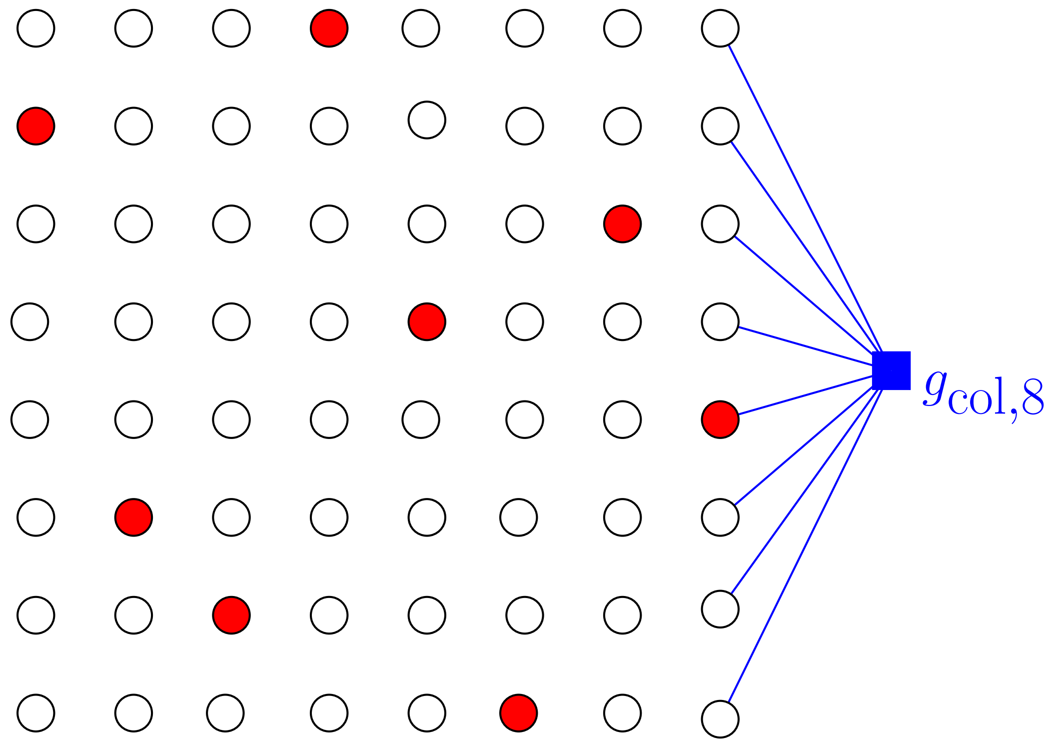


$$g_{\mathrm{row},8}(a_{8,1},\ldots,a_{8,8}) \triangleq \begin{cases} 1 & \text{exactly one rook} \\ 0 & \text{otherwise} \end{cases}$$

Towards a Graphical Model



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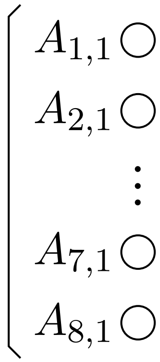

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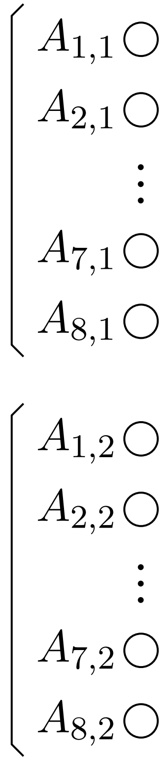

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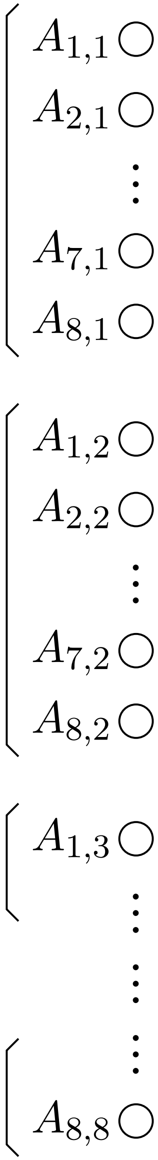

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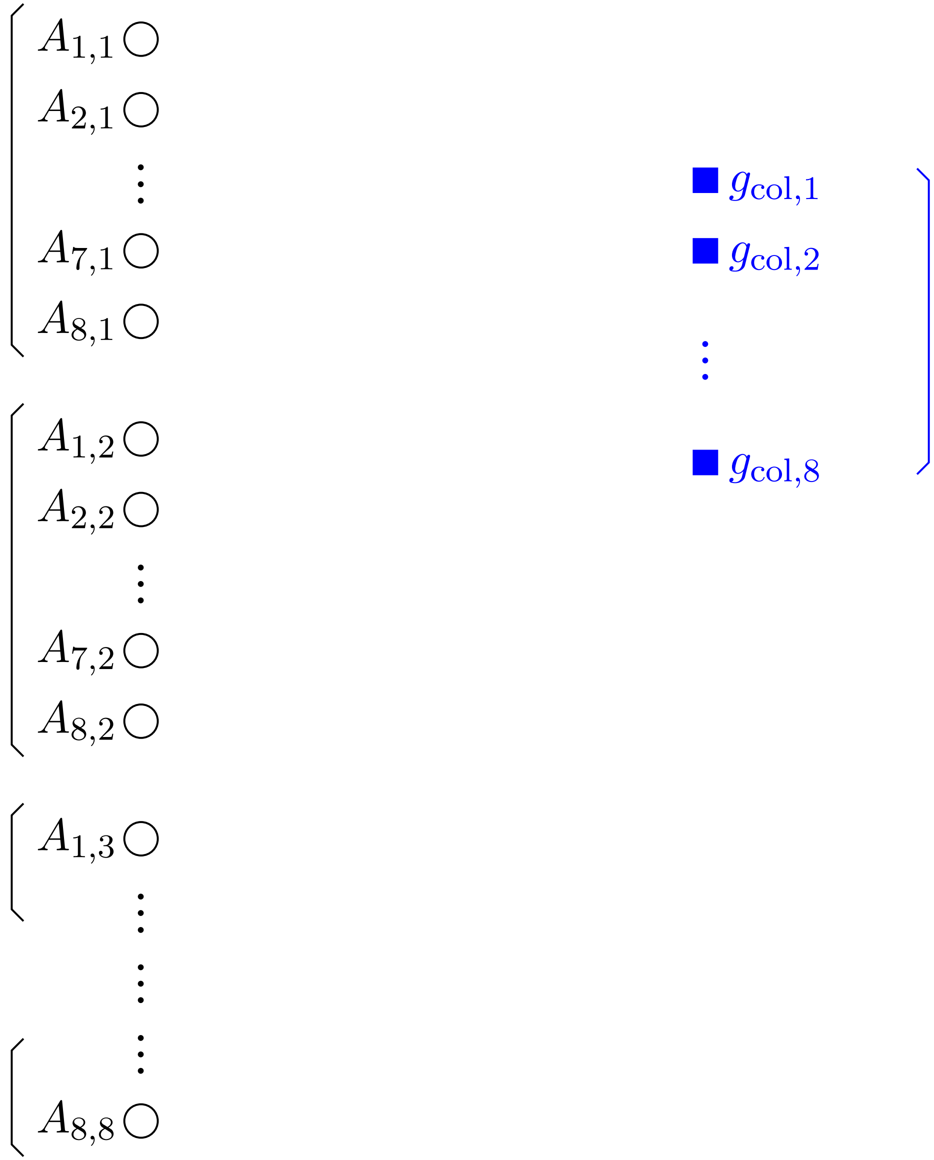

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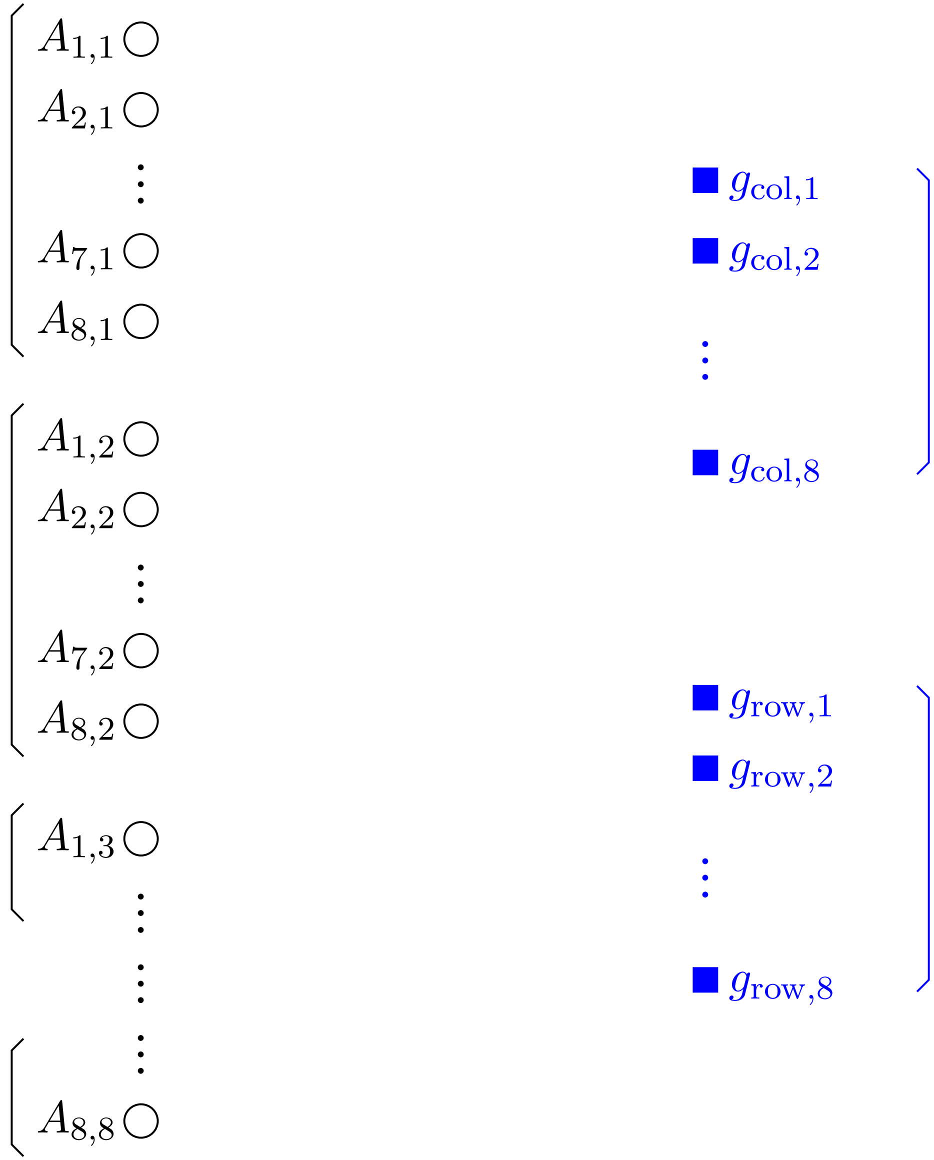

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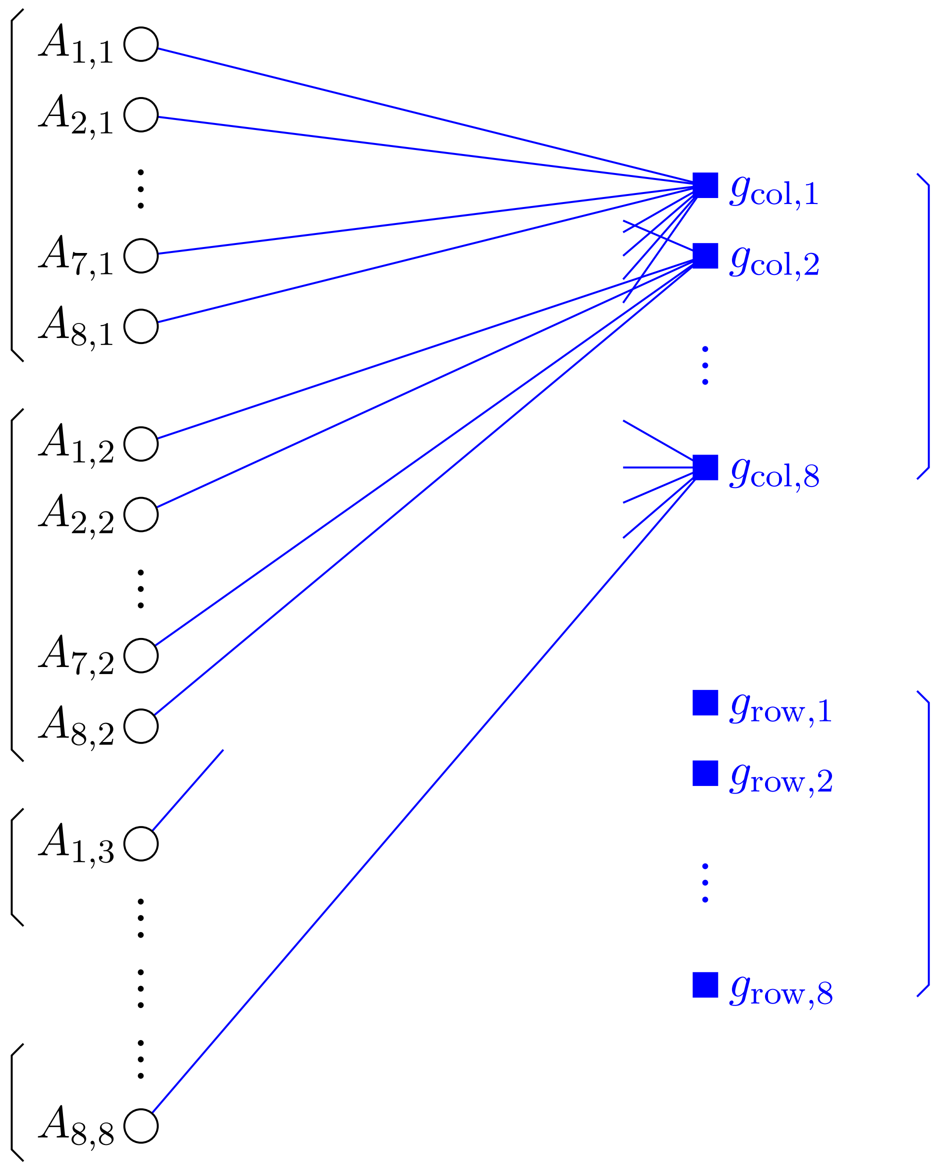

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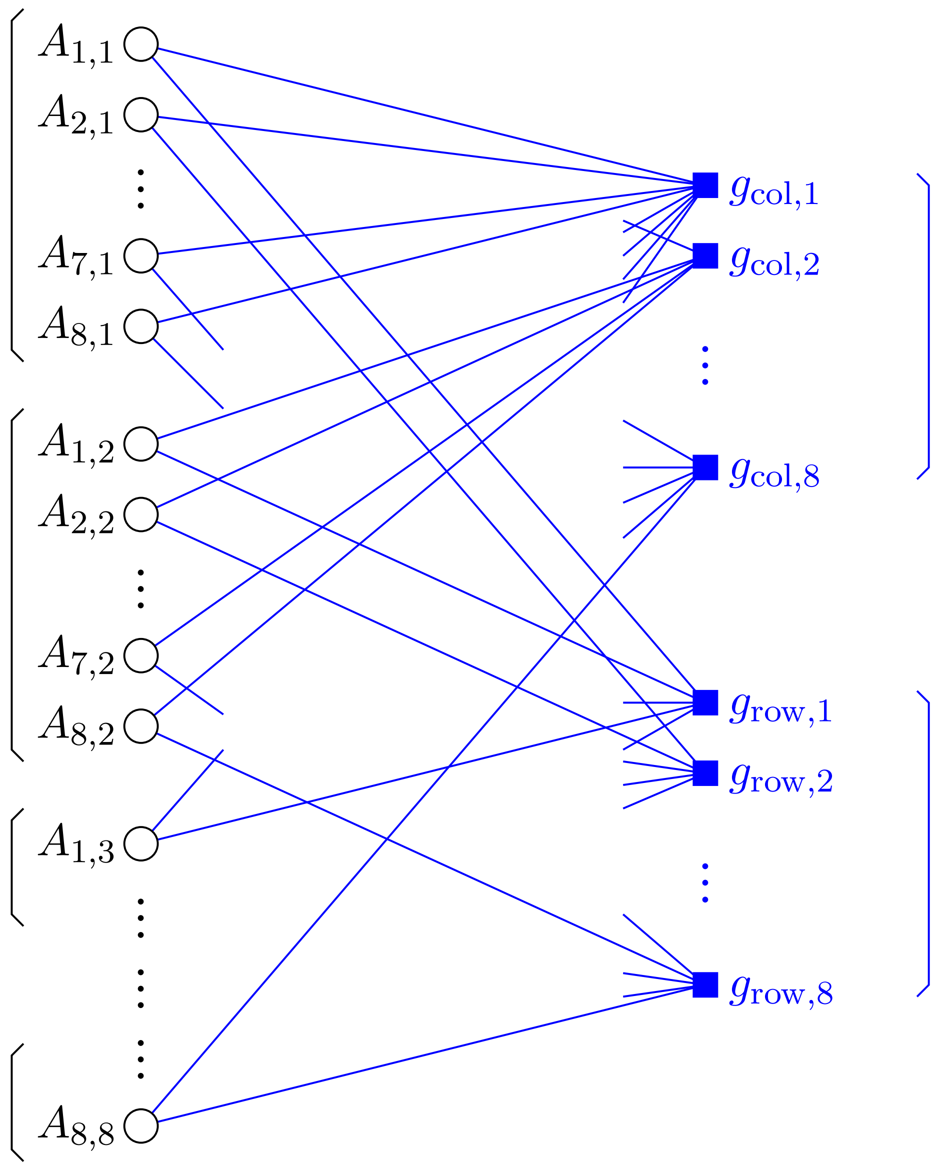

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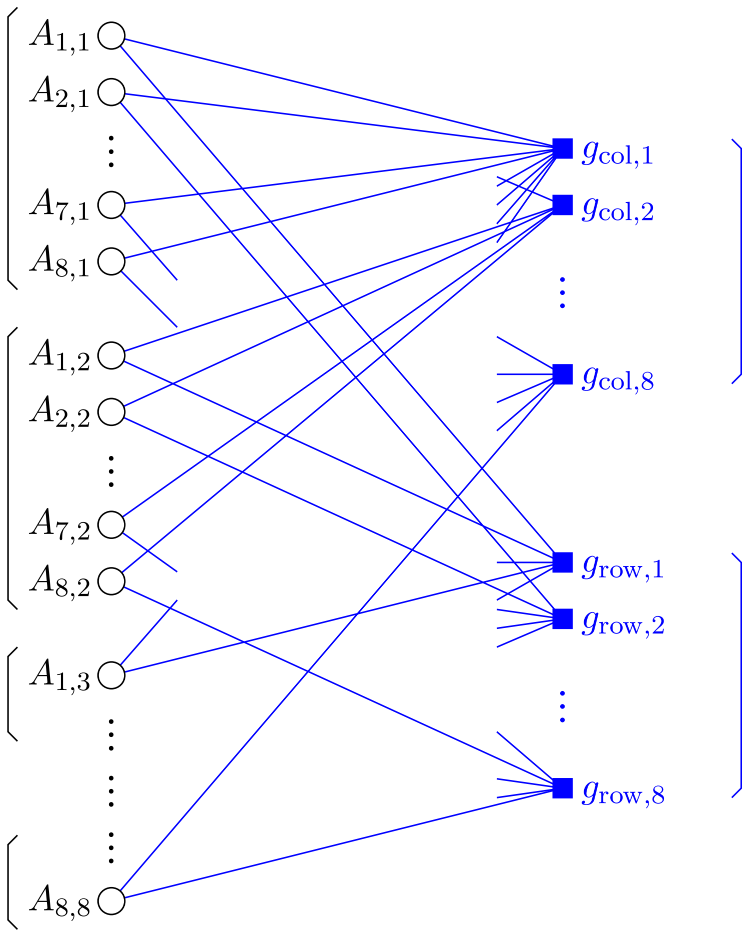

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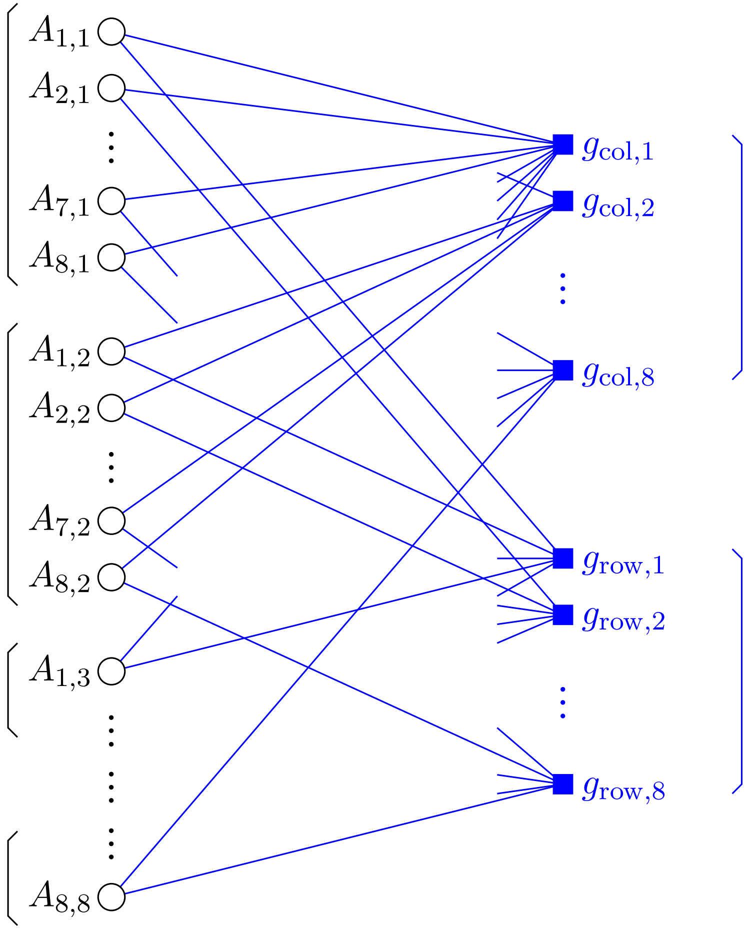
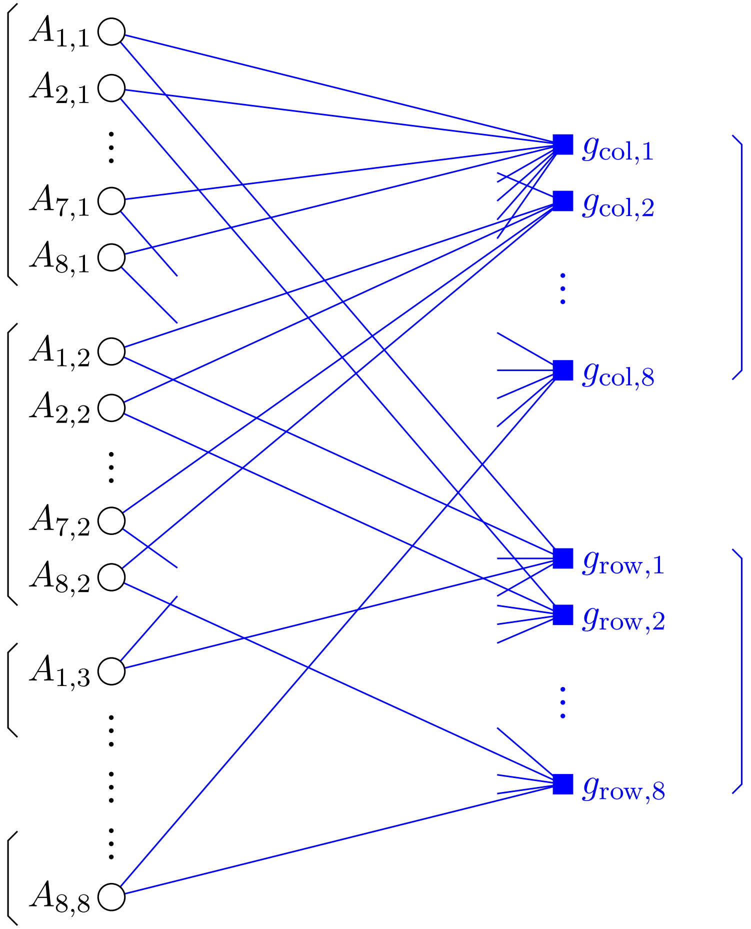


Global function:

$$
\begin{aligned}
g(a_{1,1}, \ldots, a_{8,8}) \\
&= \prod_j g_{\mathrm{col},j}(a_{1,j}, \ldots, a_{8,j}) \times \\
&\prod_i g_{\mathrm{row},i}(a_{i,1}, \ldots, a_{i,8})
\end{aligned}
$$

Towards a Graphical Model



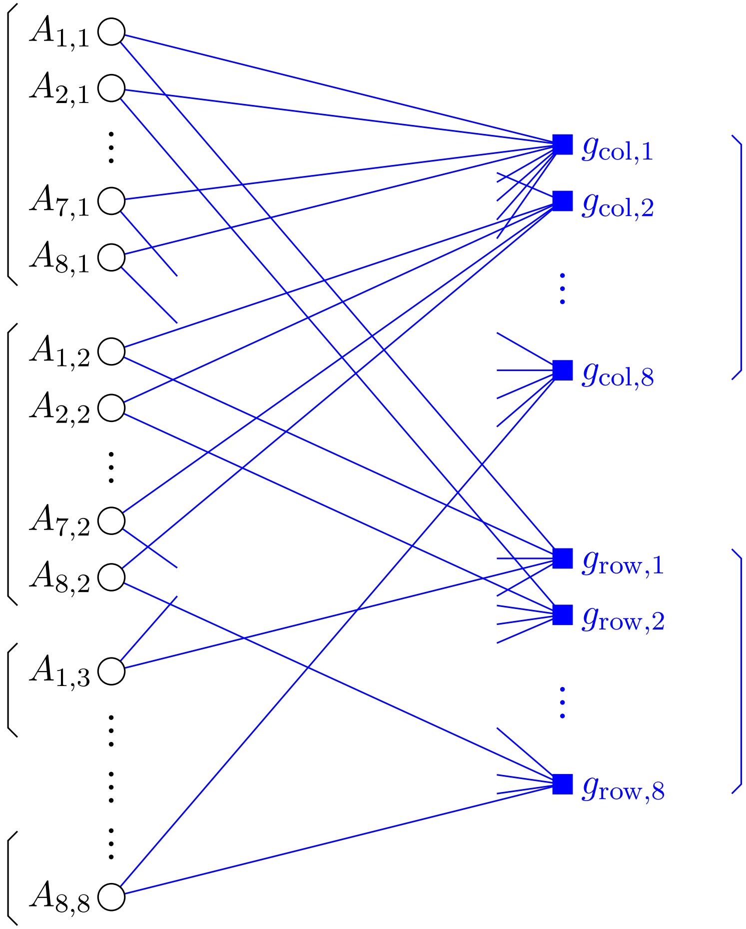
Global function:

$$g(a_{1,1}, \ldots, a_{8,8}) = \prod_j g_{\mathrm{col},j}(a_{1,j}, \ldots, a_{8,j}) \times \prod_i g_{\mathrm{row},i}(a_{i,1}, \ldots, a_{i,8})$$

Total sum:

$$Z = \sum_{a_{1,1},\ldots,a_{8,8}} g(a_{1,1}, \ldots, a_{8,8})$$

Towards a Graphical Model



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Total sum (partition function):

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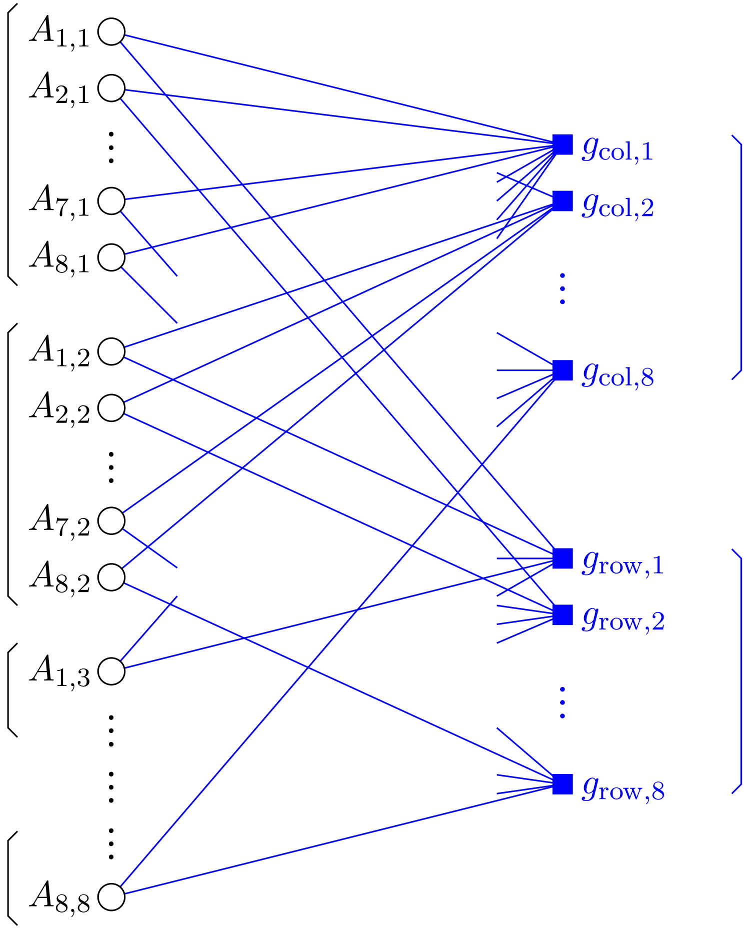
Towards a Graphical Model



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Use of loopy belief propagation for approximating Z?

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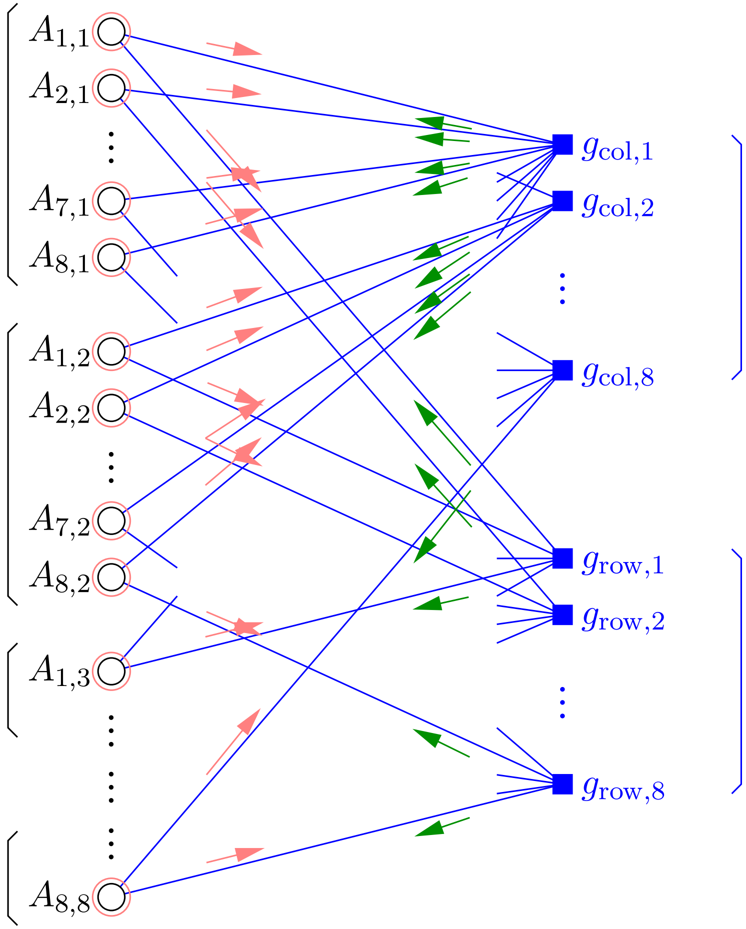

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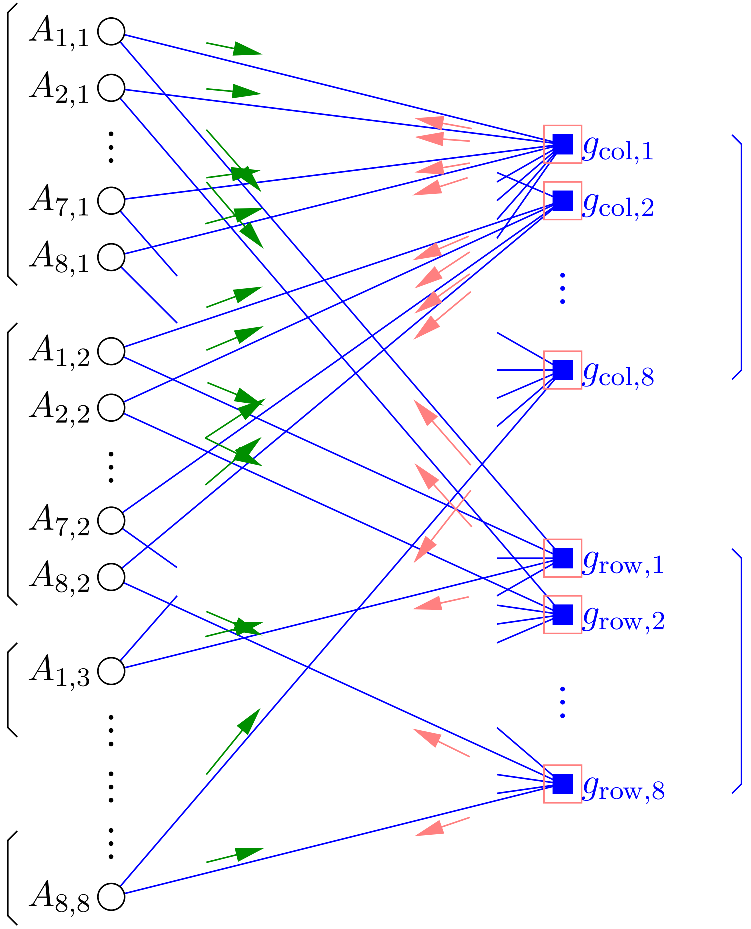

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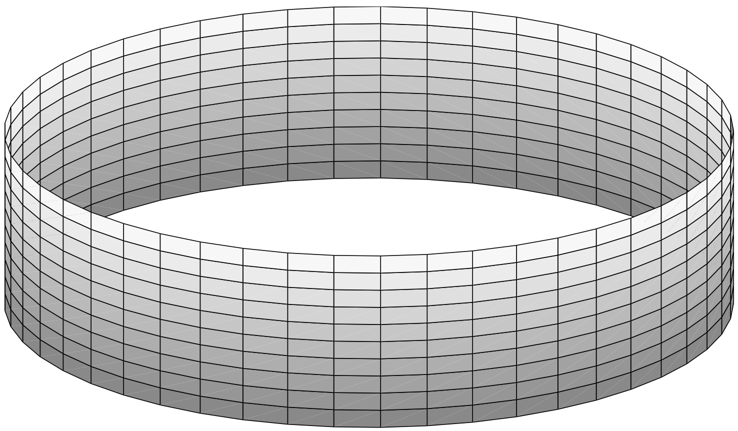
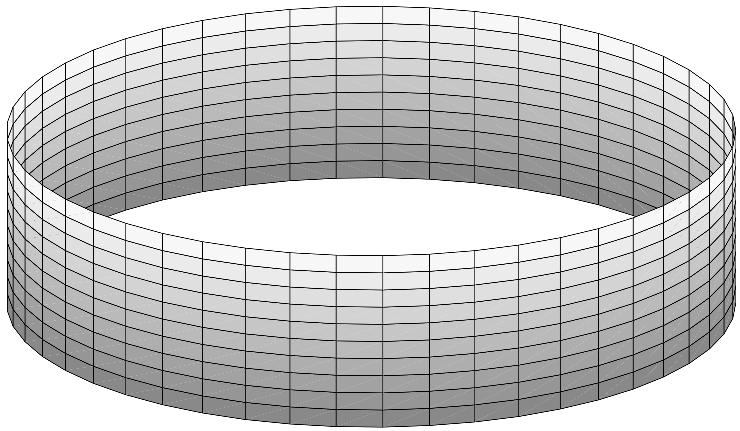
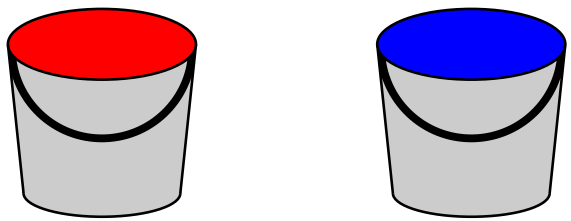
Some considerations on counting algorithms

Coloring the Surfaces of a Closed Strip



Coloring the Surfaces of a Closed Strip





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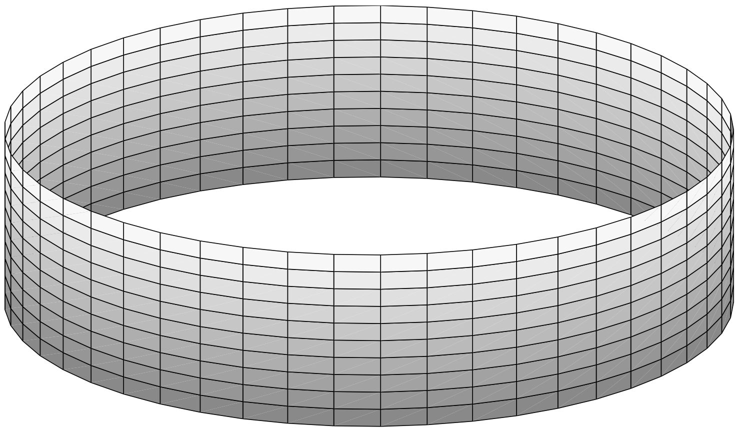

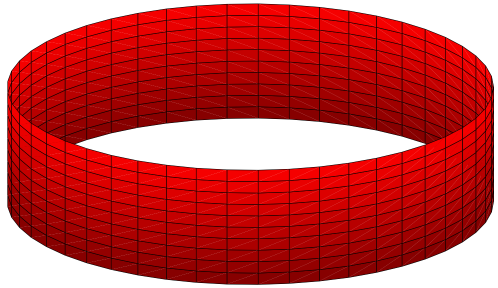

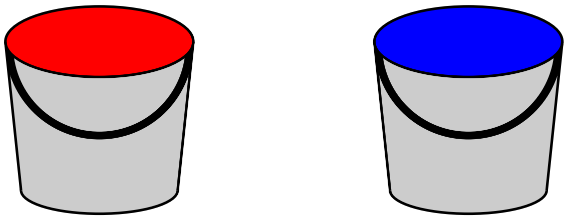

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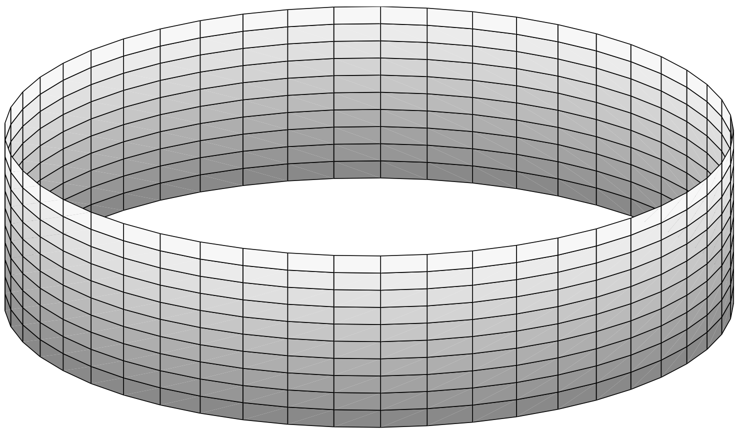

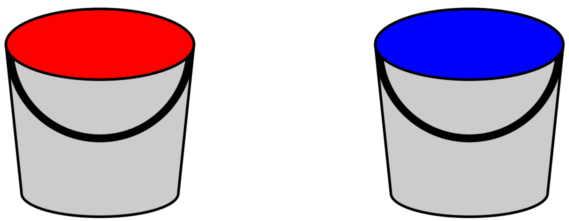

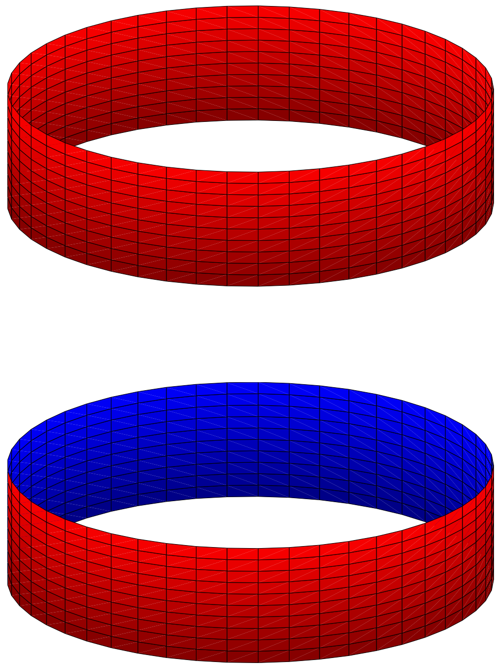

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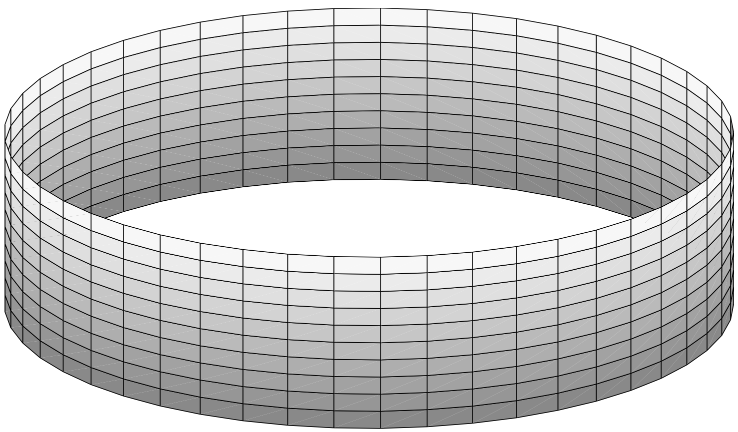

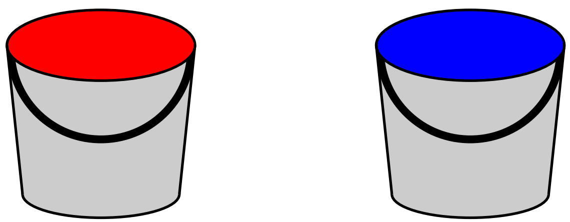

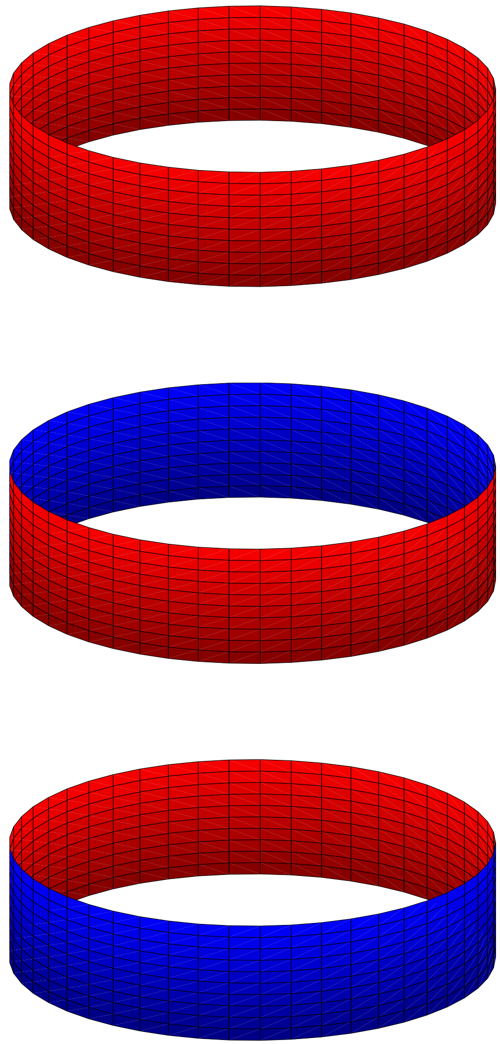

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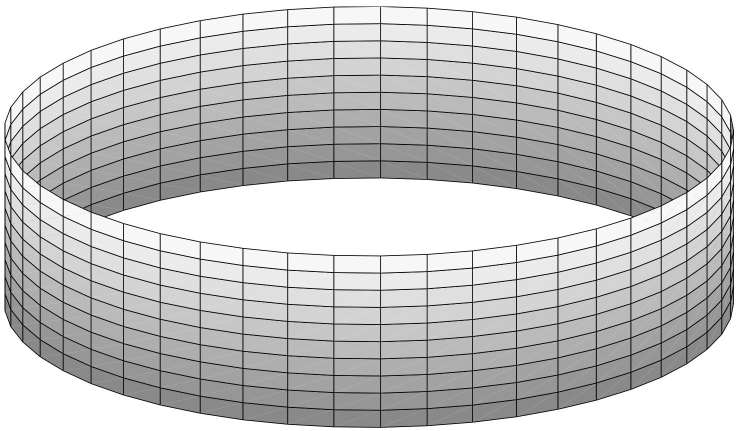

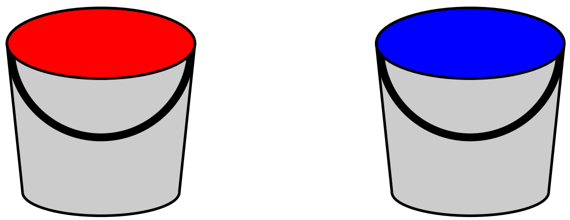

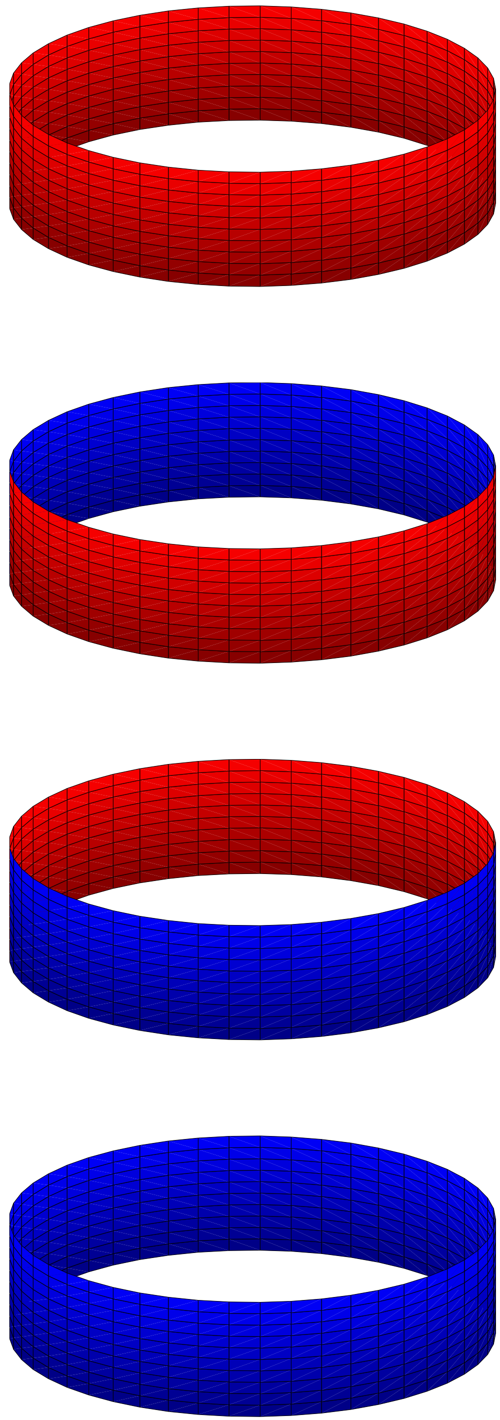

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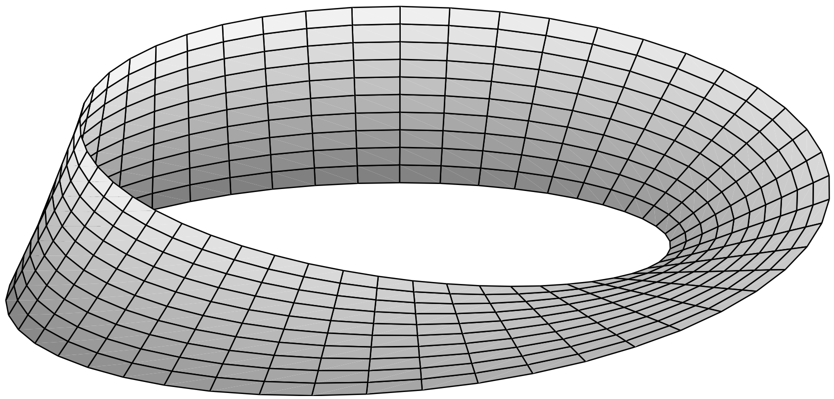

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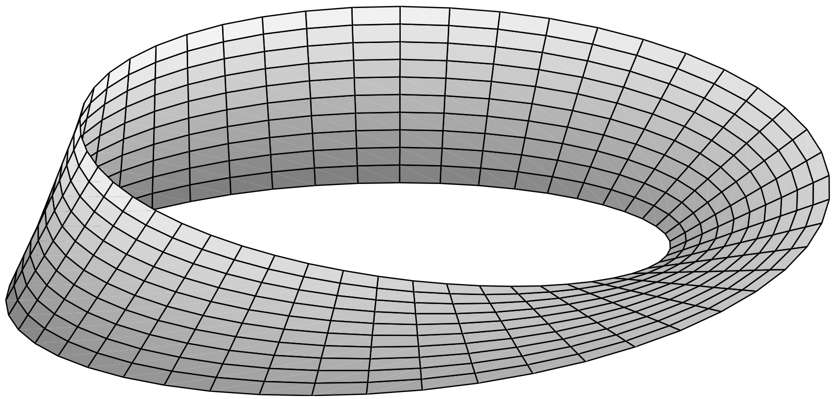

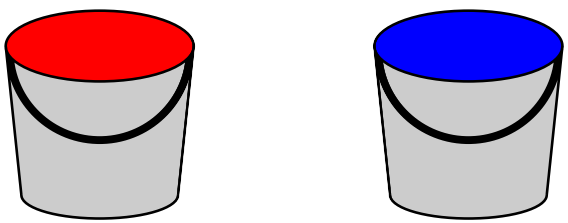

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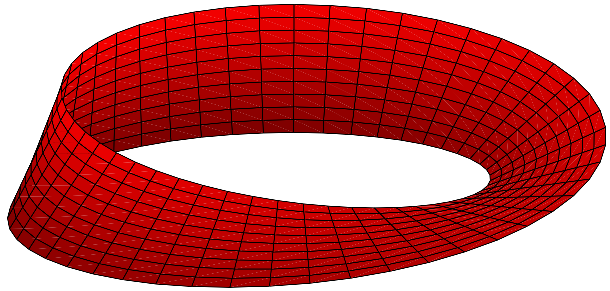

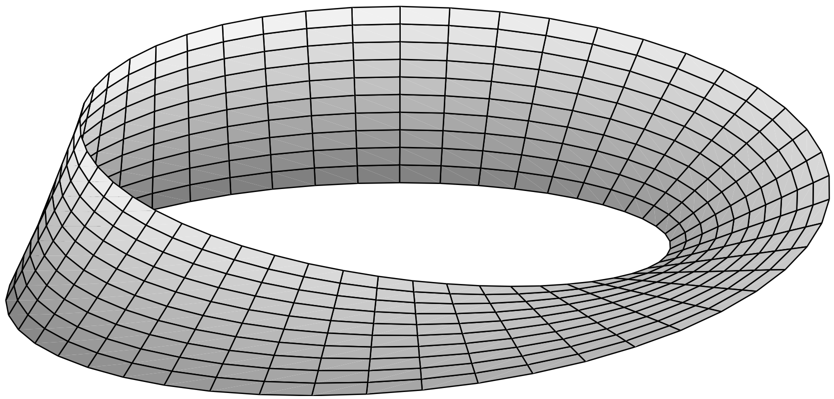

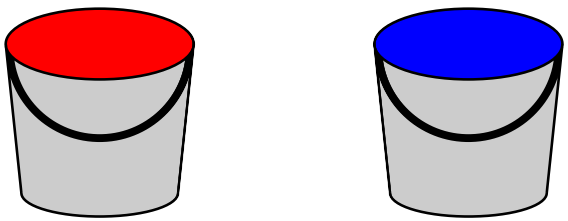

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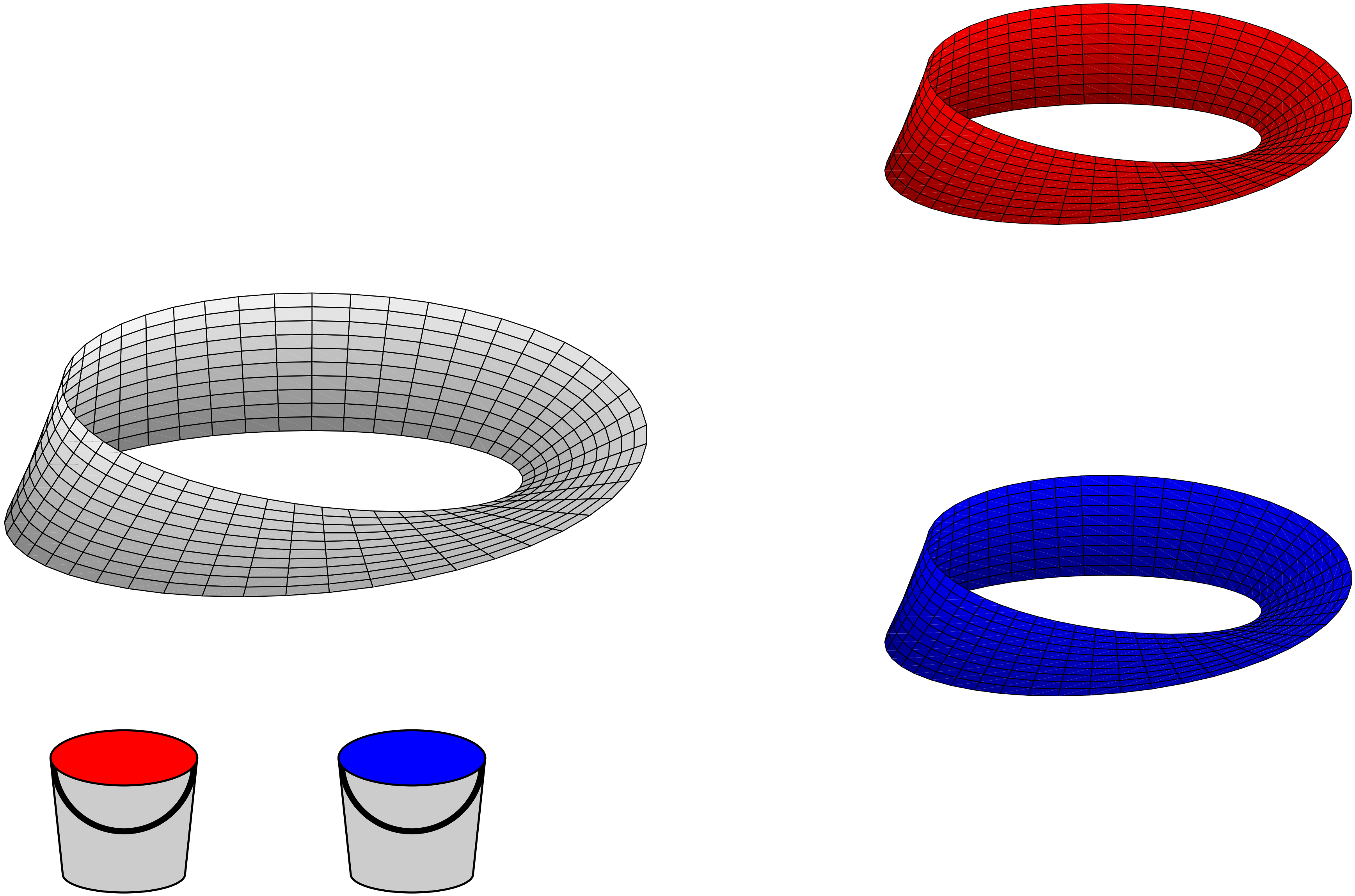
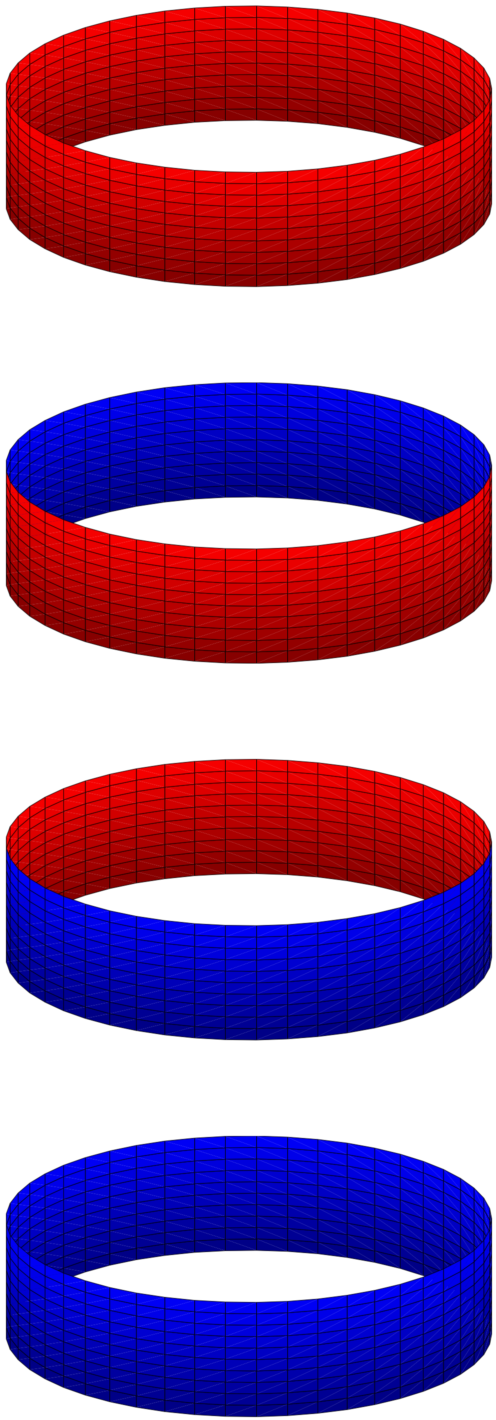
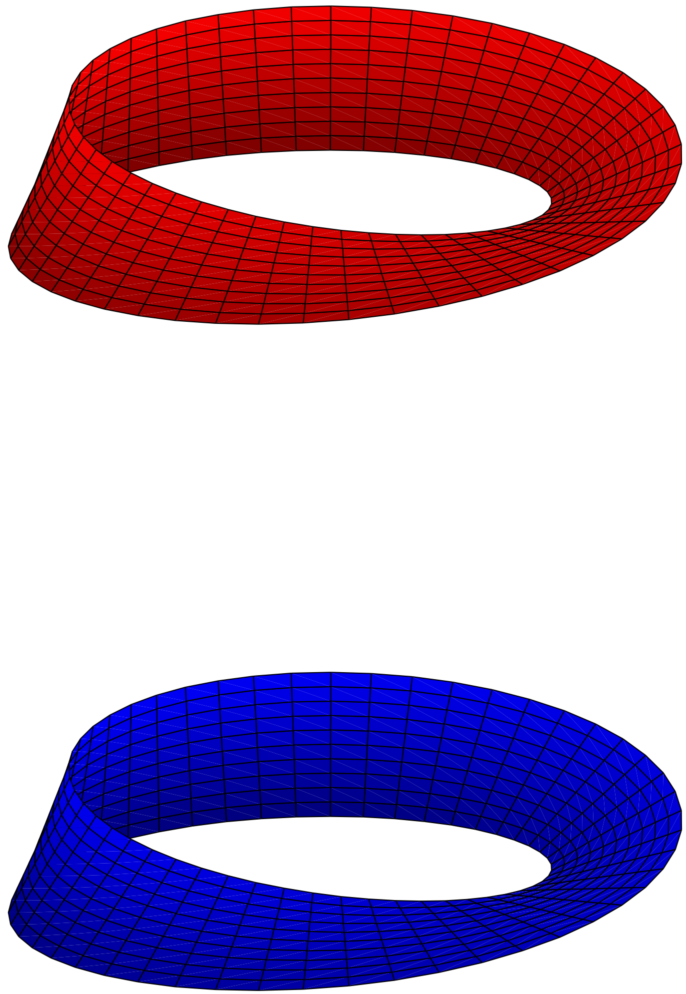
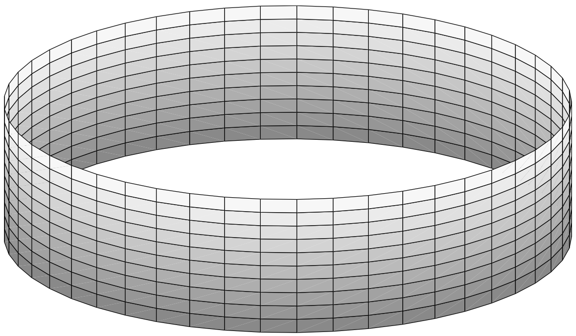
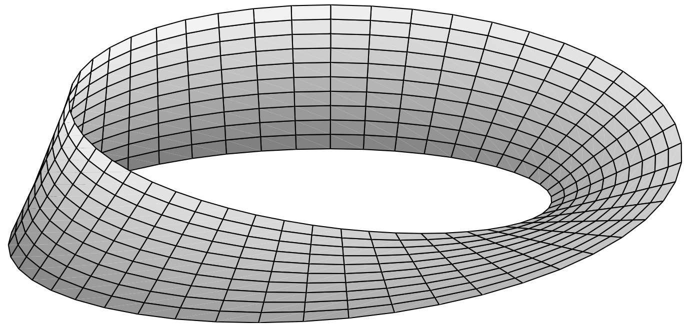
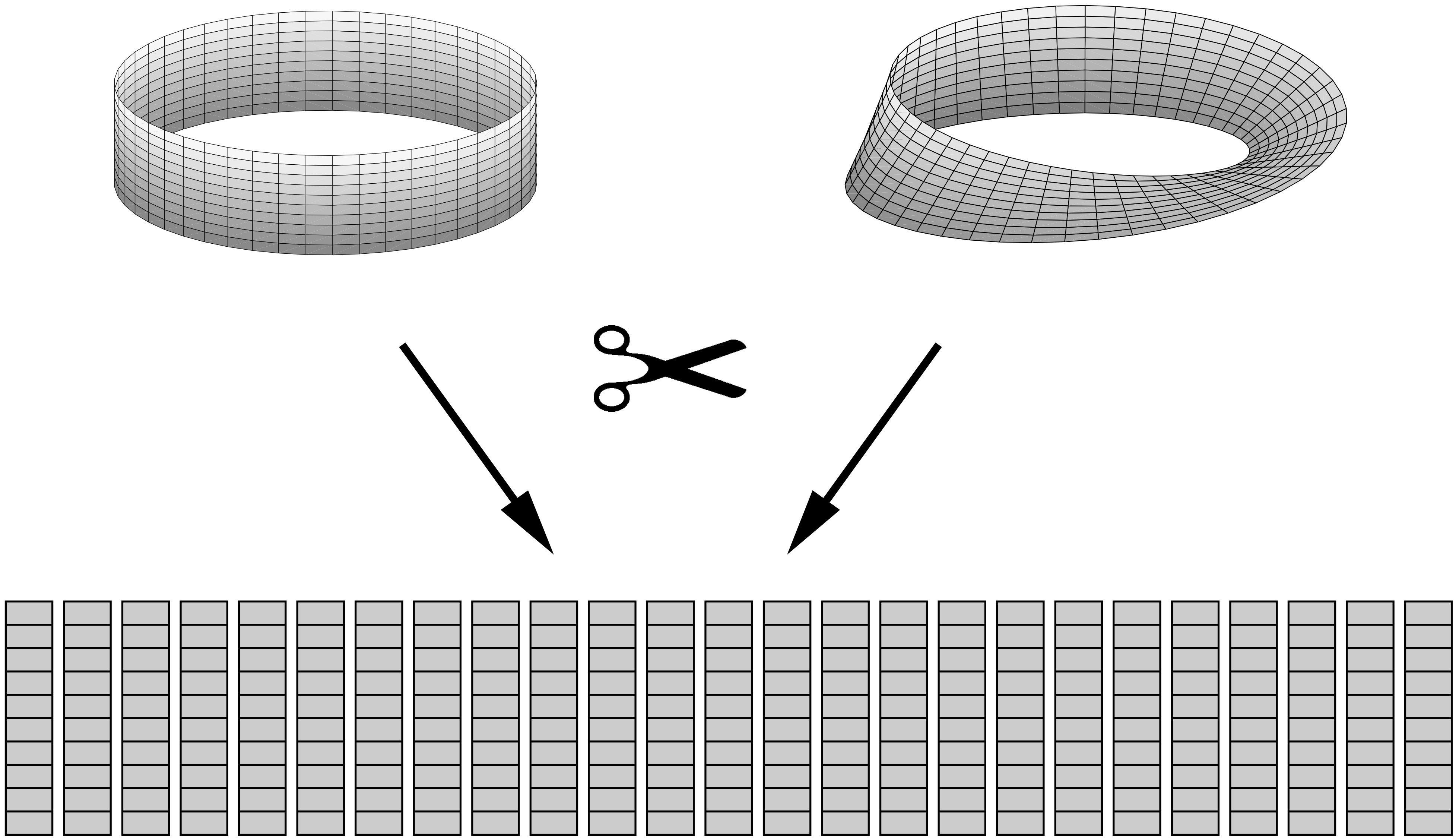




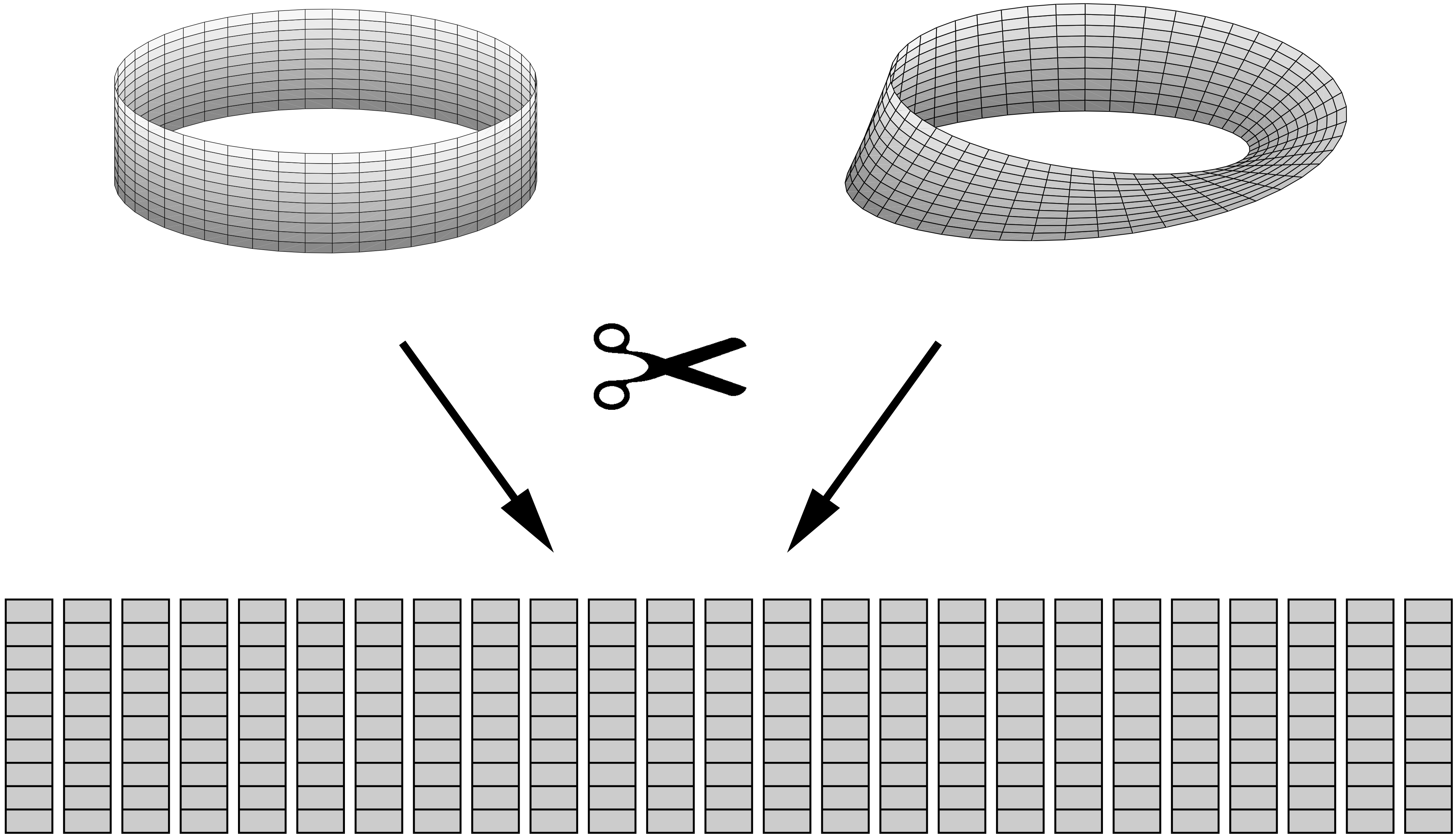


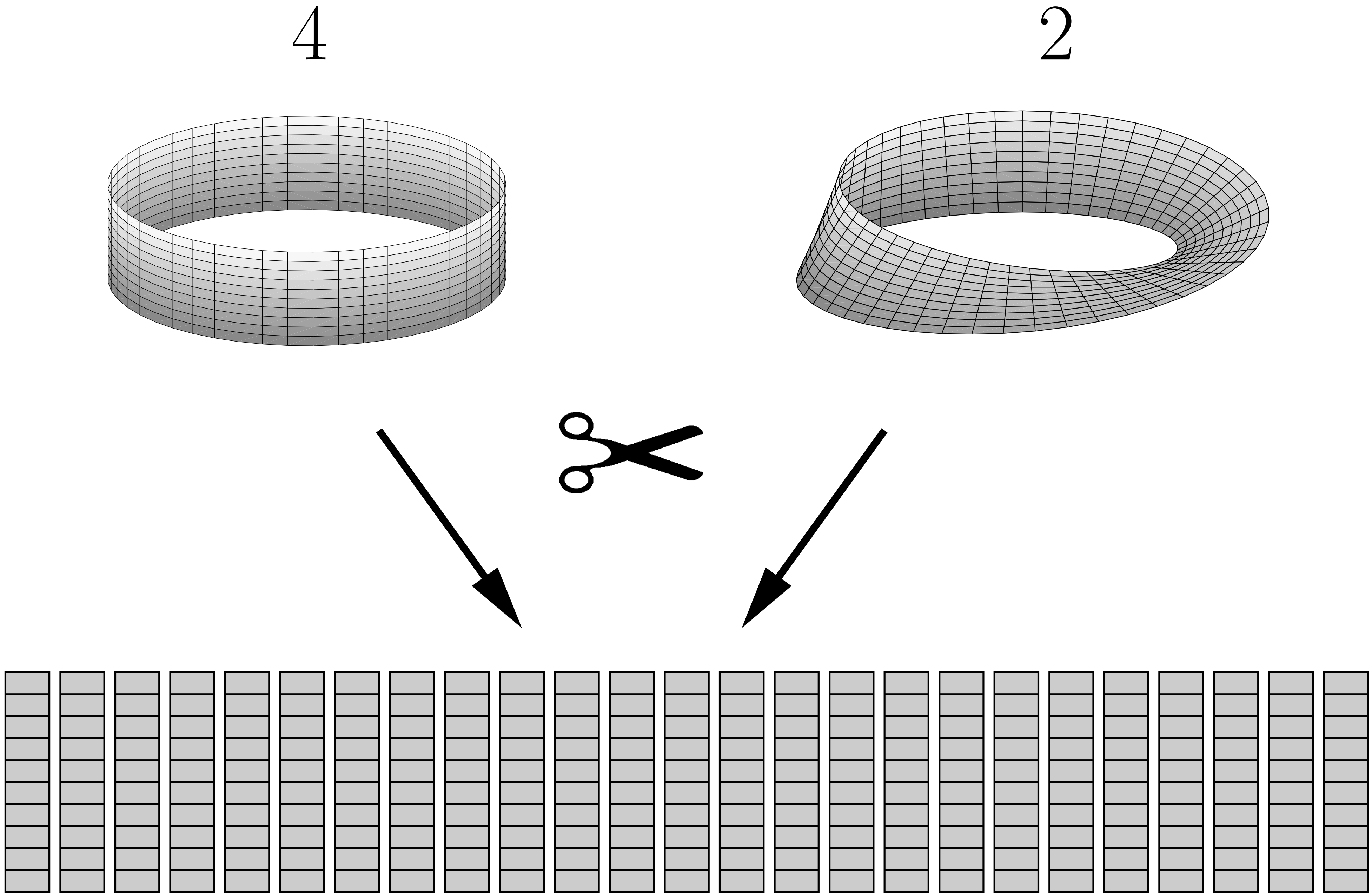




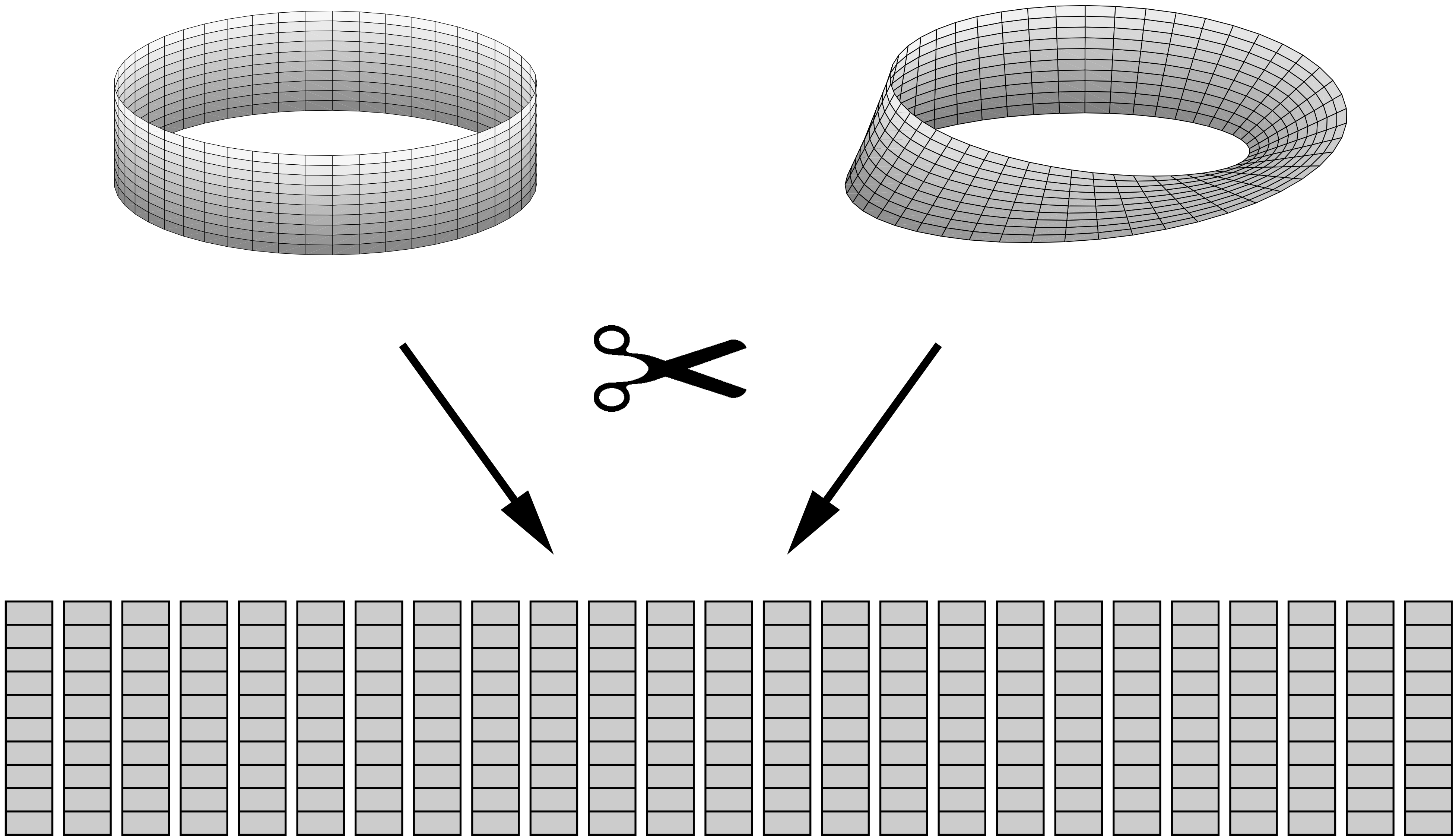


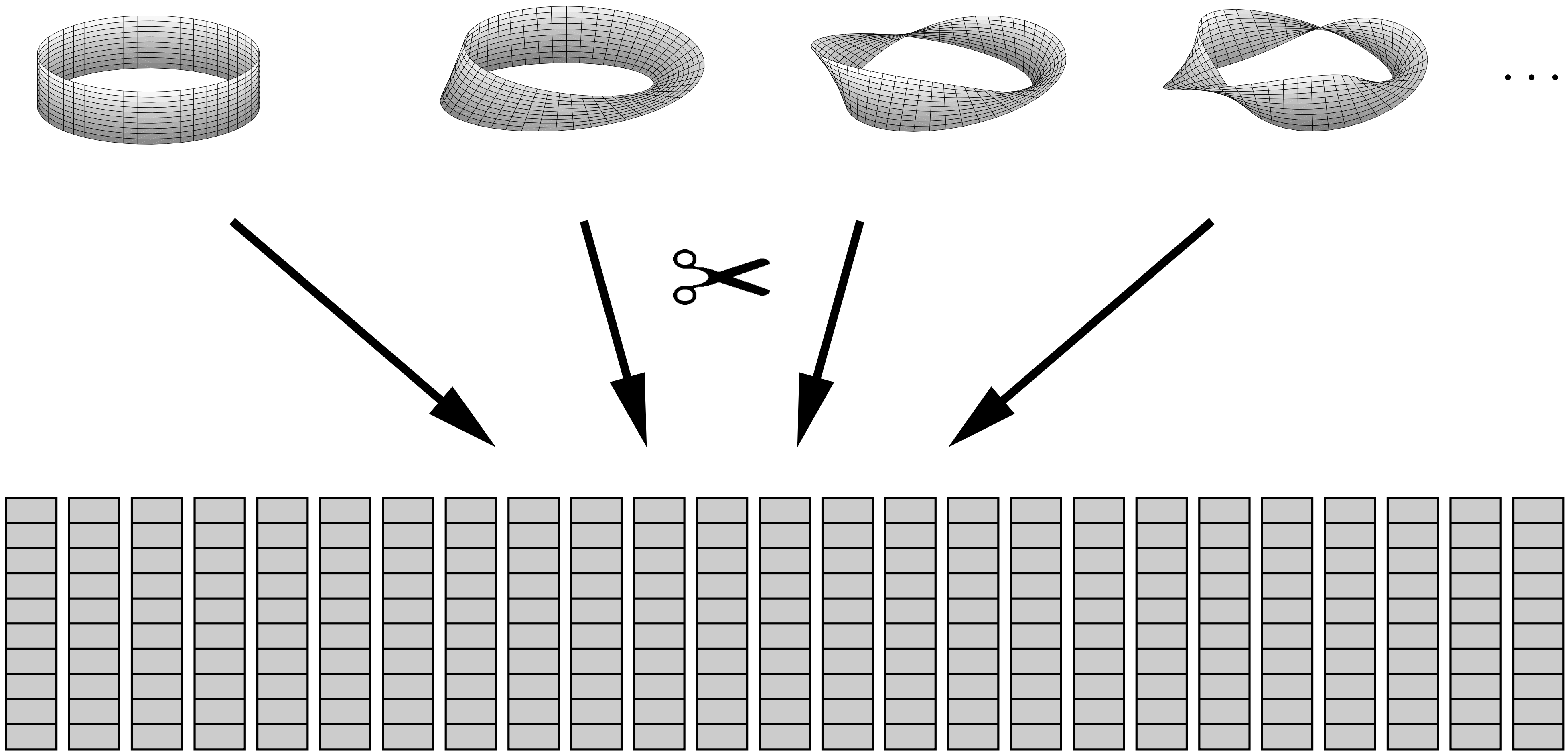
4

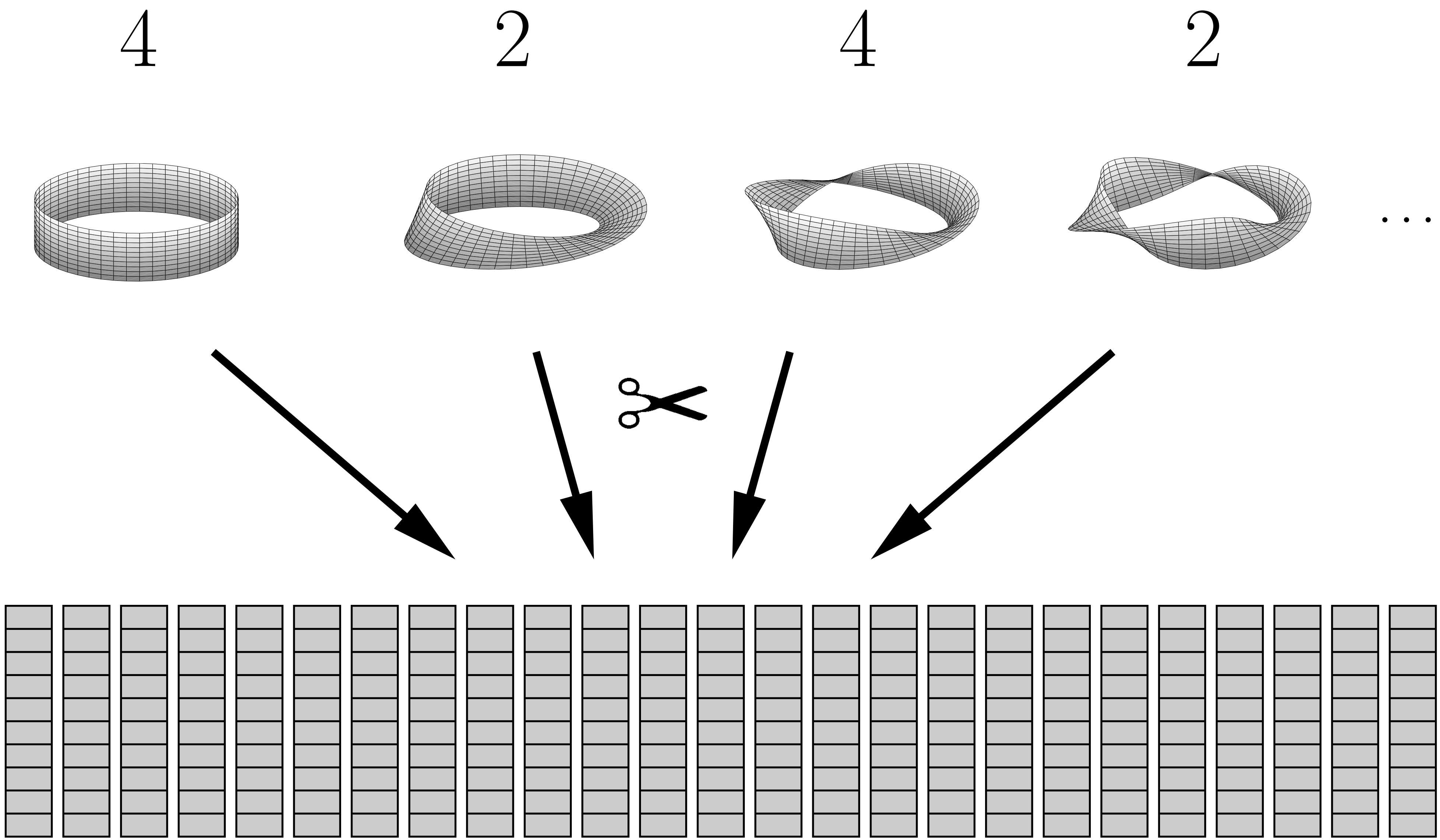



4
2

4 $\qquad \frac{4+2}{2} = 3 \qquad$ 2






4
2
4
2
...

The permanent of a matrix

Determinant vs. Permanent of a Matrix

Consider the matrix $\boldsymbol{\theta} = \begin{pmatrix} \theta_{11} & \theta_{12} & \theta_{13} \\ \theta_{21} & \theta_{22} & \theta_{23} \\ \theta_{31} & \theta_{32} & \theta_{33} \end{pmatrix}$.

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The determinant of $\boldsymbol{\theta}$:

$$\begin{aligned} \det(\boldsymbol{\theta}) = &+\theta_{11}\theta_{22}\theta_{33} + \theta_{12}\theta_{23}\theta_{31} + \theta_{13}\theta_{21}\theta_{32} \\ &- \theta_{11}\theta_{23}\theta_{32} - \theta_{12}\theta_{21}\theta_{33} - \theta_{13}\theta_{22}\theta_{31}. \end{aligned}$$

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Determinant vs. Permanent of a Matrix

The determinant of an $n \times n$-matrix $\boldsymbol{\theta}$

$$\det(\boldsymbol{\theta}) = \sum_{\sigma} \operatorname{sgn}(\sigma) \prod_{i \in [n]} \theta_{i,\sigma(i)}.$$

where the sum is over all $n!$ permutations of the set $[n] \triangleq \{1, \ldots, n\}$.

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The permanent turns up in a variety of context, especially in combinatorial problems, statistical physics (partition function), ...

Exactly Computing the Permanent

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- Complexity class [Valiant, 1979]:

#P ("sharp P" or "number P"),

where #P is the set of the counting problems associated with the decision problems in the set NP. (Note that even the computation of the permanent of zero-one matrices is #P-complete.)

Estimating the Permanent

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  $\rightarrow$ "constructive interference of terms in the summation."

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FROM NOW ON: we focus on the case where all entries of the matrix are non-negative, i.e.

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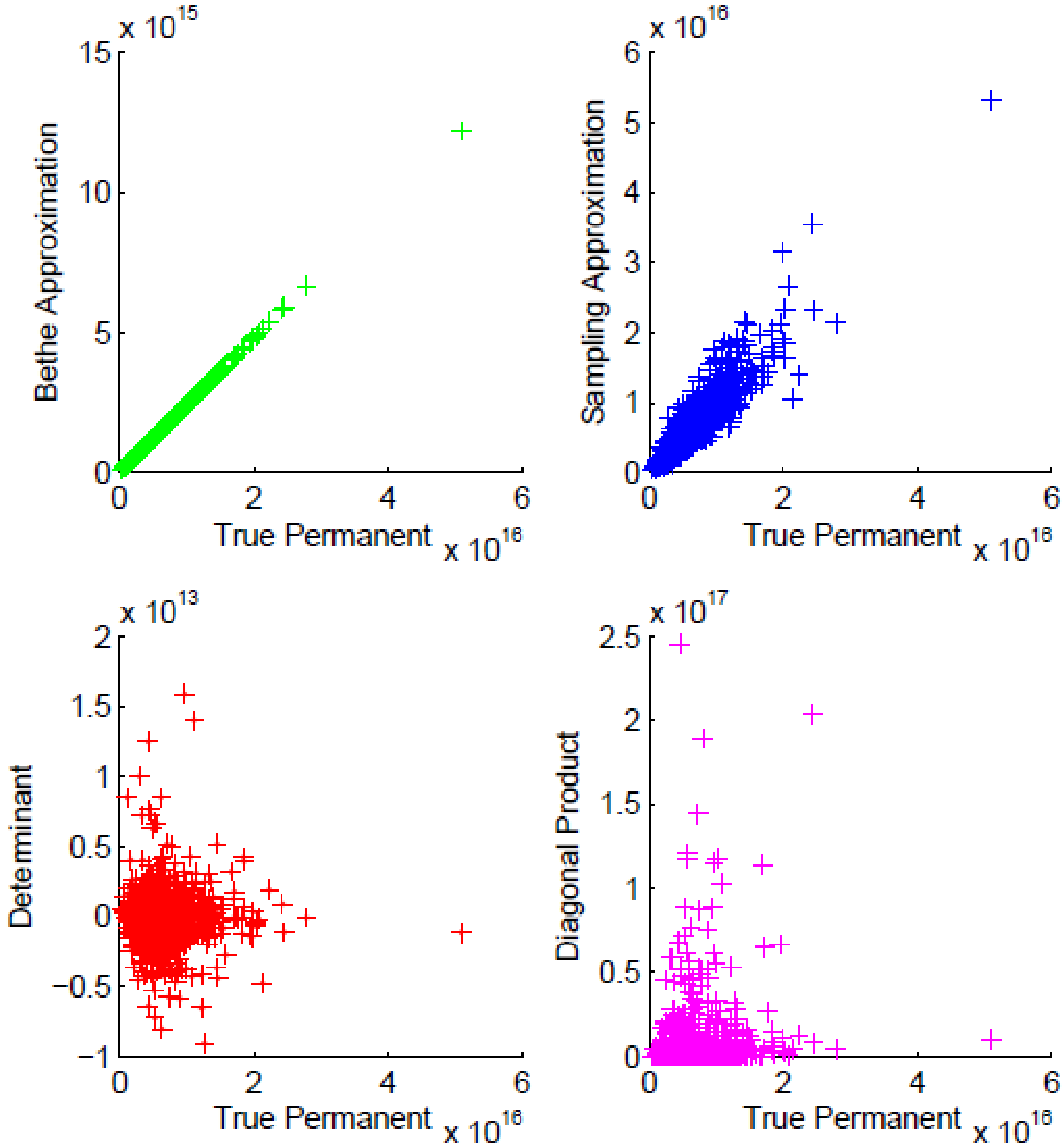
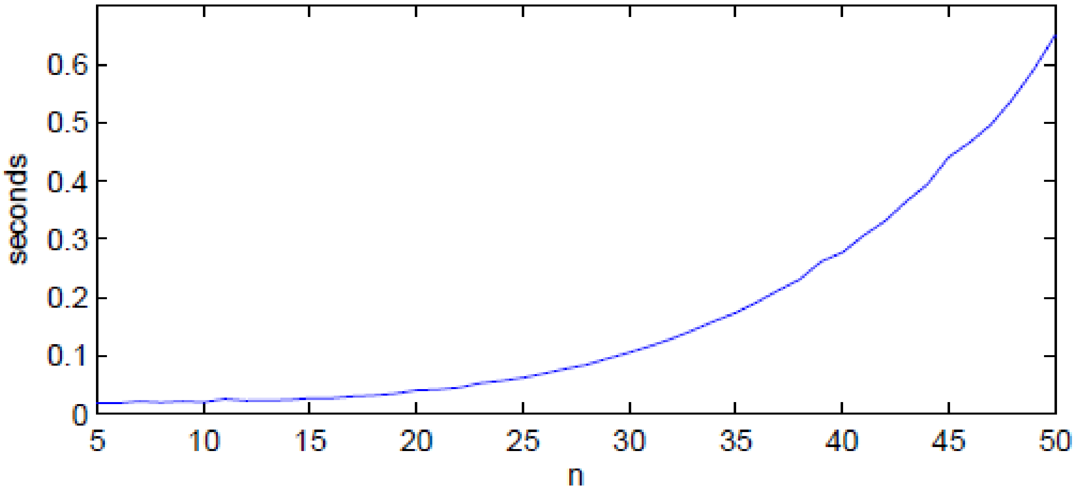
- Bethe-approximation-based / sum-product-algorithm-based methods: [Chertkov et al., 2008], [Huang and Jebara, 2009], ...

Estimating the Permanent of a Matrix

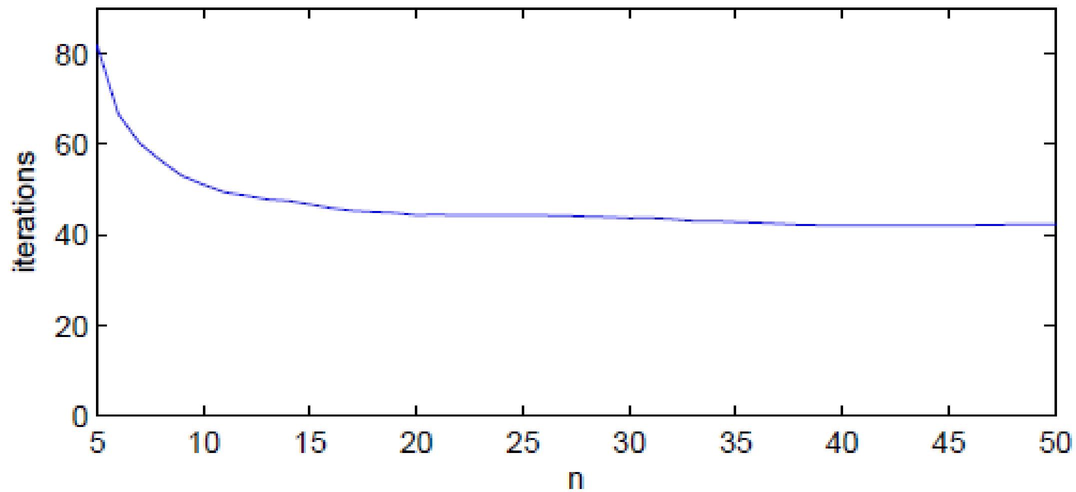


From [Huang/Jebara, 2009].

Estimating the Permanent of a Matrix



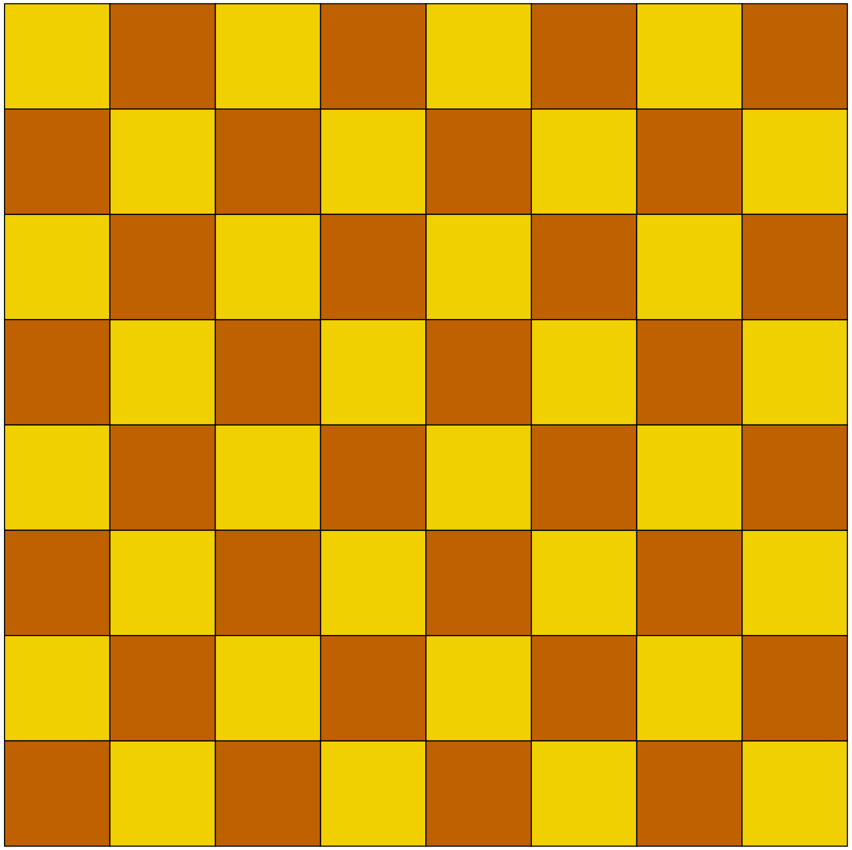
(a) Running time



(b) Iterations

From [Huang/Jebara, 2009].

Valid Rook Configurations and Permanents



Number of valid rook configurations

$$= \operatorname{perm}\begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \end{pmatrix}$$

$$= \sum_{\sigma} \prod_{i \in [8]} \theta_{i,\sigma(i)}$$

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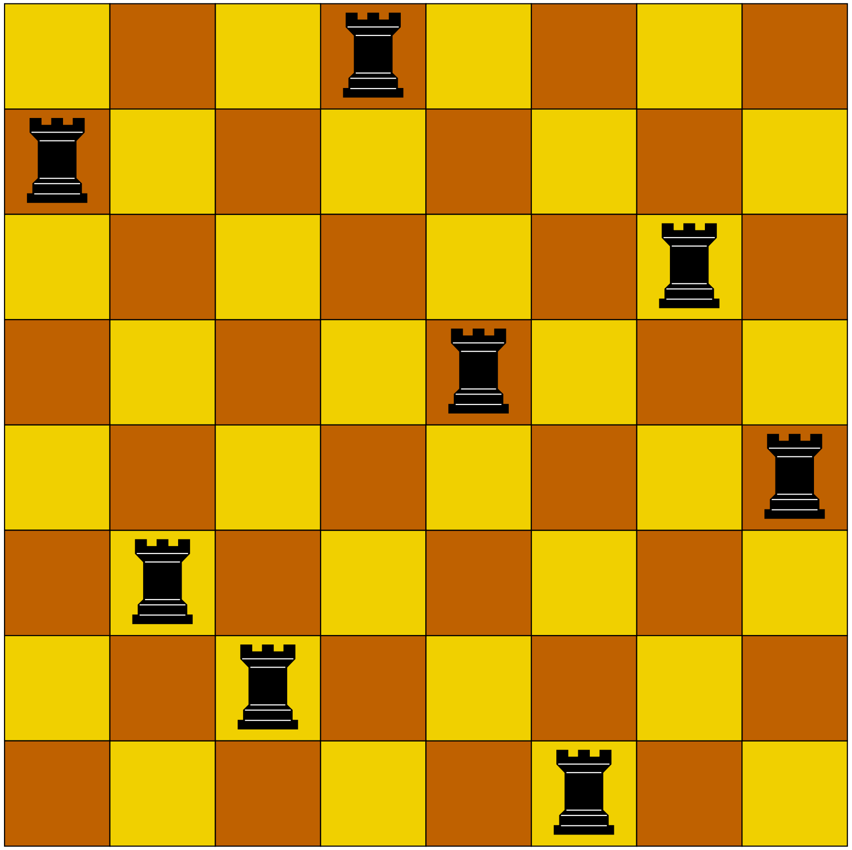
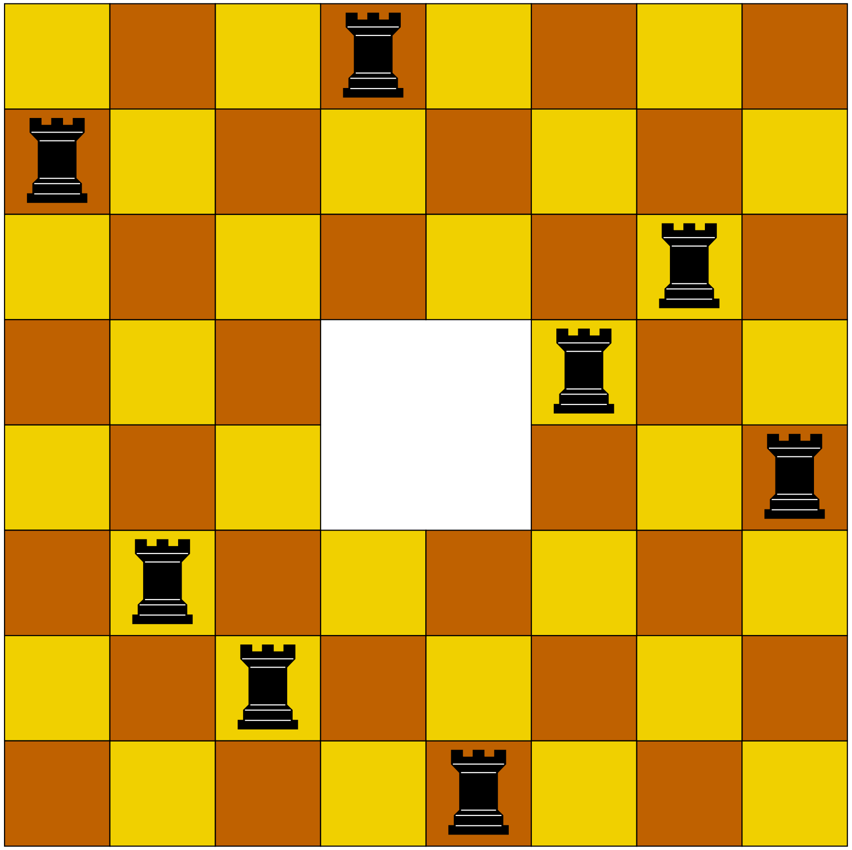


Number of valid rook configurations

$$= \operatorname{perm}\begin{pmatrix} 1 & 1 & 1 & {\color{red}1} & 1 & 1 & 1 & 1 \\ {\color{red}1} & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 & {\color{red}1} & 1 \\ 1 & 1 & 1 & 1 & {\color{red}1} & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & {\color{red}1} \\ 1 & {\color{red}1} & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & {\color{red}1} & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & {\color{red}1} & 1 & 1 \end{pmatrix}$$

$$= \sum_{\sigma} {\color{red}\prod_{i \in [8]} \theta_{i,\sigma(i)}}$$

Valid Rook Configurations and Permanents



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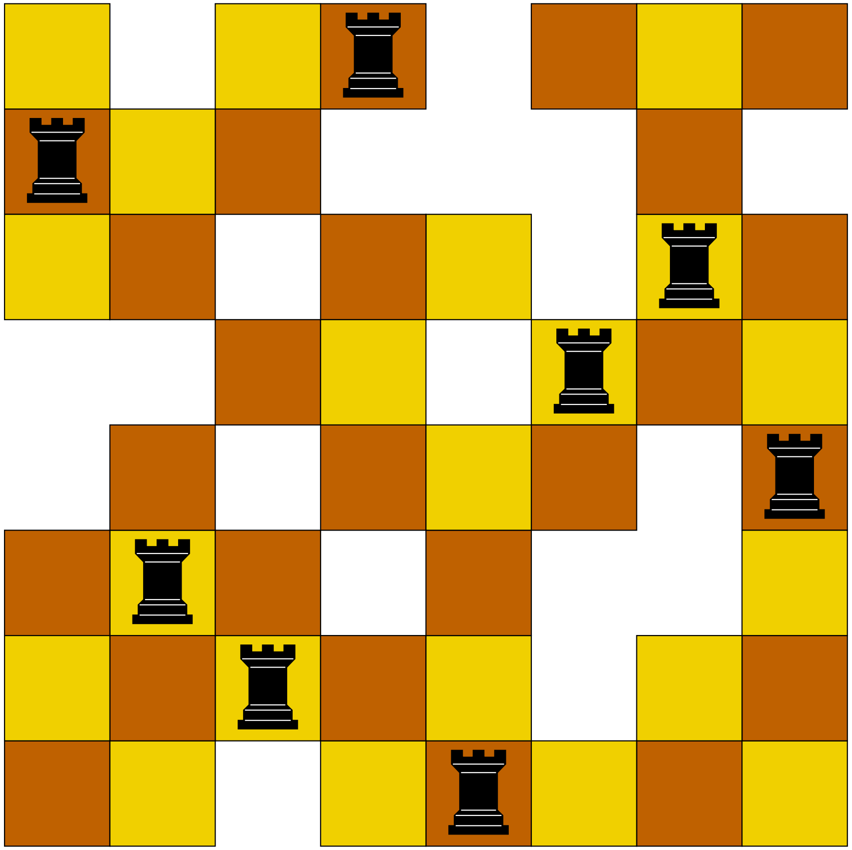

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$$= \operatorname{perm} \begin{pmatrix} 1 & 0 & 1 & 1 & 0 & 1 & 1 & 1 \\ 1 & 1 & 1 & 0 & 0 & 0 & 1 & 0 \\ 1 & 1 & 0 & 1 & 1 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 & 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 1 & 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 0 & 1 & 0 & 0 & 1 \\ 1 & 1 & 1 & 1 & 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 & 1 & 1 & 1 & 1 \end{pmatrix}$$

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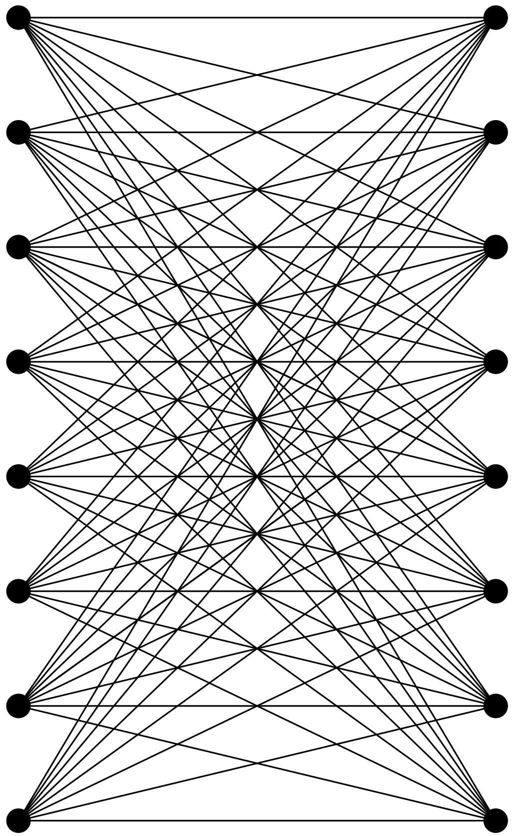
Perfect Matchings and Permanents



Number of valid perfect matchings

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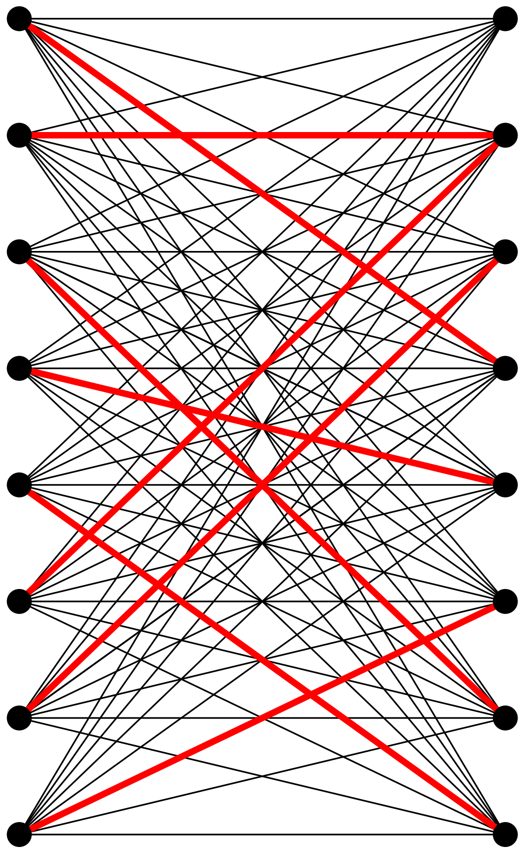

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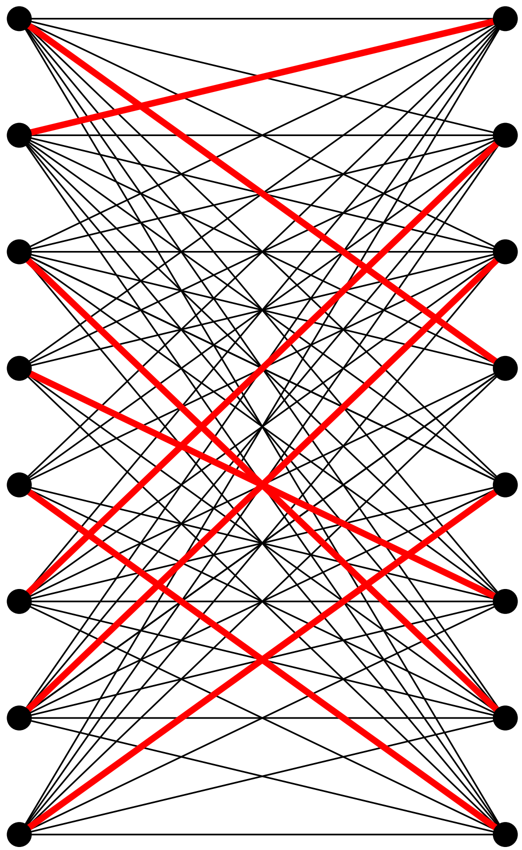

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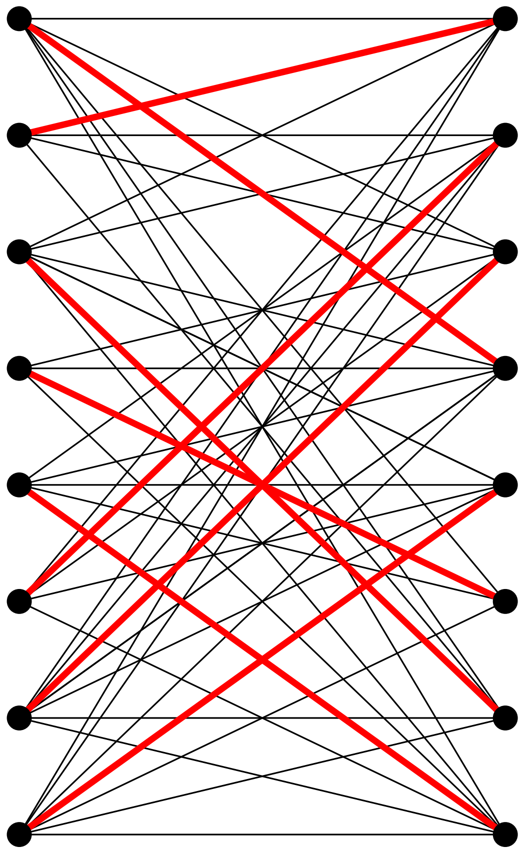
Perfect Matchings and Permanents



Number of valid perfect matchings

$$= \operatorname{perm}\begin{pmatrix} 1 & 0 & 1 & 1 & 0 & 1 & 1 & 1 \\ 1 & 1 & 1 & 0 & 0 & 0 & 1 & 0 \\ 1 & 1 & 0 & 1 & 1 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 & 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 1 & 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 0 & 1 & 0 & 0 & 1 \\ 1 & 1 & 1 & 1 & 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 & 1 & 1 & 1 & 1 \end{pmatrix}$$

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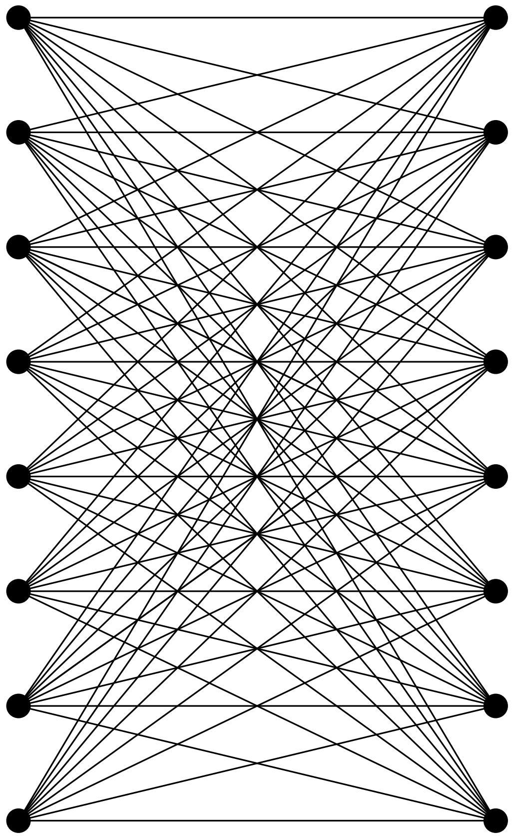

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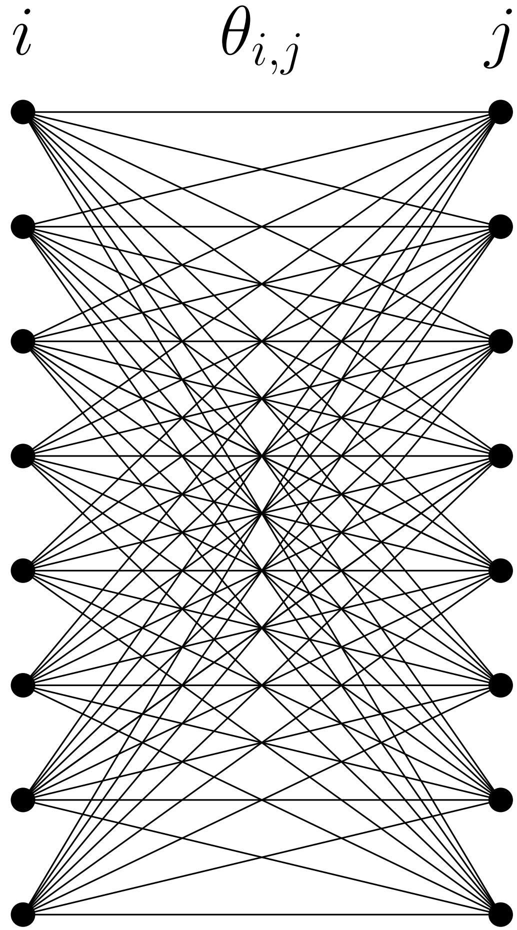

Total sum of weighted perf. matchings

$$= \operatorname{perm}\begin{pmatrix} \theta_{11} & \theta_{12} & \theta_{13} & \theta_{14} & \theta_{15} & \theta_{16} & \theta_{17} & \theta_{18} \\ \theta_{21} & \theta_{22} & \theta_{23} & \theta_{24} & \theta_{25} & \theta_{26} & \theta_{27} & \theta_{28} \\ \theta_{31} & \theta_{32} & \theta_{33} & \theta_{34} & \theta_{35} & \theta_{36} & \theta_{37} & \theta_{38} \\ \theta_{41} & \theta_{42} & \theta_{43} & \theta_{44} & \theta_{45} & \theta_{46} & \theta_{47} & \theta_{48} \\ \theta_{51} & \theta_{52} & \theta_{53} & \theta_{54} & \theta_{55} & \theta_{56} & \theta_{57} & \theta_{58} \\ \theta_{61} & \theta_{62} & \theta_{63} & \theta_{64} & \theta_{65} & \theta_{66} & \theta_{67} & \theta_{68} \\ \theta_{71} & \theta_{72} & \theta_{73} & \theta_{74} & \theta_{75} & \theta_{76} & \theta_{77} & \theta_{78} \\ \theta_{81} & \theta_{82} & \theta_{83} & \theta_{84} & \theta_{85} & \theta_{86} & \theta_{87} & \theta_{88} \end{pmatrix}$$

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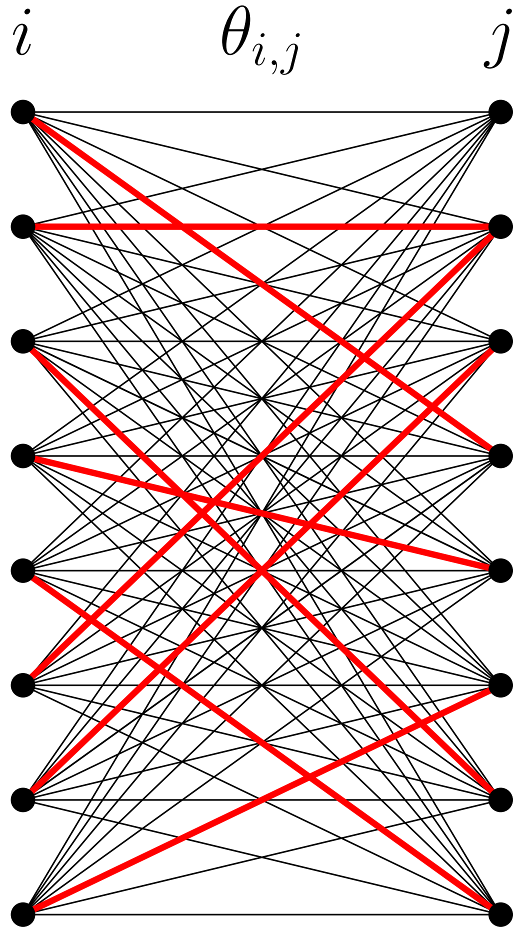
Perfect Matchings and Permanents



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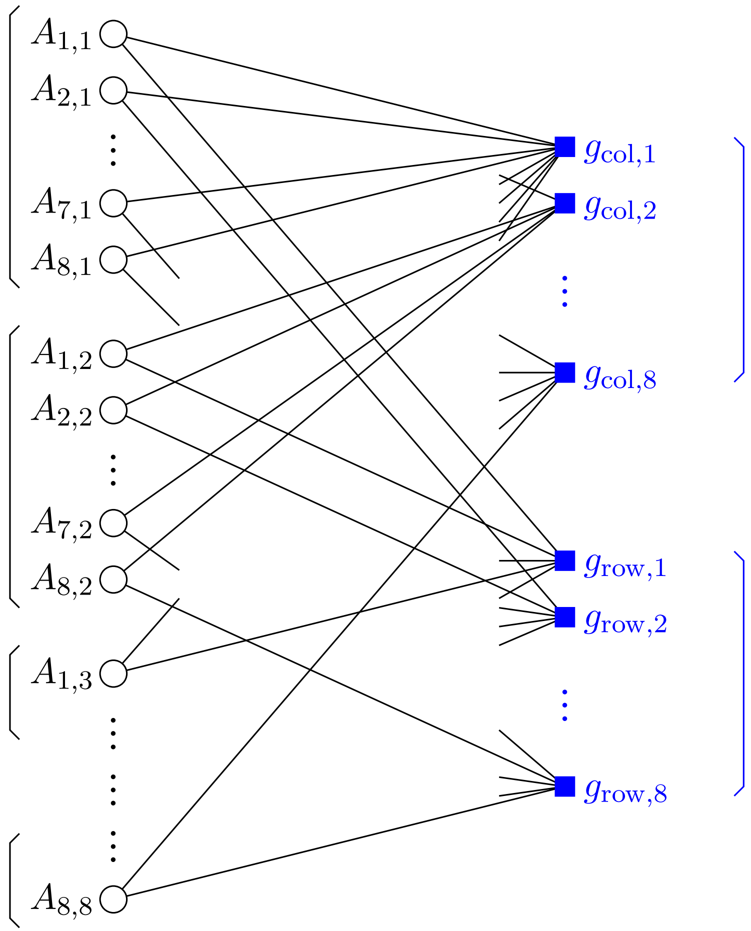
Graphical Model for Permanent



(function nodes are suitably defined based on $\boldsymbol{\theta}$)

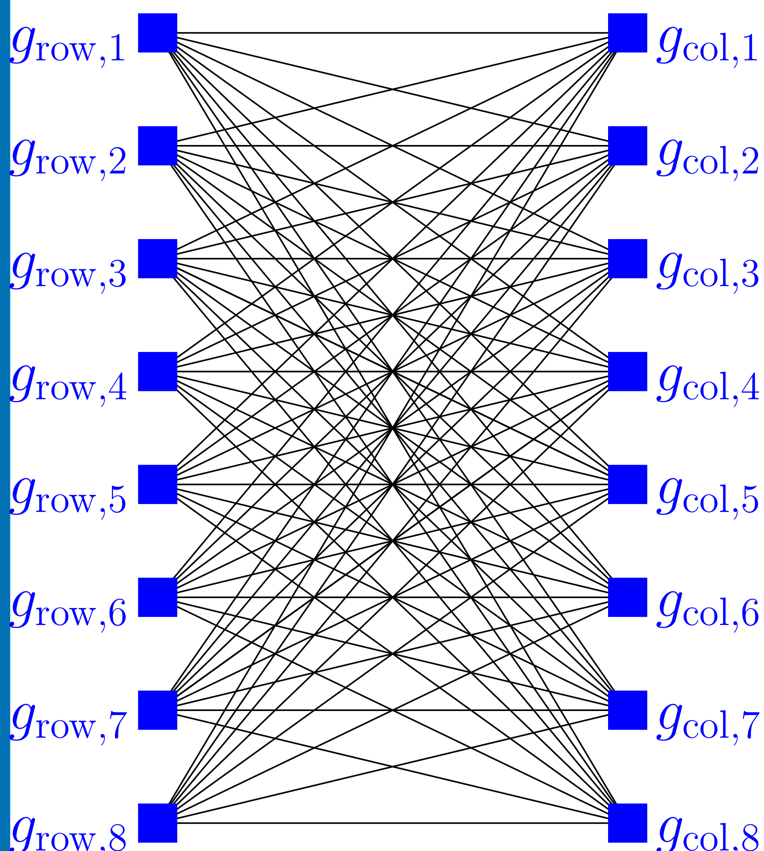
Global function:

$$g(a_{1,1}, \ldots, a_{8,8}) = \prod_j g_{\text{col},j}(a_{1,j}, \ldots, a_{8,j}) \times \prod_i g_{\text{row},i}(a_{i,1}, \ldots, a_{i,8})$$

Permanent:

$$\text{perm}(\boldsymbol{\theta}) = Z = \sum_{a_{1,1},\ldots,a_{8,8}} g(a_{1,1}, \ldots, a_{8,8})$$

Graphical Model for Permanent



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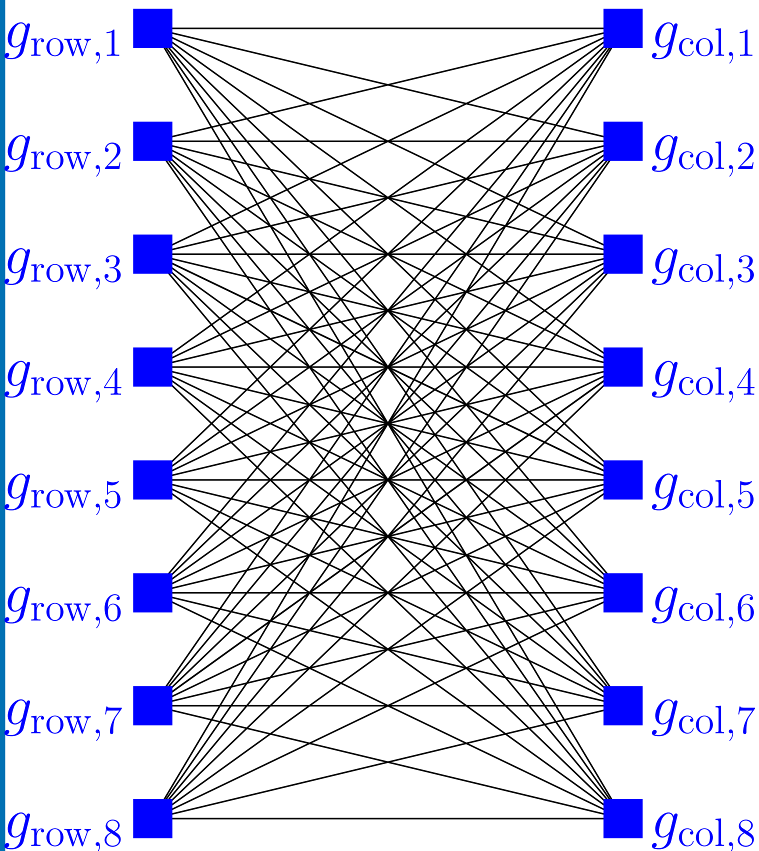

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- Many short cycles.
- The vertex degrees are high.

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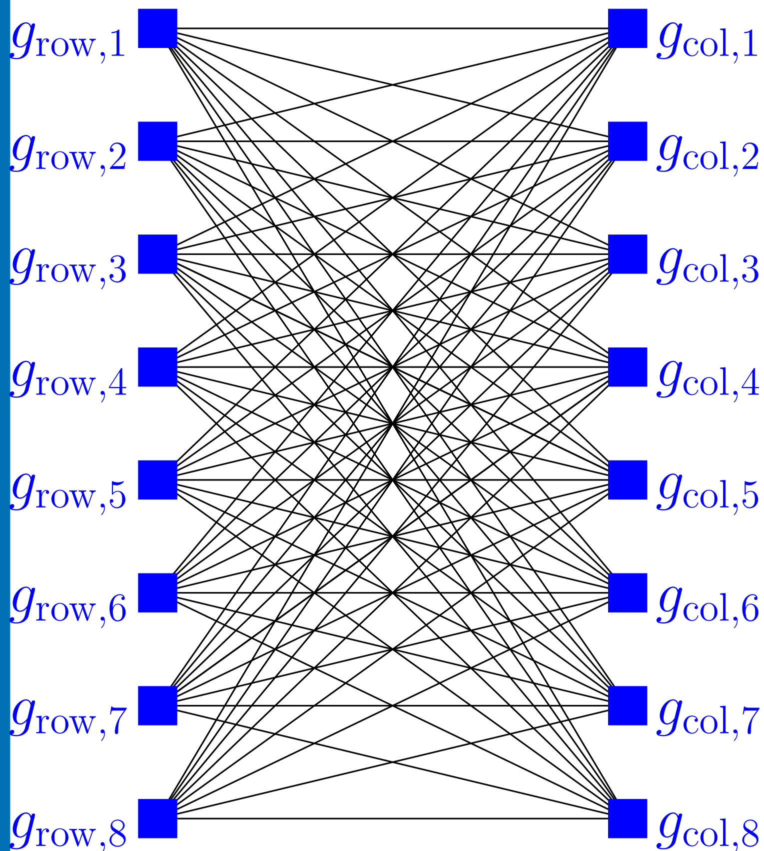

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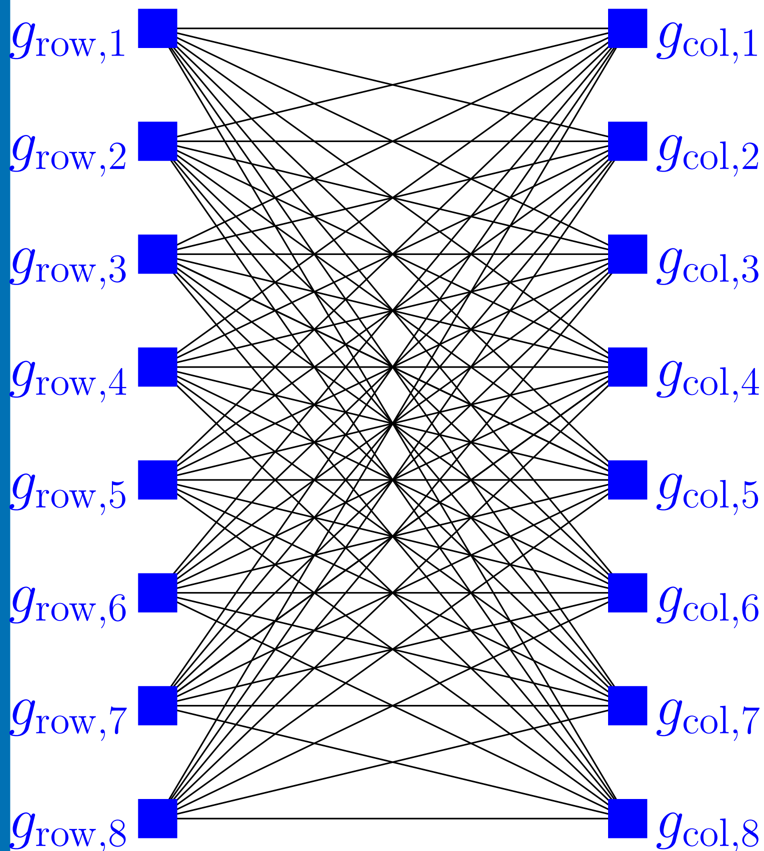
Both facts might suggest that the application of the sum-product algorithm to this factor graph is rather problematic.

Graphical Model for Permanent



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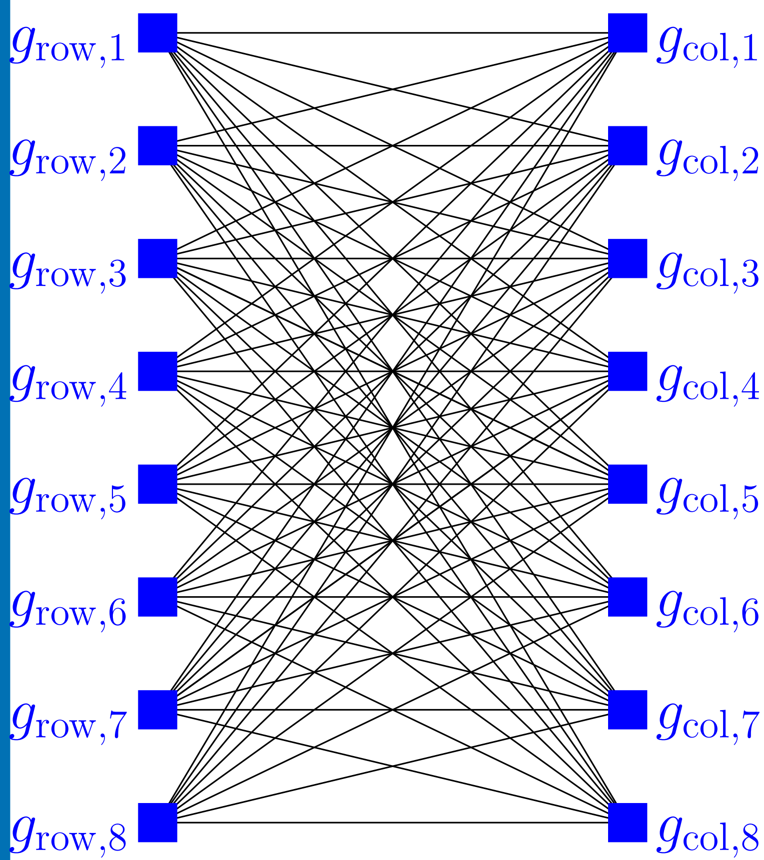

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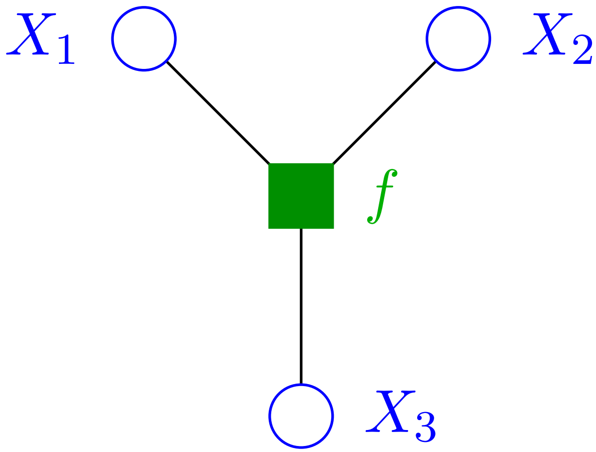
For an SPA suitability assessment, the overall cycle structure and the types of functions nodes are at least as important.

Factor graphs and the sum-product algorithm

Factor Graphs

A factor graph can be used to represent a multivariate function:

$$f(x_1, x_2, x_3)$$



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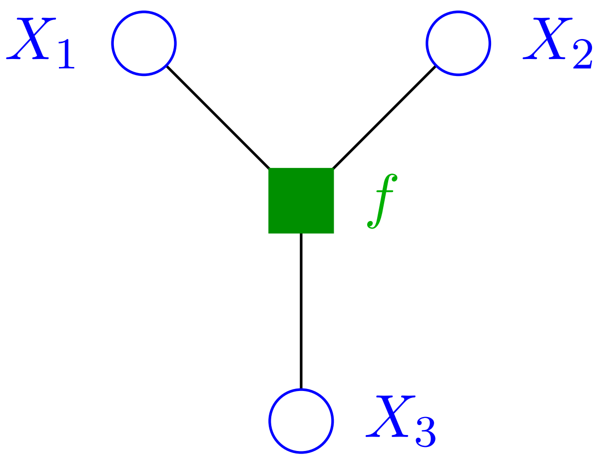

- Variable nodes: for each variable we draw a variable node (empty circles).

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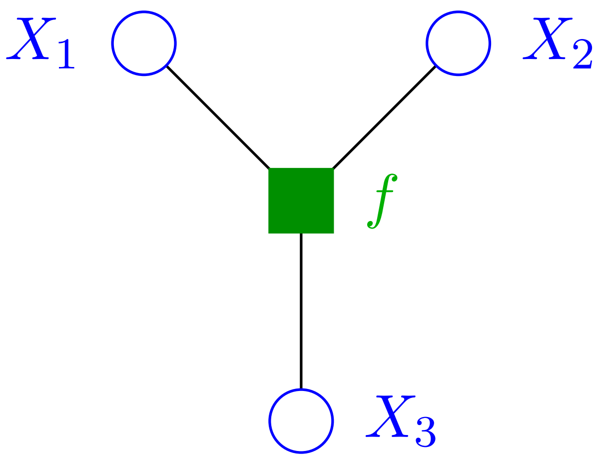

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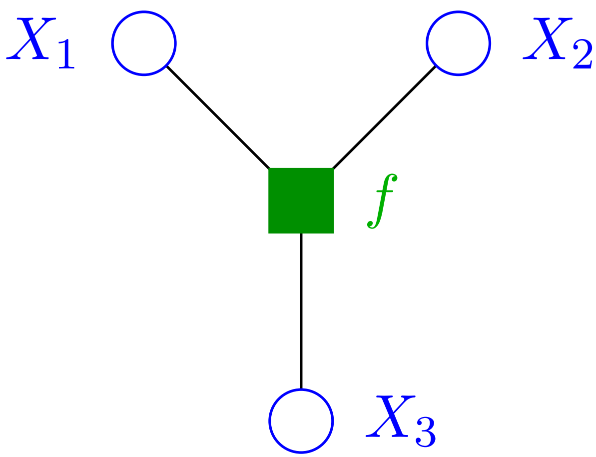

- Variable nodes: for each variable we draw a variable node (empty circles).
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- Edges: there is an edge between a variable node and a function node if the corresponding variable is an argument of the corresponding function.

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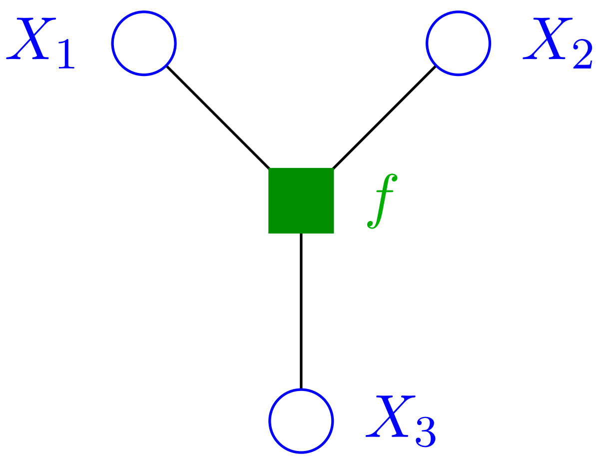

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- Bipartite graph: the resulting graph is a bipartite graph, i.e. there are only edges between vertices of different types.

Factor Graphs

General references for factor graphs are:

- F. R. Kschischang, B. J. Frey and H.-A. Loeliger, "Factor graphs and the sum-product algorithm," IEEE Trans. on Inform. Theory, IT–47, Feb. 2001.
- H.-A. Loeliger, "An introduction to factor graphs," IEEE Signal Processing Magazine, Jan. 2004.

Factor Graphs

We assume that we know more about the internal structure of the function $f(x_1, x_2, x_3)$, e.g.

$$f(x_1, x_2, x_3) = f_{\mathrm{A}}(x_1, x_2) \cdot f_{\mathrm{B}}(x_2, x_3).$$

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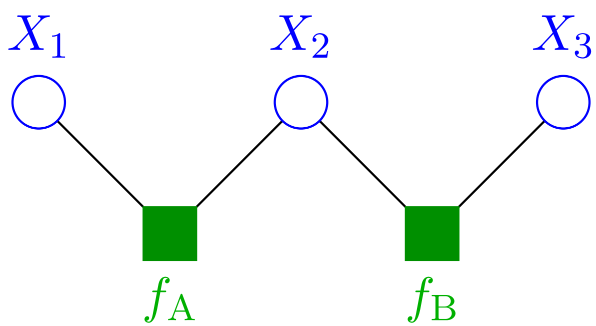
Then we can take advantage of this fact and the factor graph represents this structure.



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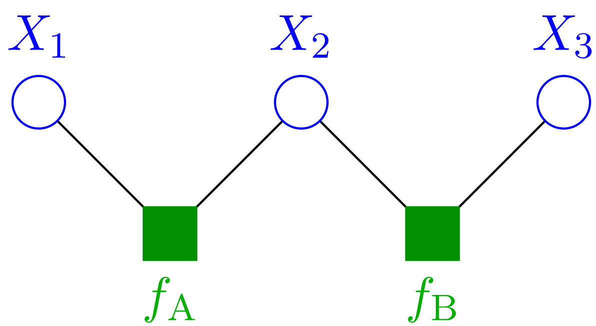

- $f(.,.,.)$ is called the global function.

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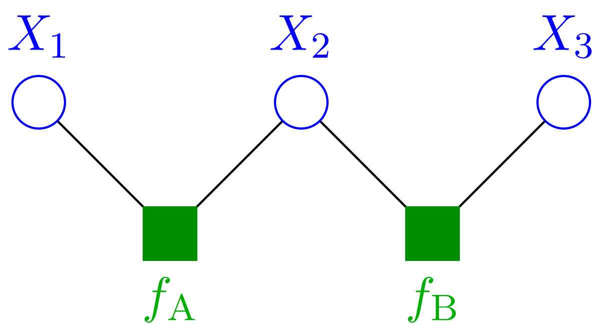

- $f(.,.,.)$ is called the global function.
- $f_{\mathrm{A}}(.,.)$ and $f_{\mathrm{B}}(.,.)$ are called local functions.

Factor Graphs

$$f(x_1, x_2, x_3, x_4, x_5)$$
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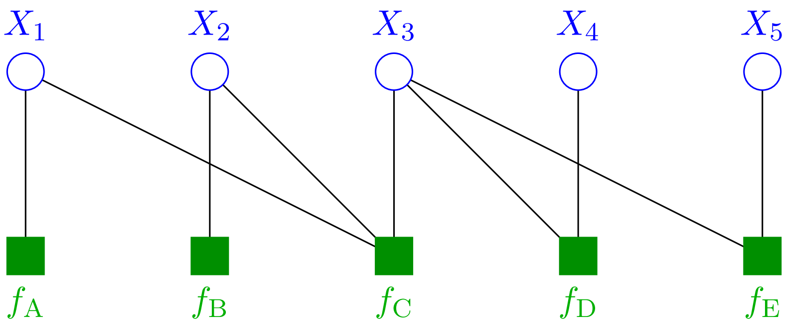
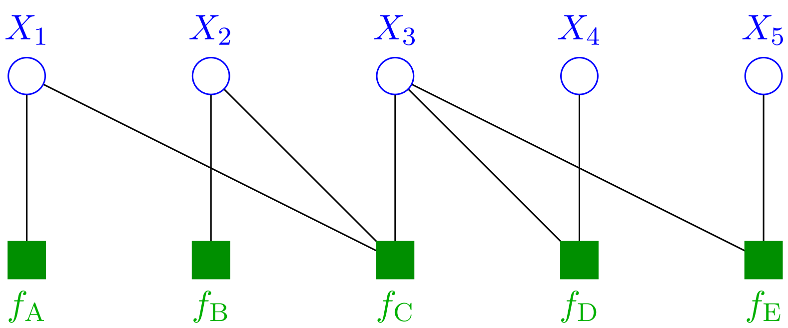
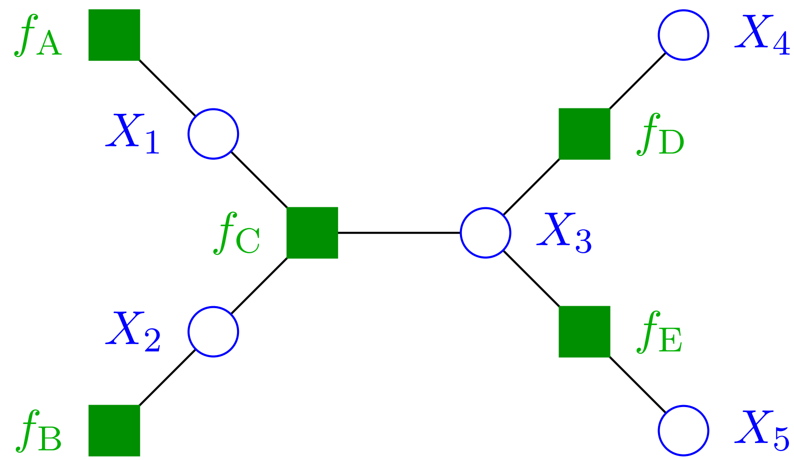


Factor Graphs

$$f(x_1, x_2, x_3, x_4, x_5) \\ = f_\mathrm{A}(x_1) \cdot f_\mathrm{B}(x_2) \cdot f_\mathrm{C}(x_1, x_2, x_3) \cdot f_\mathrm{D}(x_3, x_4) \cdot f_\mathrm{E}(x_3, x_5)$$





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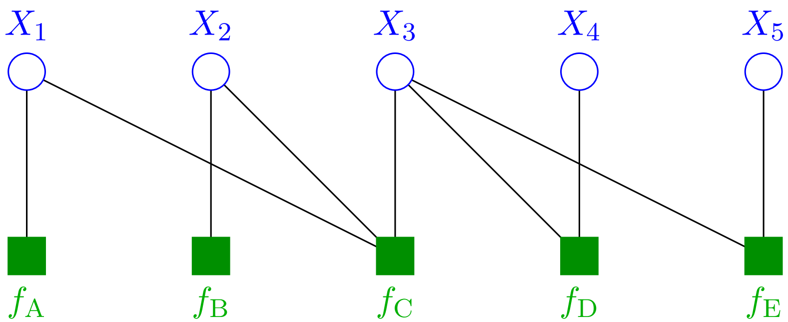
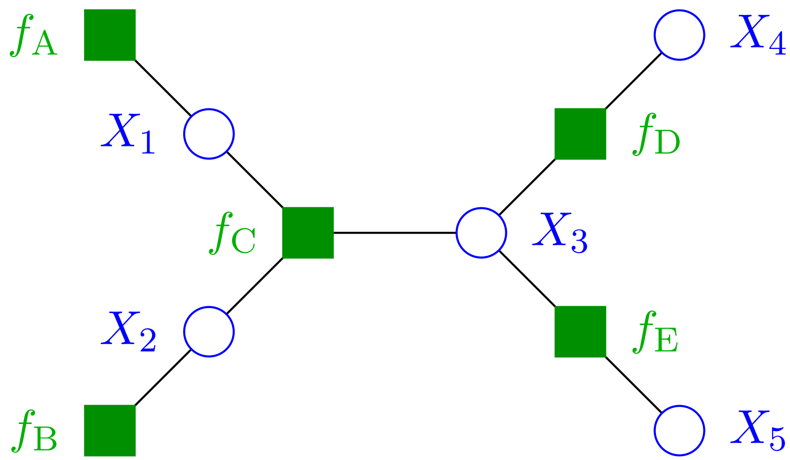
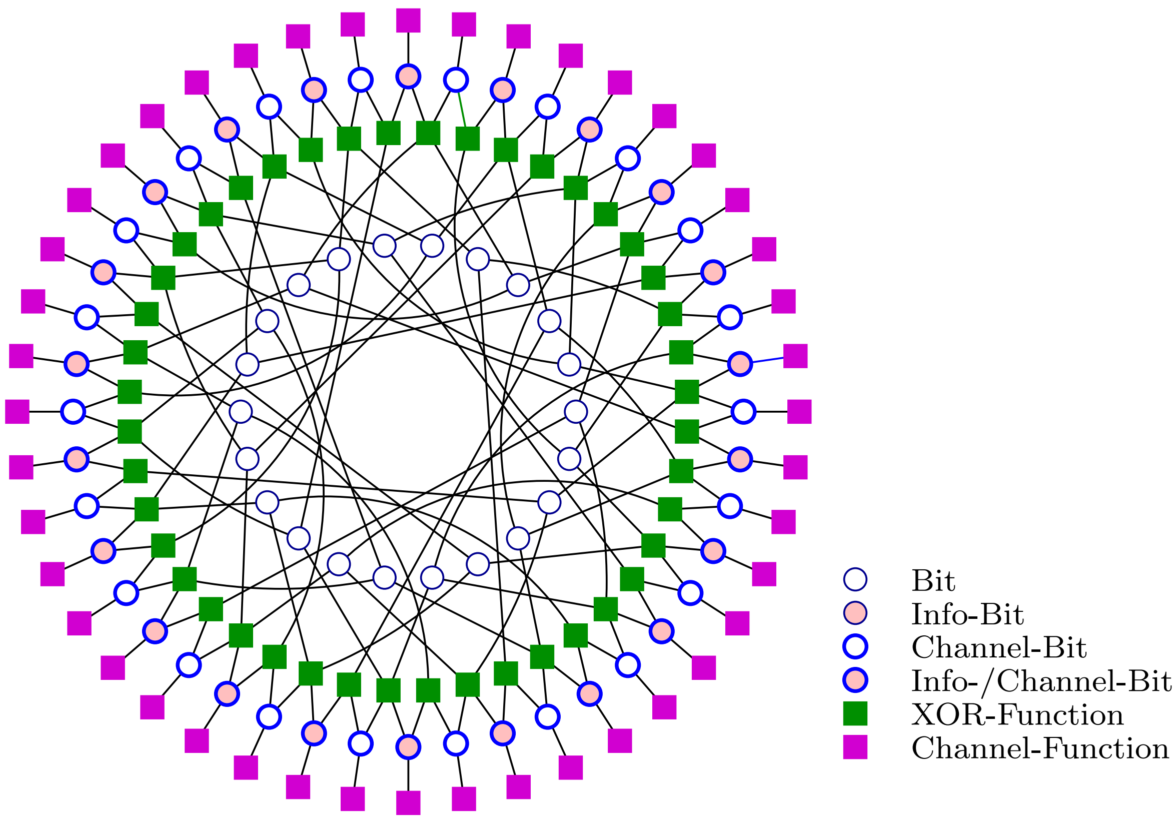




Note: One and the same function can be represented by graphs with different structures: some are more pleasing than others.

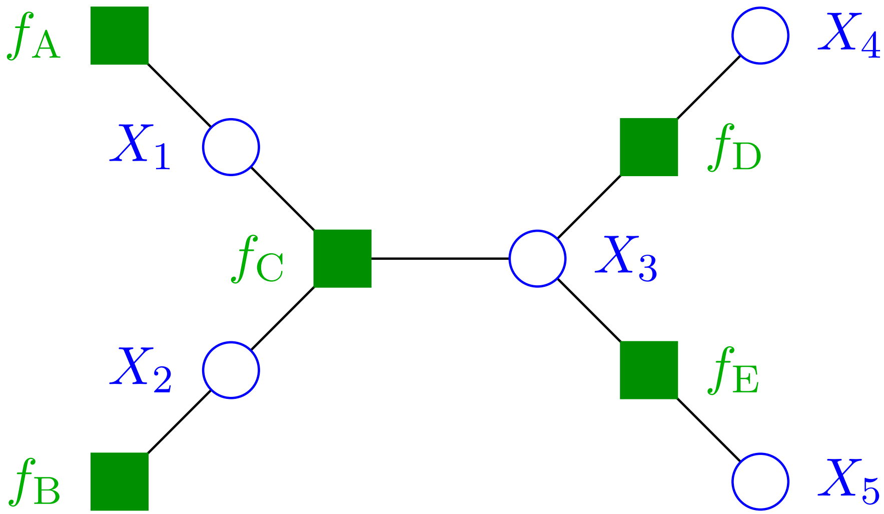
Factor Graph of an LDPC Code

In the context of channel coding, we usually take a factor graph to represent the factorization of the joint pmf/pdf of all occuring variables, i.e. uncoded symbols, coded symbols, and received symbols. Here it is shown when using a quasi-cyclic repeat-accumulate LDPC code (binary $[44, 22, 8]$ linear code).



The Sum-Product Algorithm

Let us consider again the following factor graph (which is a tree).



The global function is

$$
\begin{aligned}
& f(x_1, x_2, x_3, x_4, x_5) \\
&\qquad = f_\mathrm{A}(x_1) \cdot f_\mathrm{B}(x_2) \cdot f_\mathrm{C}(x_1, x_2, x_3) \cdot f_\mathrm{D}(x_3, x_4) \ \cdot f_\mathrm{E}(x_3, x_5).
\end{aligned}
$$

The Sum-Product Algorithm

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Very often one wants to calculate marginal functions. E.g.

$$\begin{aligned}\eta_{X_1}(x_1) &= \sum_{x_2, x_3, x_4, x_5} f(x_1, x_2, x_3, x_4, x_5) \\ &= \sum_{x_2, x_3, x_4, x_5} f_{\mathrm{A}}(x_1) \cdot f_{\mathrm{B}}(x_2) \cdot f_{\mathrm{C}}(x_1, x_2, x_3) \cdot f_{\mathrm{D}}(x_3, x_4) \cdot f_{\mathrm{E}}(x_3, x_5).\end{aligned}$$

The Sum-Product Algorithm

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\end{aligned}$$

$$\begin{aligned}
\eta_{X_3}(x_3) &= \sum_{x_1, x_2, x_4, x_5} f(x_1, x_2, x_3, x_4, x_5) \\
&= \sum_{x_1, x_2, x_4, x_5} f_{\mathrm{A}}(x_1) \cdot f_{\mathrm{B}}(x_2) \cdot f_{\mathrm{C}}(x_1, x_2, x_3) \cdot f_{\mathrm{D}}(x_3, x_4) \cdot f_{\mathrm{E}}(x_3, x_5).
\end{aligned}$$

etc.

The Sum-Product Algorithm

$$\eta_{X_1}(x_1) = \sum_{x_2}\sum_{x_3}\sum_{x_4}\sum_{x_5} f_{\mathrm{A}}(x_1)\cdot f_{\mathrm{B}}(x_2)\cdot f_{\mathrm{C}}(x_1,x_2,x_3)\cdot f_{\mathrm{D}}(x_3,x_4)\cdot f_{\mathrm{E}}(x_3,x_5)$$

The Sum-Product Algorithm

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The Sum-Product Algorithm

$$\eta_{X_1}(x_1) = \sum_{x_2}\sum_{x_3}\sum_{x_4}\sum_{x_5} f_{\mathrm{A}}(x_1)\cdot f_{\mathrm{B}}(x_2)\cdot f_{\mathrm{C}}(x_1,x_2,x_3)\cdot f_{\mathrm{D}}(x_3,x_4)\cdot f_{\mathrm{E}}(x_3,x_5)$$

$$= \underbrace{f_{\mathrm{A}}(x_1)}_{} \underbrace{\sum_{x_2}\sum_{x_3} f_{\mathrm{C}}(x_1,x_2,x_3) \underbrace{\underbrace{f_{\mathrm{B}}(x_2)}_{}}_{} \underbrace{\underbrace{\sum_{x_4} f_{\mathrm{D}}(x_3,x_4)\cdot \underbrace{1}_{\mu_{X_4\to f_{\mathrm{D}}}(x_4)}}_{} \underbrace{\sum_{x_5} f_{\mathrm{E}}(x_3,x_5)\cdot \underbrace{1}_{\mu_{X_5\to f_{\mathrm{E}}}(x_5)}}_{\mu_{f_{\mathrm{E}}\to X_3}(x_3)}}_{}}_{}$$

The Sum-Product Algorithm

$$\eta_{X_1}(x_1) = \sum_{x_2}\sum_{x_3}\sum_{x_4}\sum_{x_5} f_{\mathrm{A}}(x_1)\cdot f_{\mathrm{B}}(x_2)\cdot f_{\mathrm{C}}(x_1,x_2,x_3)\cdot f_{\mathrm{D}}(x_3,x_4)\cdot f_{\mathrm{E}}(x_3,x_5)$$

$$= \underbrace{f_{\mathrm{A}}(x_1)}\ \underbrace{\sum_{x_2}\sum_{x_3} f_{\mathrm{C}}(x_1,x_2,x_3)\ \underbrace{\underbrace{f_{\mathrm{B}}(x_2)}}\ \underbrace{\underbrace{\sum_{x_4} f_{\mathrm{D}}(x_3,x_4)\cdot \underbrace{1}_{\mu_{X_4\to f_{\mathrm{D}}}(x_4)}}_{\mu_{f_{\mathrm{D}}\to X_3}(x_3)}\ \underbrace{\sum_{x_5} f_{\mathrm{E}}(x_3,x_5)\cdot \underbrace{1}_{\mu_{X_5\to f_{\mathrm{E}}}(x_5)}}_{\mu_{f_{\mathrm{E}}\to X_3}(x_3)}}}$$

The Sum-Product Algorithm

$$
\begin{aligned}
\eta_{X_1}(x_1) &= \sum_{x_2}\sum_{x_3}\sum_{x_4}\sum_{x_5} f_{\mathrm{A}}(x_1)\cdot f_{\mathrm{B}}(x_2)\cdot f_{\mathrm{C}}(x_1,x_2,x_3)\cdot f_{\mathrm{D}}(x_3,x_4)\cdot f_{\mathrm{E}}(x_3,x_5)\\
&= \underbrace{f_{\mathrm{A}}(x_1)}\;\underbrace{\sum_{x_2}\sum_{x_3} f_{\mathrm{C}}(x_1,x_2,x_3)\;\underbrace{\underbrace{f_{\mathrm{B}}(x_2)}}\;\underbrace{\underbrace{\sum_{x_4} f_{\mathrm{D}}(x_3,x_4)\cdot\underbrace{1}_{\mu_{X_4\to f_{\mathrm{D}}}(x_4)}}_{\mu_{f_{\mathrm{D}}\to X_3}(x_3)}\;\underbrace{\sum_{x_5} f_{\mathrm{E}}(x_3,x_5)\cdot\underbrace{1}_{\mu_{X_5\to f_{\mathrm{E}}}(x_5)}}_{\mu_{f_{\mathrm{E}}\to X_3}(x_3)}}_{\mu_{f_{X_3\to f_{\mathrm{C}}}}(x_3)}}
\end{aligned}
$$

The Sum-Product Algorithm

$$\eta_{X_1}(x_1) = \sum_{x_2}\sum_{x_3}\sum_{x_4}\sum_{x_5} f_{\mathrm{A}}(x_1)\cdot f_{\mathrm{B}}(x_2)\cdot f_{\mathrm{C}}(x_1,x_2,x_3)\cdot f_{\mathrm{D}}(x_3,x_4)\cdot f_{\mathrm{E}}(x_3,x_5)$$

$$= \underbrace{f_{\mathrm{A}}(x_1)} \quad \underbrace{\sum_{x_2}\sum_{x_3} f_{\mathrm{C}}(x_1,x_2,x_3) \quad \underbrace{\underbrace{f_{\mathrm{B}}(x_2)}_{\mu_{f_{\mathrm{B}}\to X_2}(x_2)}} \quad \underbrace{\underbrace{\sum_{x_4} f_{\mathrm{D}}(x_3,x_4)\cdot \underbrace{1}_{\mu_{X_4\to f_{\mathrm{D}}}(x_4)}}_{\mu_{f_{\mathrm{D}}\to X_3}(x_3)} \quad \underbrace{\sum_{x_5} f_{\mathrm{E}}(x_3,x_5)\cdot \underbrace{1}_{\mu_{X_5\to f_{\mathrm{E}}}(x_5)}}_{\mu_{f_{\mathrm{E}}\to X_3}(x_3)}}_{\mu_{f_{X_3\to f_{\mathrm{C}}}}(x_3)}}$$

The Sum-Product Algorithm

$$
\begin{aligned}
\eta_{X_1}(x_1) &= \sum_{x_2}\sum_{x_3}\sum_{x_4}\sum_{x_5} f_{\mathrm{A}}(x_1)\cdot f_{\mathrm{B}}(x_2)\cdot f_{\mathrm{C}}(x_1,x_2,x_3)\cdot f_{\mathrm{D}}(x_3,x_4)\cdot f_{\mathrm{E}}(x_3,x_5) \\
&= \underbrace{f_{\mathrm{A}}(x_1)} \quad \underbrace{\sum_{x_2}\sum_{x_3} f_{\mathrm{C}}(x_1,x_2,x_3) \quad \underbrace{\underbrace{f_{\mathrm{B}}(x_2)}_{\mu_{f_{\mathrm{B}}\to X_2}(x_2)}}_{\mu_{X_2\to f_{\mathrm{C}}}(x_2)} \quad \underbrace{\underbrace{\sum_{x_4} f_{\mathrm{D}}(x_3,x_4)\cdot \underbrace{1}_{\mu_{X_4\to f_{\mathrm{D}}}(x_4)}}_{\mu_{f_{\mathrm{D}}\to X_3}(x_3)} \underbrace{\sum_{x_5} f_{\mathrm{E}}(x_3,x_5)\cdot \underbrace{1}_{\mu_{X_5\to f_{\mathrm{E}}}(x_5)}}_{\mu_{f_{\mathrm{E}}\to X_3}(x_3)}}_{\mu_{f_{X_3\to f_{\mathrm{C}}}}(x_3)}}
\end{aligned}
$$

The Sum-Product Algorithm

$$\eta_{X_1}(x_1) = \sum_{x_2}\sum_{x_3}\sum_{x_4}\sum_{x_5} f_{\mathrm{A}}(x_1)\cdot f_{\mathrm{B}}(x_2)\cdot f_{\mathrm{C}}(x_1,x_2,x_3)\cdot f_{\mathrm{D}}(x_3,x_4)\cdot f_{\mathrm{E}}(x_3,x_5)$$

$$= \underbrace{f_{\mathrm{A}}(x_1)} \underbrace{\sum_{x_2}\sum_{x_3} f_{\mathrm{C}}(x_1,x_2,x_3) \underbrace{\underbrace{f_{\mathrm{B}}(x_2)}_{\mu_{f_{\mathrm{B}}\to X_2}(x_2)}}_{\mu_{X_2\to f_{\mathrm{C}}}(x_2)} \underbrace{\underbrace{\sum_{x_4} f_{\mathrm{D}}(x_3,x_4)\cdot \underbrace{1}_{\mu_{X_4\to f_{\mathrm{D}}}(x_4)}}_{\mu_{f_{\mathrm{D}}\to X_3}(x_3)} \underbrace{\sum_{x_5} f_{\mathrm{E}}(x_3,x_5)\cdot \underbrace{1}_{\mu_{X_5\to f_{\mathrm{E}}}(x_5)}}_{\mu_{f_{\mathrm{E}}\to X_3}(x_3)}}_{\mu_{f_{X_3\to f_{\mathrm{C}}}}(x_3)}}_{\mu_{f_{\mathrm{C}}\to X_1}(x_1)}$$

The Sum-Product Algorithm

$$\eta_{X_1}(x_1) = \sum_{x_2}\sum_{x_3}\sum_{x_4}\sum_{x_5} f_{\mathrm{A}}(x_1)\cdot f_{\mathrm{B}}(x_2)\cdot f_{\mathrm{C}}(x_1,x_2,x_3)\cdot f_{\mathrm{D}}(x_3,x_4)\cdot f_{\mathrm{E}}(x_3,x_5)$$

$$= \underbrace{f_{\mathrm{A}}(x_1)}_{\mu_{f_{\mathrm{A}}\to X_1}(x_1)} \underbrace{\sum_{x_2}\sum_{x_3} f_{\mathrm{C}}(x_1,x_2,x_3) \underbrace{\underbrace{f_{\mathrm{B}}(x_2)}_{\mu_{f_{\mathrm{B}}\to X_2}(x_2)}}_{\mu_{X_2\to f_{\mathrm{C}}}(x_2)} \underbrace{\underbrace{\sum_{x_4} f_{\mathrm{D}}(x_3,x_4)\cdot \underbrace{1}_{\mu_{X_4\to f_{\mathrm{D}}}(x_4)}}_{\mu_{f_{\mathrm{D}}\to X_3}(x_3)} \underbrace{\sum_{x_5} f_{\mathrm{E}}(x_3,x_5)\cdot \underbrace{1}_{\mu_{X_5\to f_{\mathrm{E}}}(x_5)}}_{\mu_{f_{\mathrm{E}}\to X_3}(x_3)}}_{\mu_{f_{X_3}\to f_{\mathrm{C}}}(x_3)}}_{\mu_{f_{\mathrm{C}}\to X_1}(x_1)}$$

The Sum-Product Algorithm

$$\eta_{X_1}(x_1) = \sum_{x_2}\sum_{x_3}\sum_{x_4}\sum_{x_5} f_{\mathrm{A}}(x_1)\cdot f_{\mathrm{B}}(x_2)\cdot f_{\mathrm{C}}(x_1,x_2,x_3)\cdot f_{\mathrm{D}}(x_3,x_4)\cdot f_{\mathrm{E}}(x_3,x_5)$$

$$= \underbrace{f_{\mathrm{A}}(x_1)}_{\mu_{f_{\mathrm{A}}\to X_1}(x_1)} \underbrace{\sum_{x_2}\sum_{x_3} f_{\mathrm{C}}(x_1,x_2,x_3) \underbrace{\underbrace{f_{\mathrm{B}}(x_2)}_{\mu_{f_{\mathrm{B}}\to X_2}(x_2)}}_{\mu_{X_2\to f_{\mathrm{C}}}(x_2)} \underbrace{\underbrace{\sum_{x_4} f_{\mathrm{D}}(x_3,x_4)\cdot \underbrace{1}_{\mu_{X_4\to f_{\mathrm{D}}}(x_4)}}_{\mu_{f_{\mathrm{D}}\to X_3}(x_3)} \underbrace{\sum_{x_5} f_{\mathrm{E}}(x_3,x_5)\cdot \underbrace{1}_{\mu_{X_5\to f_{\mathrm{E}}}(x_5)}}_{\mu_{f_{\mathrm{E}}\to X_3}(x_3)}}_{\mu_{f_{X_3\to f_{\mathrm{C}}}}(x_3)}}_{\mu_{f_{\mathrm{C}}\to X_1}(x_1)}$$

The objects $\mu_{X_i\to f_j}(x_i)$ and $\mu_{f_j\to X_i}(x_i)$:

The Sum-Product Algorithm

$$\eta_{X_1}(x_1) = \sum_{x_2}\sum_{x_3}\sum_{x_4}\sum_{x_5} f_{\mathrm{A}}(x_1)\cdot f_{\mathrm{B}}(x_2)\cdot f_{\mathrm{C}}(x_1,x_2,x_3)\cdot f_{\mathrm{D}}(x_3,x_4)\cdot f_{\mathrm{E}}(x_3,x_5)$$

$$= \underbrace{f_{\mathrm{A}}(x_1)}_{\mu_{f_{\mathrm{A}}\to X_1}(x_1)} \underbrace{\sum_{x_2}\sum_{x_3} f_{\mathrm{C}}(x_1,x_2,x_3) \underbrace{\underbrace{f_{\mathrm{B}}(x_2)}_{\mu_{f_{\mathrm{B}}\to X_2}(x_2)}}_{\mu_{X_2\to f_{\mathrm{C}}}(x_2)} \underbrace{\underbrace{\sum_{x_4} f_{\mathrm{D}}(x_3,x_4)\cdot \underbrace{1}_{\mu_{X_4\to f_{\mathrm{D}}}(x_4)}}_{\mu_{f_{\mathrm{D}}\to X_3}(x_3)} \underbrace{\sum_{x_5} f_{\mathrm{E}}(x_3,x_5)\cdot \underbrace{1}_{\mu_{X_5\to f_{\mathrm{E}}}(x_5)}}_{\mu_{f_{\mathrm{E}}\to X_3}(x_3)}}_{\mu_{f_{X_3\to f_{\mathrm{C}}}}(x_3)}}_{\mu_{f_{\mathrm{C}}\to X_1}(x_1)}$$

The objects $\mu_{X_i\to f_j}(x_i)$ and $\mu_{f_j\to X_i}(x_i)$:

- They are called messages.

The Sum-Product Algorithm

$$\eta_{X_1}(x_1) = \sum_{x_2}\sum_{x_3}\sum_{x_4}\sum_{x_5} f_{\mathrm{A}}(x_1)\cdot f_{\mathrm{B}}(x_2)\cdot f_{\mathrm{C}}(x_1,x_2,x_3)\cdot f_{\mathrm{D}}(x_3,x_4)\cdot f_{\mathrm{E}}(x_3,x_5)$$

$$= \underbrace{f_{\mathrm{A}}(x_1)}_{\mu_{f_{\mathrm{A}}\to X_1}(x_1)} \underbrace{\sum_{x_2}\sum_{x_3} f_{\mathrm{C}}(x_1,x_2,x_3) \underbrace{\underbrace{f_{\mathrm{B}}(x_2)}_{\mu_{f_{\mathrm{B}}\to X_2}(x_2)}}_{\mu_{X_2\to f_{\mathrm{C}}}(x_2)} \underbrace{\underbrace{\sum_{x_4} f_{\mathrm{D}}(x_3,x_4)\cdot \underbrace{1}_{\mu_{X_4\to f_{\mathrm{D}}}(x_4)}}_{\mu_{f_{\mathrm{D}}\to X_3}(x_3)} \underbrace{\sum_{x_5} f_{\mathrm{E}}(x_3,x_5)\cdot \underbrace{1}_{\mu_{X_5\to f_{\mathrm{E}}}(x_5)}}_{\mu_{f_{\mathrm{E}}\to X_3}(x_3)}}_{\mu_{f_{X_3}\to f_{\mathrm{C}}}(x_3)}}_{\mu_{f_{\mathrm{C}}\to X_1}(x_1)}$$

The objects $\mu_{X_i\to f_j}(x_i)$ and $\mu_{f_j\to X_i}(x_i)$:

- They are called messages.
- They can be associated with the edge between the vertex X_i and the vertex f_j.

The Sum-Product Algorithm

$$\eta_{X_1}(x_1) = \sum_{x_2}\sum_{x_3}\sum_{x_4}\sum_{x_5} f_{\mathrm{A}}(x_1)\cdot f_{\mathrm{B}}(x_2)\cdot f_{\mathrm{C}}(x_1,x_2,x_3)\cdot f_{\mathrm{D}}(x_3,x_4)\cdot f_{\mathrm{E}}(x_3,x_5)$$

$$= \underbrace{f_{\mathrm{A}}(x_1)}_{\mu_{f_{\mathrm{A}}\to X_1}(x_1)} \underbrace{\sum_{x_2}\sum_{x_3} f_{\mathrm{C}}(x_1,x_2,x_3)\; \underbrace{\underbrace{f_{\mathrm{B}}(x_2)}_{\mu_{f_{\mathrm{B}}\to X_2}(x_2)}}_{\mu_{X_2\to f_{\mathrm{C}}}(x_2)} \underbrace{\underbrace{\sum_{x_4} f_{\mathrm{D}}(x_3,x_4)\cdot \underbrace{1}_{\mu_{X_4\to f_{\mathrm{D}}}(x_4)}}_{\mu_{f_{\mathrm{D}}\to X_3}(x_3)} \underbrace{\sum_{x_5} f_{\mathrm{E}}(x_3,x_5)\cdot \underbrace{1}_{\mu_{X_5\to f_{\mathrm{E}}}(x_5)}}_{\mu_{f_{\mathrm{E}}\to X_3}(x_3)}}_{\mu_{f_{X_3}\to f_{\mathrm{C}}}(x_3)}}_{\mu_{f_{\mathrm{C}}\to X_1}(x_1)}$$

The objects $\mu_{X_i\to f_j}(x_i)$ and $\mu_{f_j\to X_i}(x_i)$:

- They are called messages.
- They can be associated with the edge between the vertex X_i and the vertex f_j.
- They are functions of x_i, i.e. their domain is the alphabet of X_i.

The Sum-Product Algorithm

$$\eta_{X_1}(x_1) = \sum_{x_2}\sum_{x_3}\sum_{x_4}\sum_{x_5} f_{\mathrm{A}}(x_1)\cdot f_{\mathrm{B}}(x_2)\cdot f_{\mathrm{C}}(x_1,x_2,x_3)\cdot f_{\mathrm{D}}(x_3,x_4)\cdot f_{\mathrm{E}}(x_3,x_5)$$

$$= \underbrace{f_{\mathrm{A}}(x_1)}_{\mu_{f_{\mathrm{A}}\to X_1}(x_1)} \underbrace{\sum_{x_2}\sum_{x_3} f_{\mathrm{C}}(x_1,x_2,x_3) \underbrace{\underbrace{f_{\mathrm{B}}(x_2)}_{\mu_{f_{\mathrm{B}}\to X_2}(x_2)}}_{\mu_{X_2\to f_{\mathrm{C}}}(x_2)} \underbrace{\underbrace{\sum_{x_4} f_{\mathrm{D}}(x_3,x_4)\cdot \underbrace{1}_{\mu_{X_4\to f_{\mathrm{D}}}(x_4)}}_{\mu_{f_{\mathrm{D}}\to X_3}(x_3)} \underbrace{\sum_{x_5} f_{\mathrm{E}}(x_3,x_5)\cdot \underbrace{1}_{\mu_{X_5\to f_{\mathrm{E}}}(x_5)}}_{\mu_{f_{\mathrm{E}}\to X_3}(x_3)}}_{\mu_{f_{X_3\to f_{\mathrm{C}}}}(x_3)}}_{\mu_{f_{\mathrm{C}}\to X_1}(x_1)}$$

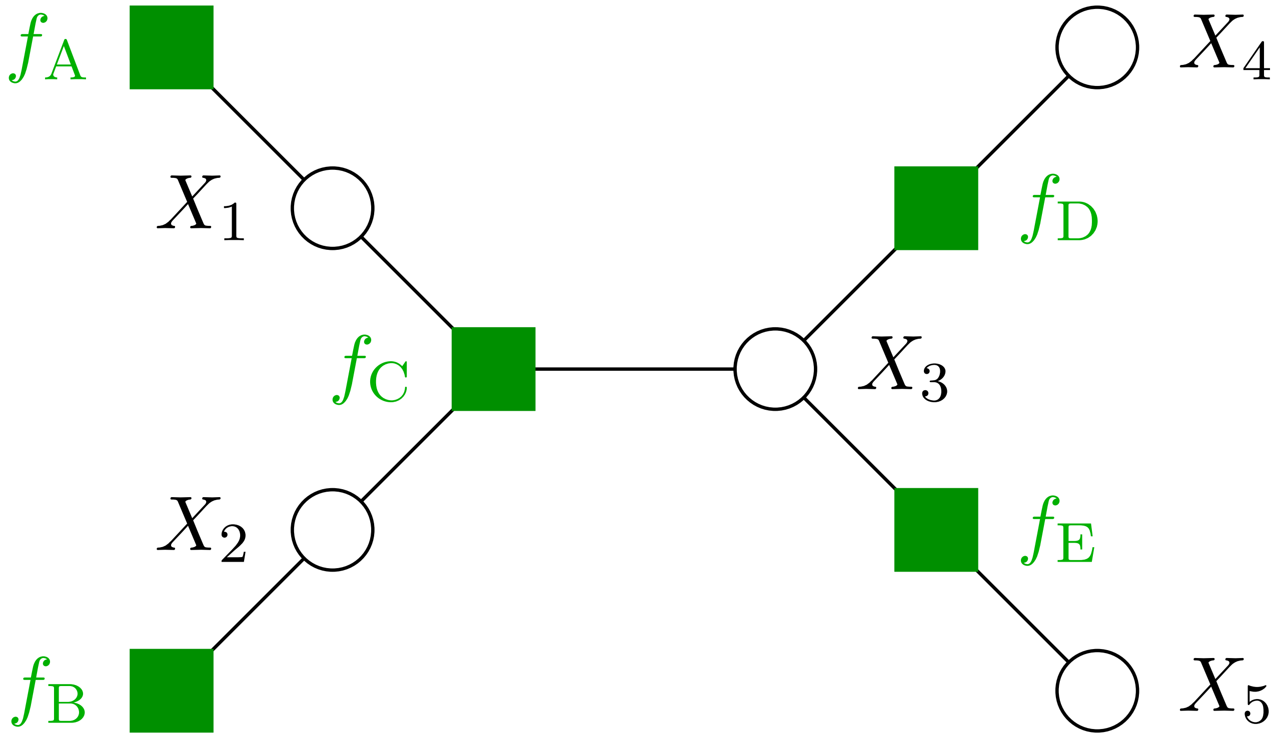
The objects $\mu_{X_i\to f_j}(x_i)$ and $\mu_{f_j\to X_i}(x_i)$:

- They are called messages.
- They can be associated with the edge between the vertex X_i and the vertex f_j.
- They are functions of x_i, i.e. their domain is the alphabet of X_i.

Note: similar manipulations can be performed for calculating $\eta_{X_2}(x_2)$, $\eta_{X_3}(x_3)$, $\eta_{X_4}(x_4)$, $\eta_{X_5}(x_5)$.

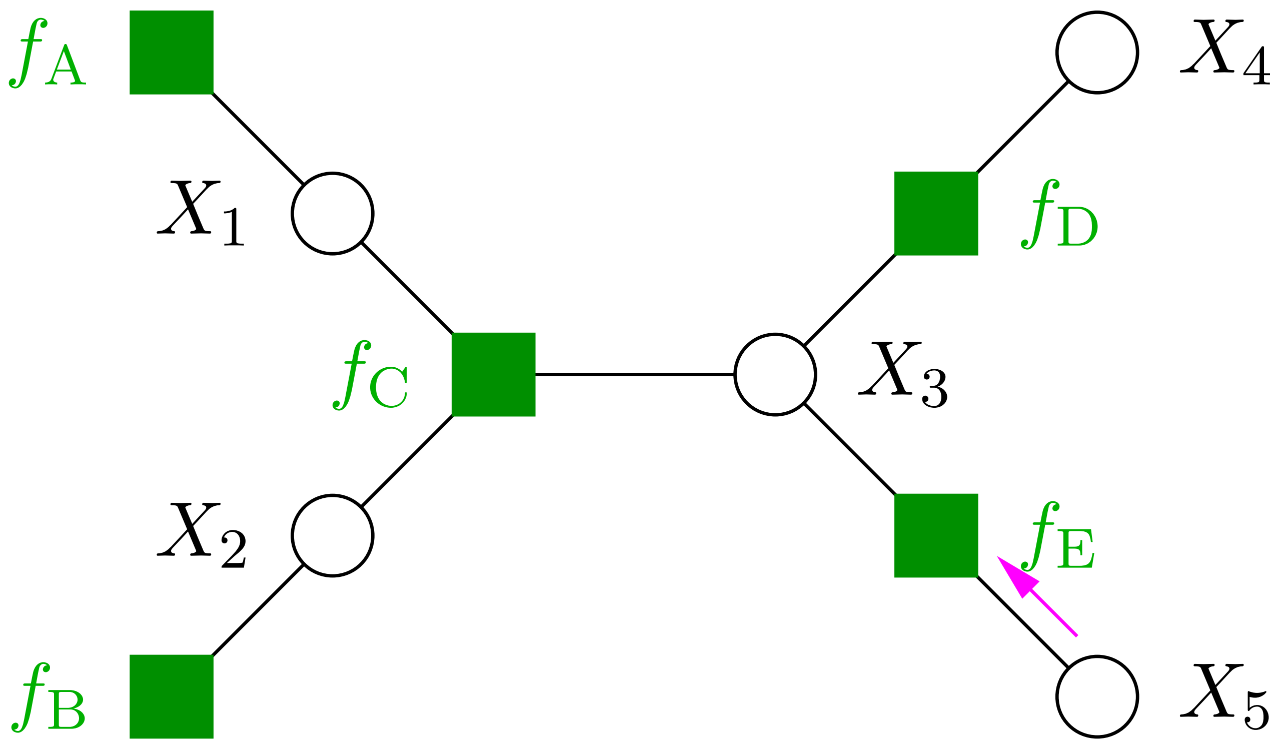
The Sum-Product Algorithm

Messages necessary for calculating $\eta_{X_1}(x_1)$.



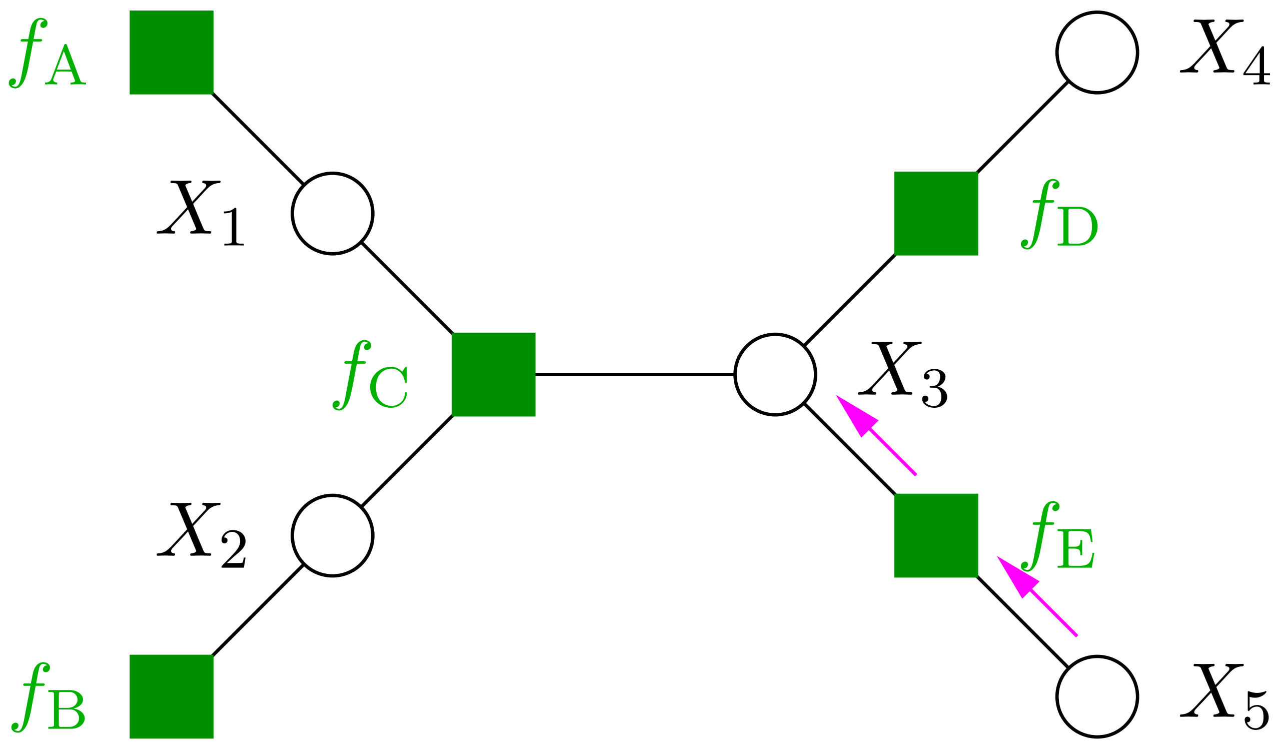
The Sum-Product Algorithm

Messages necessary for calculating $\eta_{X_1}(x_1)$.



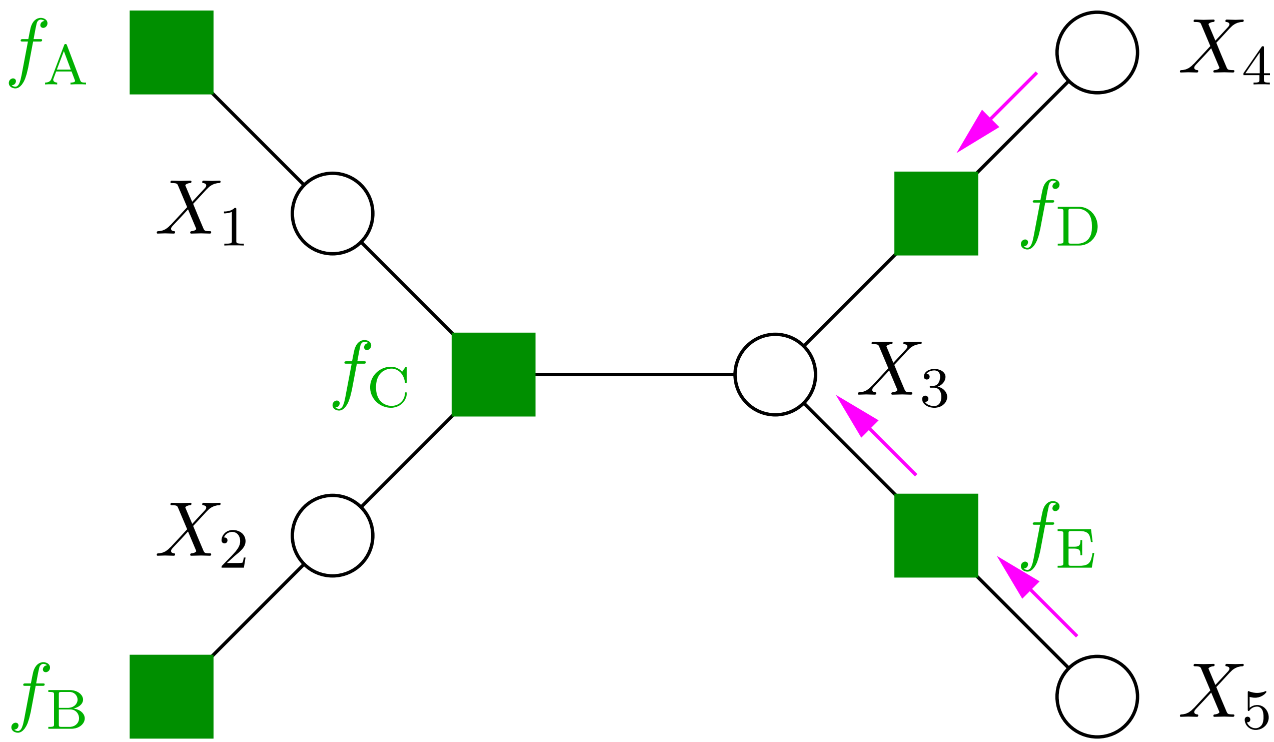
The Sum-Product Algorithm

Messages necessary for calculating $\eta_{X_1}(x_1)$.



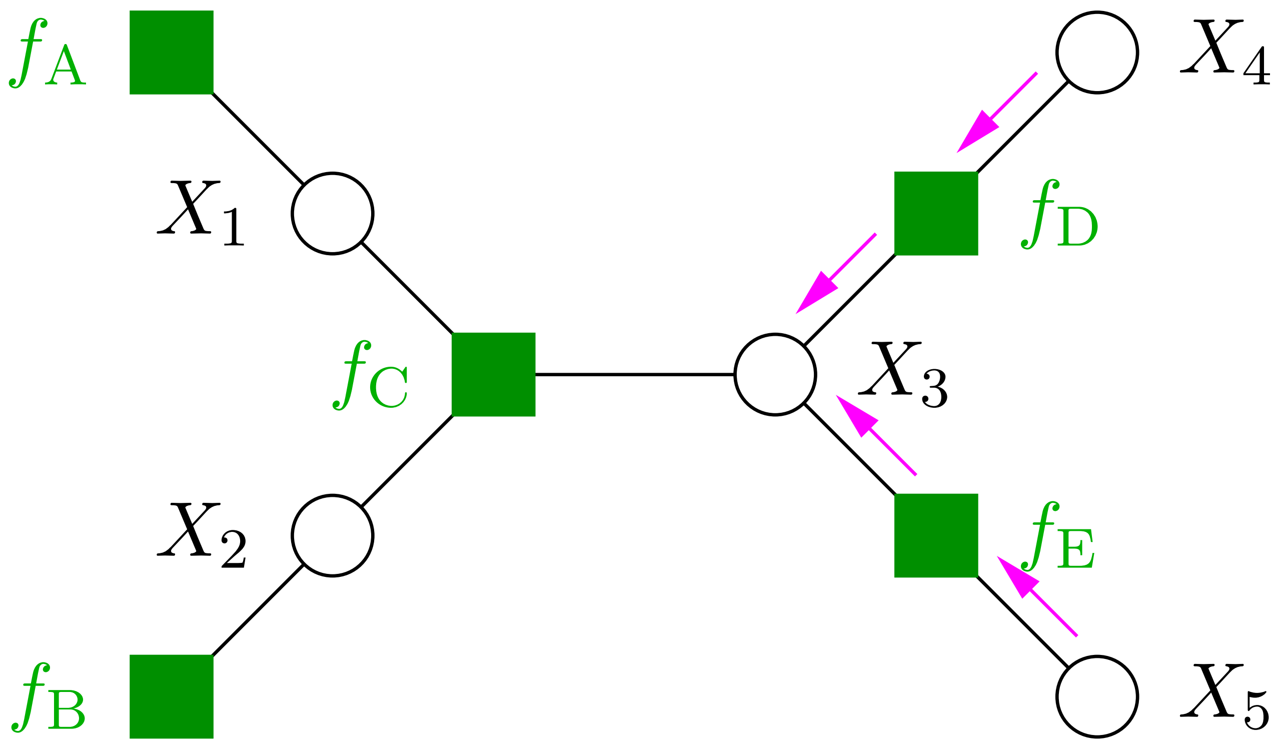
The Sum-Product Algorithm

Messages necessary for calculating $\eta_{X_1}(x_1)$.



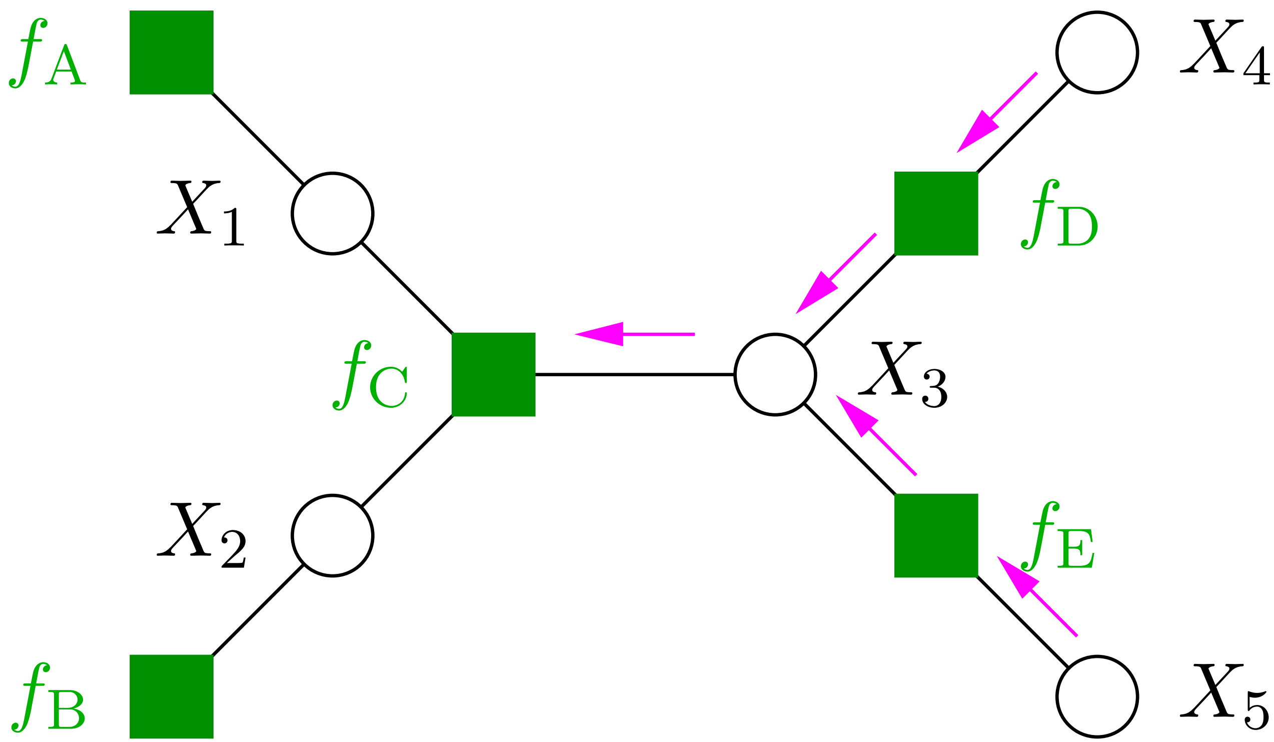
The Sum-Product Algorithm

Messages necessary for calculating $\eta_{X_1}(x_1)$.



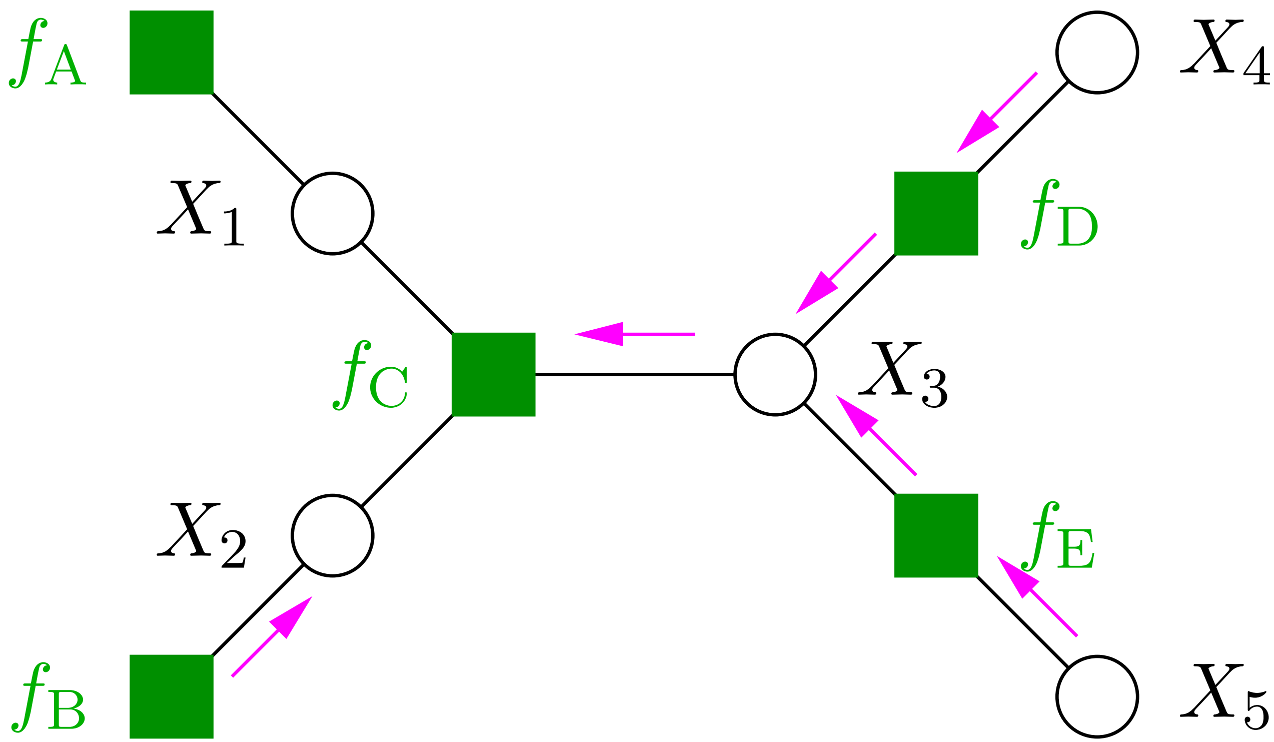
The Sum-Product Algorithm

Messages necessary for calculating $\eta_{X_1}(x_1)$.



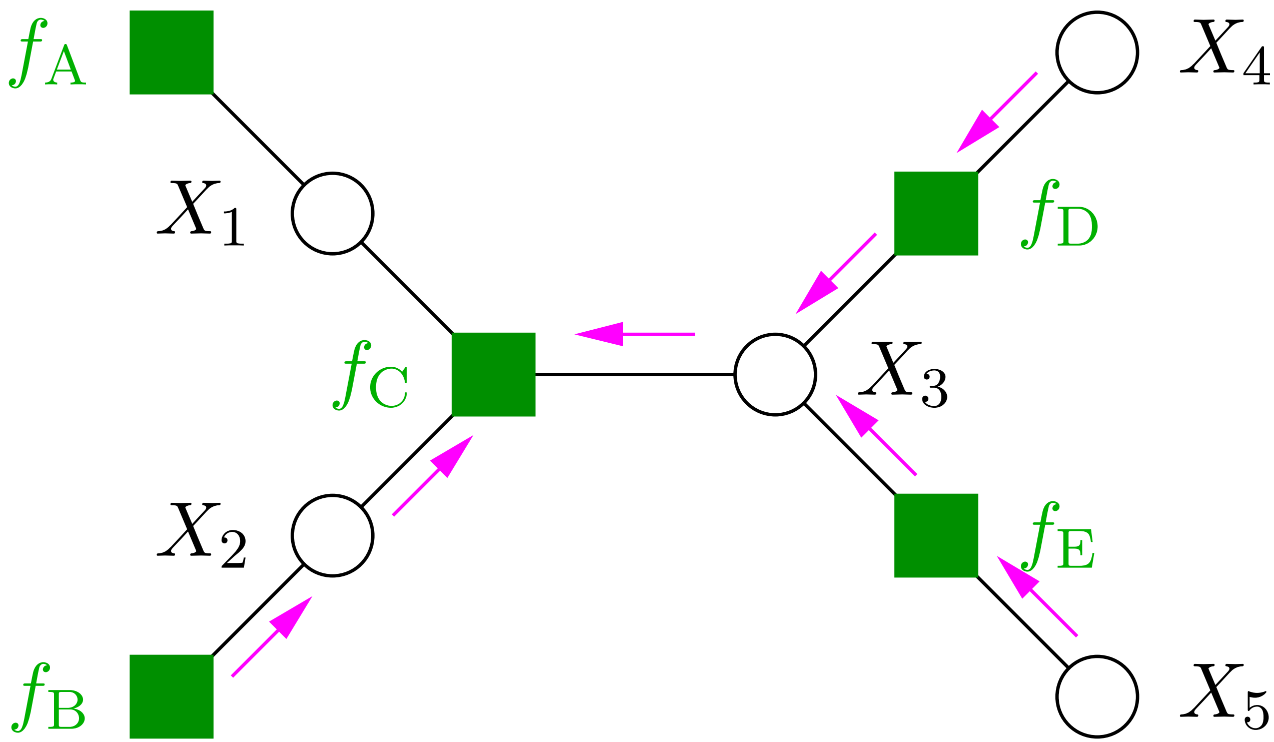
The Sum-Product Algorithm

Messages necessary for calculating $\eta_{X_1}(x_1)$.



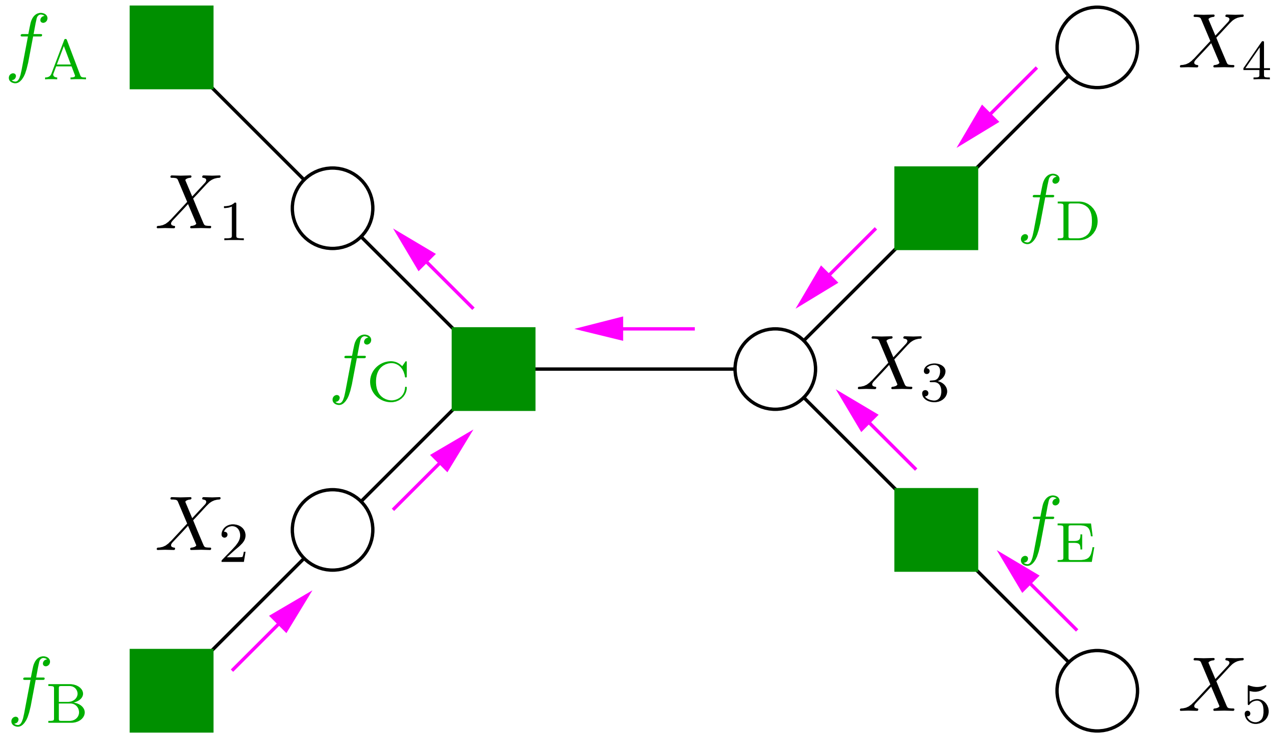
The Sum-Product Algorithm

Messages necessary for calculating $\eta_{X_1}(x_1)$.



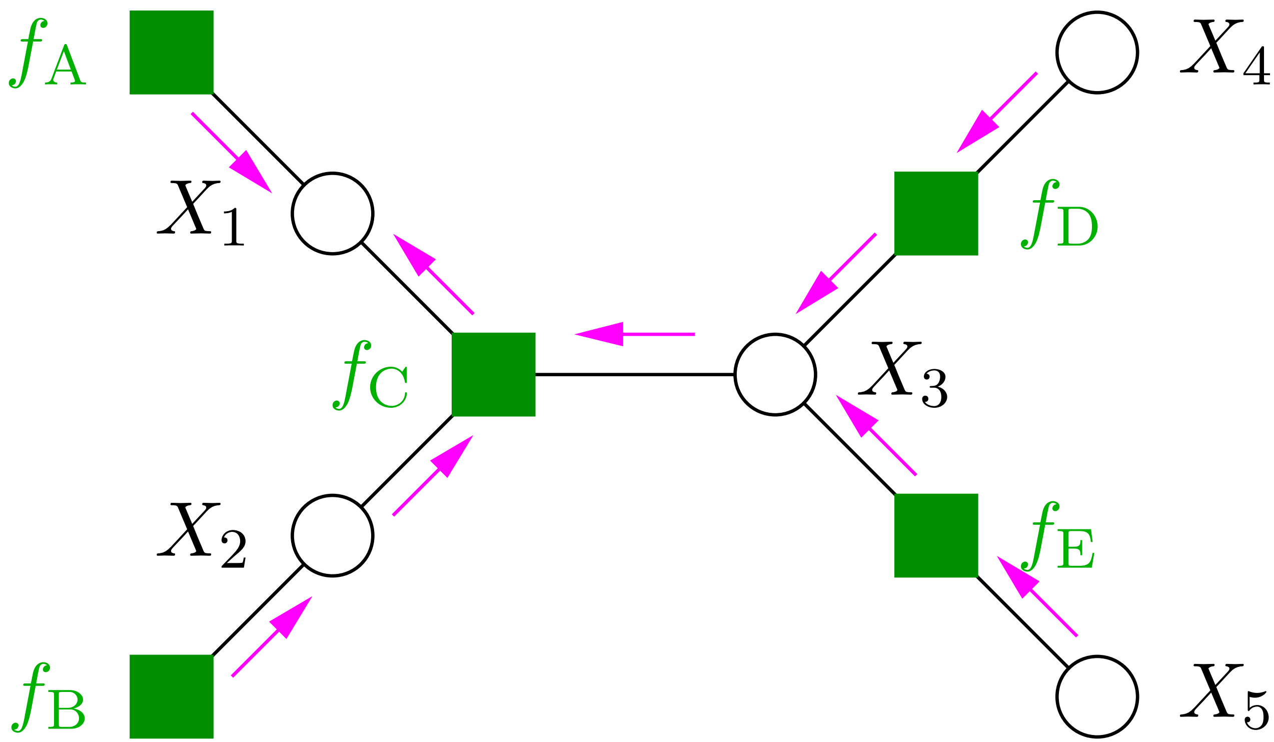
The Sum-Product Algorithm

Messages necessary for calculating $\eta_{X_1}(x_1)$.



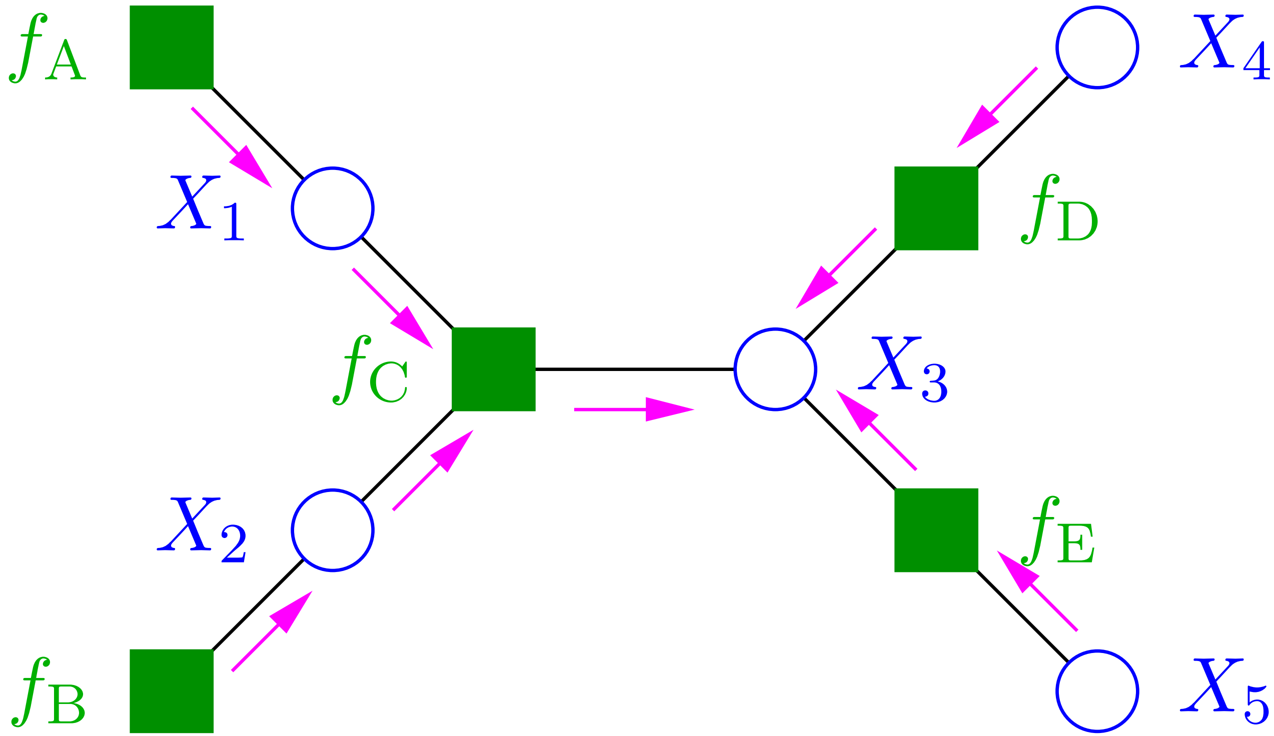
The Sum-Product Algorithm

Messages necessary for calculating $\eta_{X_1}(x_1)$.



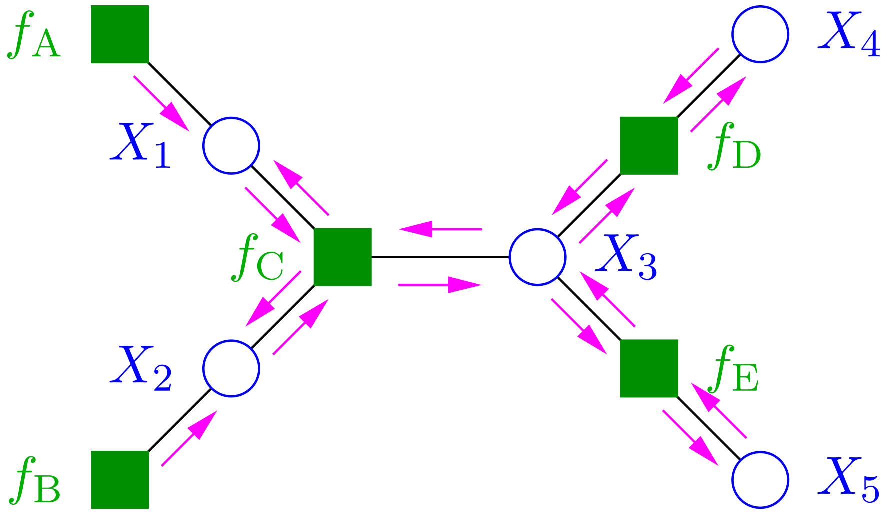
The Sum-Product Algorithm

Messages necessary for calculating $\eta_{X_3}(x_3)$.



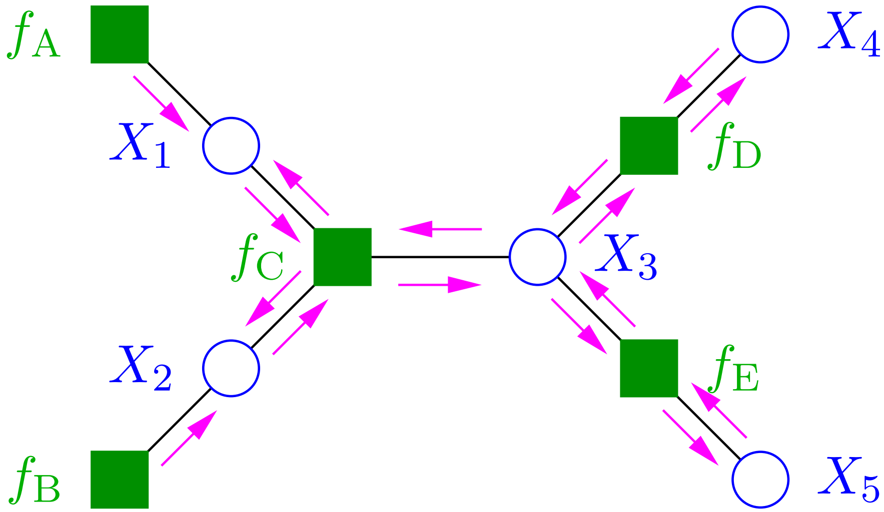
The Sum-Product Algorithm

The figure shows the messages that are necessary for calculating $\eta_{X_1}(x_1)$, $\eta_{X_2}(x_2)$, $\eta_{X_3}(x_3)$, $\eta_{X_4}(x_4)$, and $\eta_{X_5}(x_5)$.



The Sum-Product Algorithm

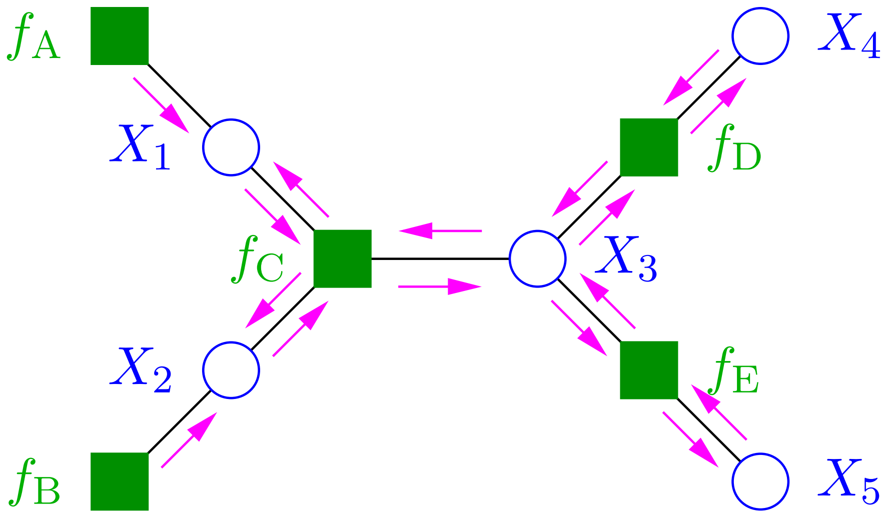
The figure shows the messages that are necessary for calculating $\eta_{X_1}(x_1)$, $\eta_{X_2}(x_2)$, $\eta_{X_3}(x_3)$, $\eta_{X_4}(x_4)$, and $\eta_{X_5}(x_5)$.



- Edges: Messages are sent along edges.

The Sum-Product Algorithm

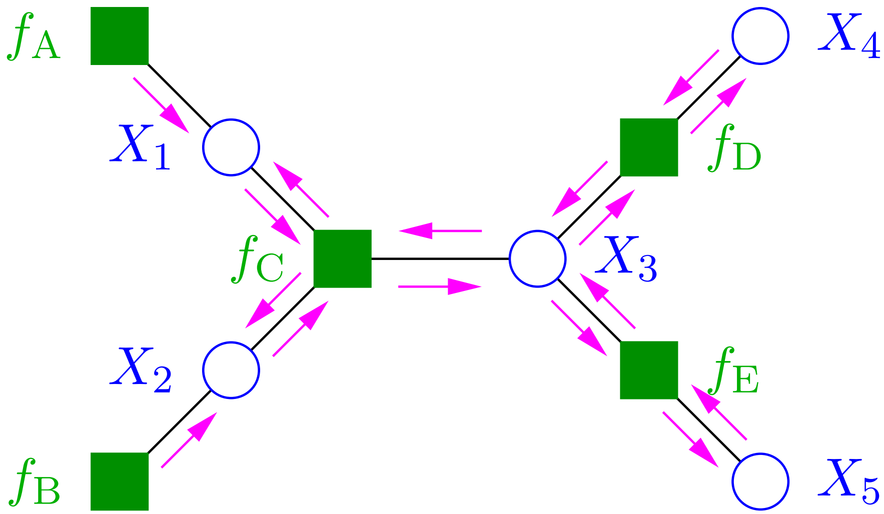
The figure shows the messages that are necessary for calculating $\eta_{X_1}(x_1)$, $\eta_{X_2}(x_2)$, $\eta_{X_3}(x_3)$, $\eta_{X_4}(x_4)$, and $\eta_{X_5}(x_5)$.



- Edges: Messages are sent along edges.
- Processing: Taking products and doing summations is done in the vertices.

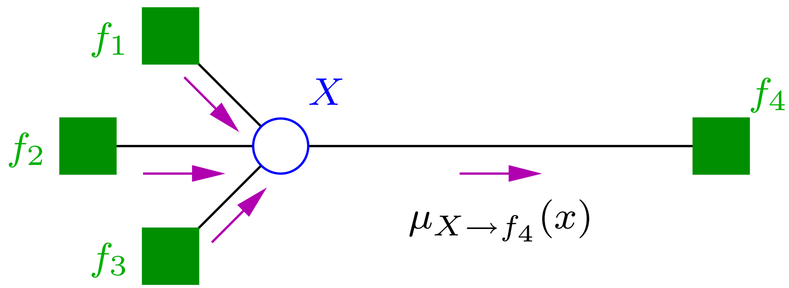
The Sum-Product Algorithm

The figure shows the messages that are necessary for calculating $\eta_{X_1}(x_1)$, $\eta_{X_2}(x_2)$, $\eta_{X_3}(x_3)$, $\eta_{X_4}(x_4)$, and $\eta_{X_5}(x_5)$.



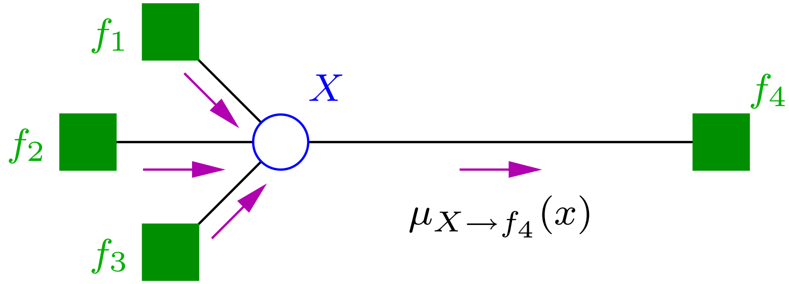
- Edges: Messages are sent along edges.
- Processing: Taking products and doing summations is done in the vertices.
- Reuse of messages: We see that messages can be "reused" in the sense that many partial calculations are the same; so it suffices to perform them only once.

The Sum-Product Algorithm

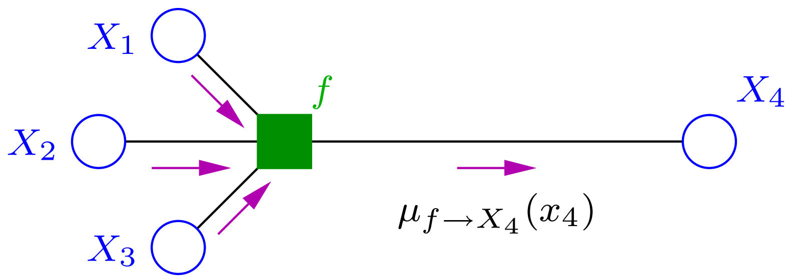


$$\mu_{X\to f_4}(x) = \mu_{f_1\to X}(x)\cdot\mu_{f_2\to X}(x)\cdot\mu_{f_3\to X}(x)$$

The Sum-Product Algorithm

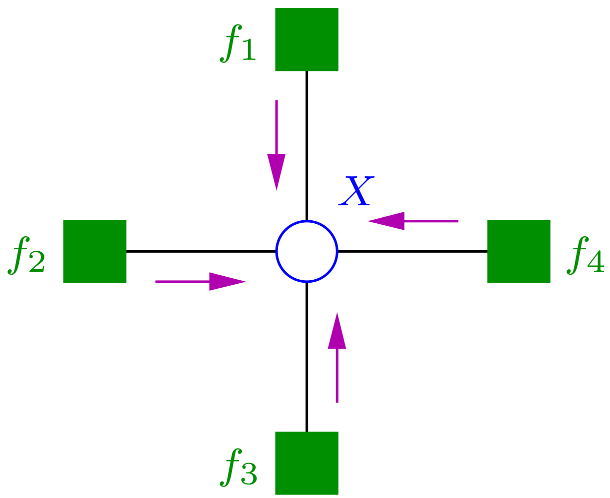


$$\mu_{X\to f_4}(x) = \mu_{f_1\to X}(x)\cdot\mu_{f_2\to X}(x)\cdot\mu_{f_3\to X}(x)$$



$$\mu_{f\to X_4}(x_4) = \sum_{x_1}\sum_{x_2}\sum_{x_3} f(x_1,x_2,x_3,x_4)\cdot\mu_{X_1\to f}(x_1)\cdot\mu_{X_2\to f}(x_2)\cdot\mu_{X_3\to f}(x_3)$$

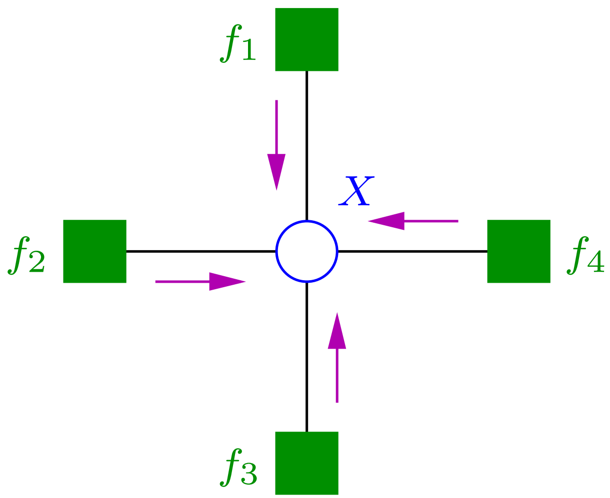
The Sum-Product Algorithm



Computation of marginal at variable node:

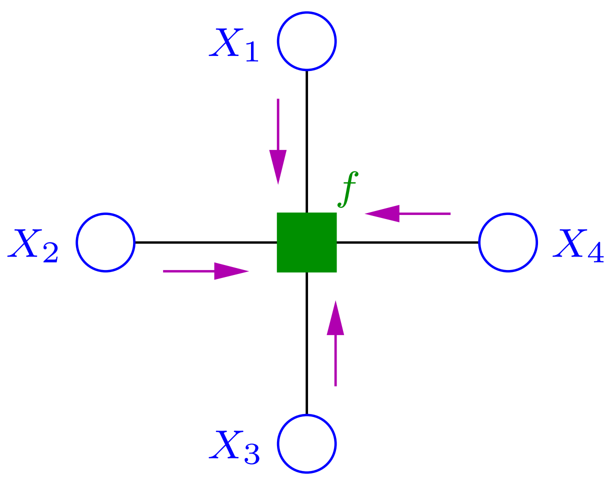
$$\eta_X(x) = \mu_{f_1 \to X}(x) \cdot \mu_{f_2 \to X}(x) \cdot \mu_{f_3 \to X}(x) \cdot \mu_{f_4 \to X}(x)$$

The Sum-Product Algorithm



Computation of marginal at variable node:

$$\eta_X(x) = \mu_{f_1 \to X}(x) \cdot \mu_{f_2 \to X}(x) \cdot \mu_{f_3 \to X}(x) \cdot \mu_{f_4 \to X}(x)$$



Computation of marginal at function node:

$$\eta_f(x_1, x_2, x_3, x_4) = f(x_1, x_2, x_3, x_4) \cdot \mu_{X_1 \to f}(x_1) \cdot \mu_{X_2 \to f}(x_2) \cdot \mu_{X_3 \to f}(x_3) \cdot \mu_{X_4 \to f}(x_4)$$

The Sum-Product Algorithm

- Factor graph without loops: in this case it is obvious what messages have to be calculated when.

$\Rightarrow$ Mode of operation 1

The Sum-Product Algorithm

- Factor graph without loops: in this case it is obvious what messages have to be calculated when.

  $\Rightarrow$ Mode of operation 1

- Factor graph with loops: one has to decide what update schedule to take.

  $\Rightarrow$ Mode of operation 2

Comments on the Sum-Product Algorithm

- If the factor graph has no loops then it is obvious what messages have to be calculated when.
- If the factor graphs has loops one has to decide what update schedule to take.
- Depending on the underlying semi-ring one gets different versions of the summary-product algorithm.
  - For $\langle \mathbb{R}, +, \cdot \rangle$ one gets the sum-product algorithm. (This is the case discussed above.)
  - For $\langle \mathbb{R}^{+}, \max, \cdot \rangle$ one gets the max-product algorithm.
  - For $\langle \mathbb{R}, \min, + \rangle$ one gets the min-sum algorithm.
  - etc.

Partition function (total sum)

Partition Function

$$Z = \sum_{x_1, x_2, x_3, x_4, x_5} f(x_1, x_2, x_3, x_4, x_5)$$

Partition Function

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Recall:

$$\eta_{X_1}(x_1) = \sum_{x_2, x_3, x_4, x_5} f(x_1, x_2, x_3, x_4, x_5)$$

$$\eta_{X_2}(x_2) = \sum_{x_1, x_3, x_4, x_5} f(x_1, x_2, x_3, x_4, x_5)$$

$$\vdots \qquad\qquad \vdots$$

Partition Function

$$Z = \sum_{x_1,x_2,x_3,x_4,x_5} f(x_1, x_2, x_3, x_4, x_5)$$

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$$\vdots \qquad \vdots$$

Define:

$$Z_{X_1} = \sum_{x_1} \eta_{X_1}(x_1)$$

$$Z_{X_2} = \sum_{x_2} \eta_{X_2}(x_2)$$

$$\vdots \qquad \vdots$$

Partition Function

$$Z = \sum_{x_1,x_2,x_3,x_4,x_5} f(x_1, x_2, x_3, x_4, x_5)$$

Recall:

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$$\vdots \qquad \vdots$$

Define:

$$Z_{X_1} = \sum_{x_1} \eta_{X_1}(x_1) = Z$$

$$Z_{X_2} = \sum_{x_2} \eta_{X_2}(x_2) = Z$$

$$\vdots \qquad \vdots$$

Partition Function

$$Z = \sum_{x_1, x_2, x_3, x_4, x_5} f(x_1, x_2, x_3, x_4, x_5)$$

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$$Z = \sum_{x_1,x_2,x_3,x_4,x_5} f(x_1, x_2, x_3, x_4, x_5)$$

Recall:

$$\begin{aligned} &\vdots && \vdots \\ \eta_{f_C}(x_1, x_2, x_3) &= \sum_{x_4,x_5} f(x_1, x_2, x_3, x_4, x_5) \\ &\vdots && \vdots \end{aligned}$$

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Define:

$$\begin{aligned} &\vdots && \vdots \\ Z_{f_C} &= \sum_{x_1,x_2,x_3} \eta_{f_C}(x_1, x_2, x_3) \\ &\vdots && \vdots \end{aligned}$$

Partition Function

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Define:

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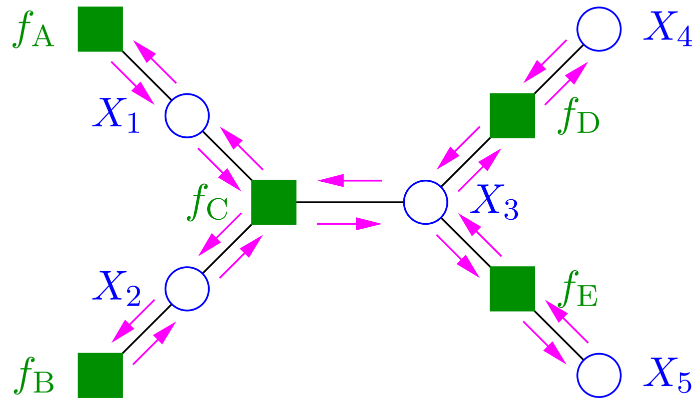
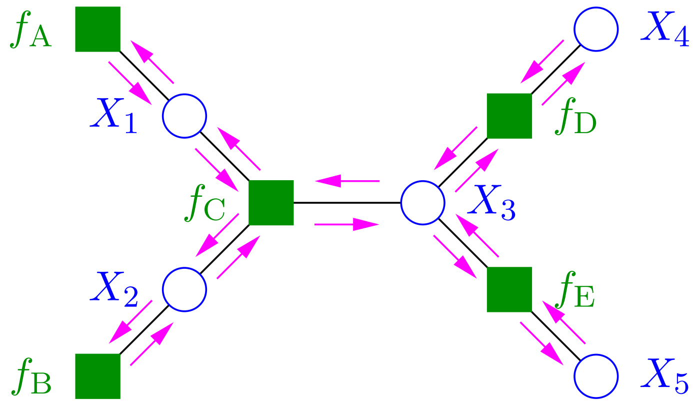
Partition Function

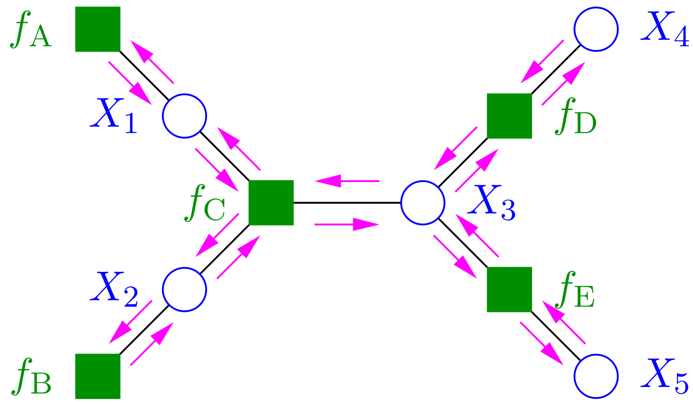


Partition Function



$$Z = Z_{X_1} = Z_{X_2} = Z_{X_3} = Z_{X_4} = Z_{X_5} = Z_{f_A} = Z_{f_B} = Z_{f_C} = Z_{f_D} = Z_{f_E}$$

Partition Function



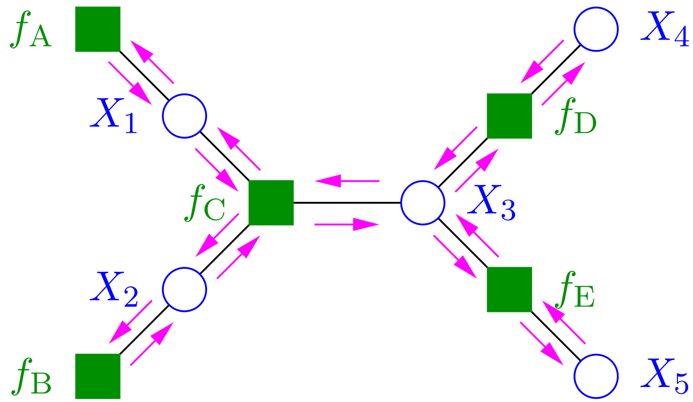
$$Z = Z_{X_1} = Z_{X_2} = Z_{X_3} = Z_{X_4} = Z_{X_5} = Z_{f_{\mathrm{A}}} = Z_{f_{\mathrm{B}}} = Z_{f_{\mathrm{C}}} = Z_{f_{\mathrm{D}}} = Z_{f_{\mathrm{E}}}$$

Claim:

$$Z = \frac{Z_{f_{\mathrm{A}}} \cdot Z_{f_{\mathrm{B}}} \cdot Z_{f_{\mathrm{C}}} \cdot Z_{f_{\mathrm{D}}} \cdot Z_{f_{\mathrm{E}}} \cdot Z_{X_1} \cdot Z_{X_2} \cdot Z_{X_3} \cdot Z_{X_4} \cdot Z_{X_5}}{Z_{X_1}^2 \cdot Z_{X_2}^2 \cdot Z_{X_3}^3 \cdot Z_{X_4}^1 \cdot Z_{X_5}^1}$$

(Note: exponents in denominator equal variable node degrees.)

Partition Function



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$$Z = \frac{Z^{\#\text{vertices}}}{Z^{\#\text{edges}}} = \frac{Z_{f_{\mathrm{A}}} \cdot Z_{f_{\mathrm{B}}} \cdot Z_{f_{\mathrm{C}}} \cdot Z_{f_{\mathrm{D}}} \cdot Z_{f_{\mathrm{E}}} \cdot Z_{X_1} \cdot Z_{X_2} \cdot Z_{X_3} \cdot Z_{X_4} \cdot Z_{X_5}}{Z_{X_1}^2 \cdot Z_{X_2}^2 \cdot Z_{X_3}^3 \cdot Z_{X_4}^1 \cdot Z_{X_5}^1}$$

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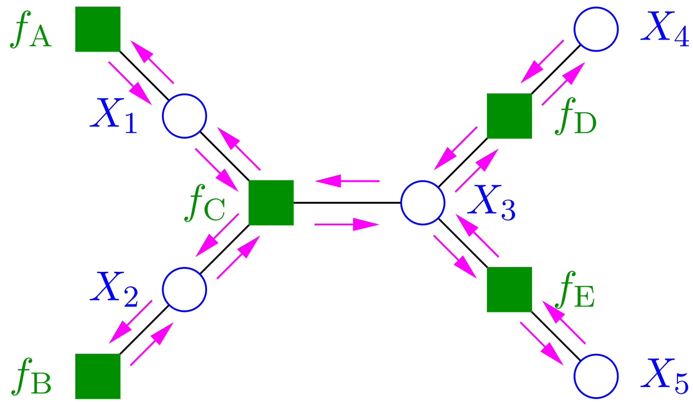
Partition Function



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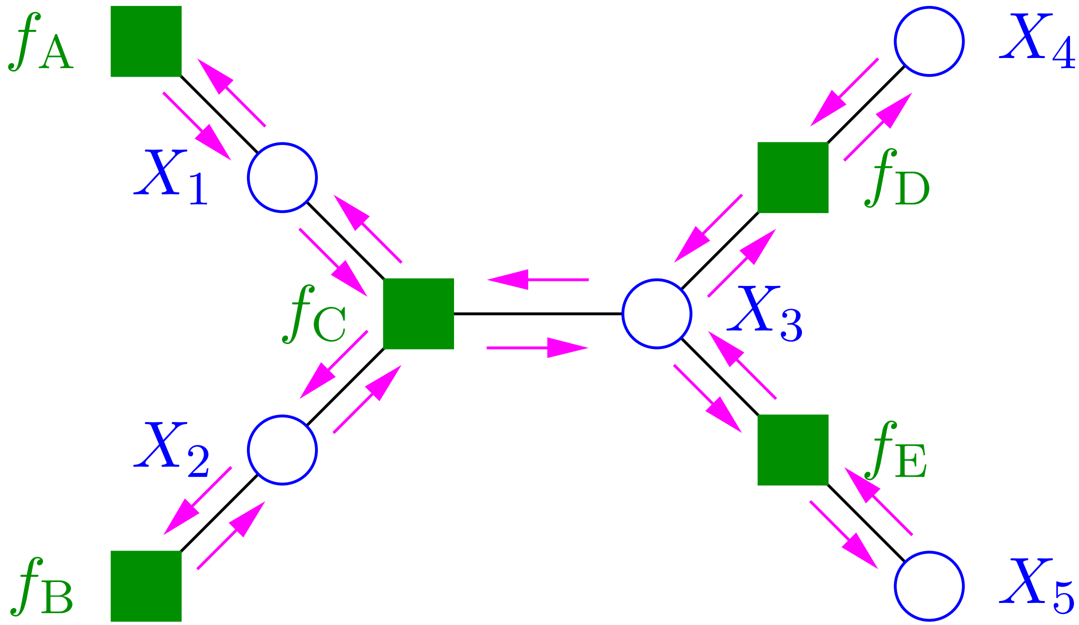
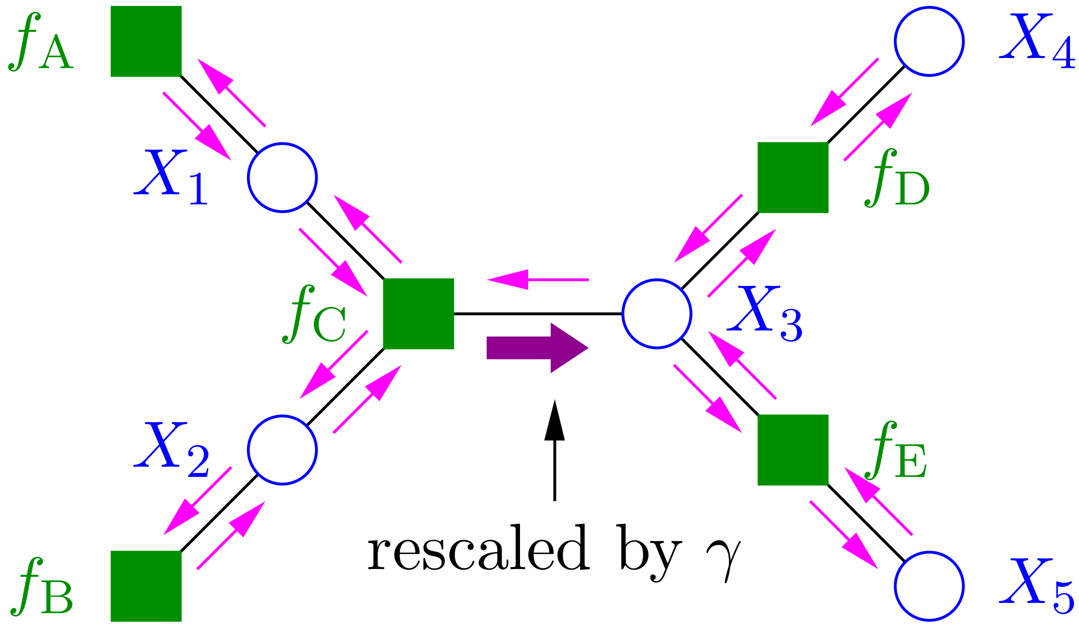
(Here we used the fact that for a graph with one component and no cycles it holds that $\#\text{vertices} = \#\text{edges} + 1$.)

Partition Function



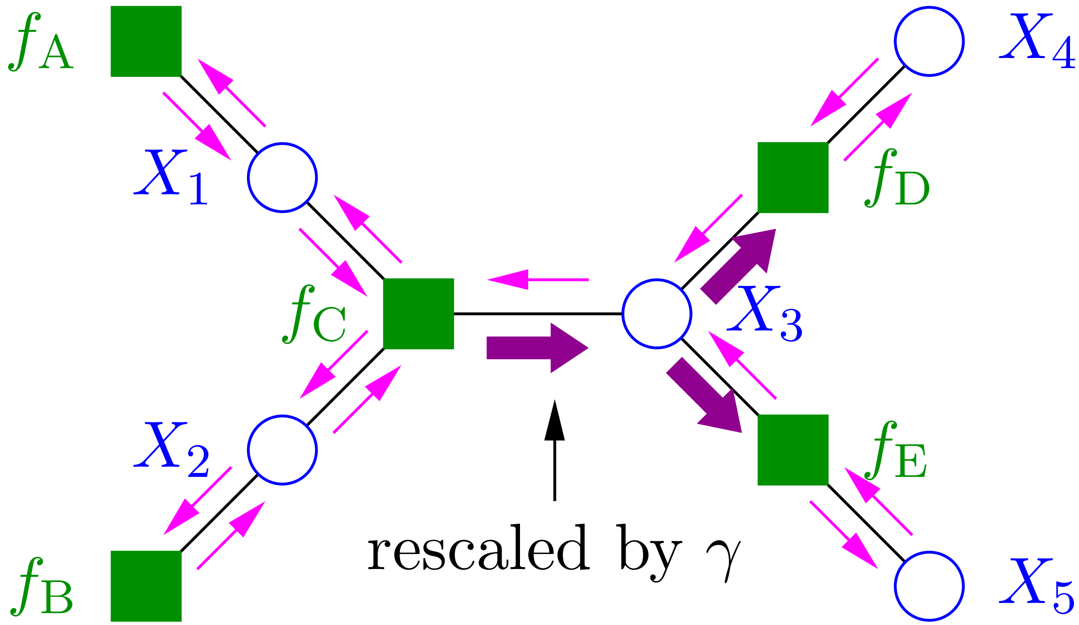
$$Z = \frac{Z_{f_A} \cdot Z_{f_B} \cdot Z_{f_C} \cdot Z_{f_D} \cdot Z_{f_E} \cdot Z_{X_1} \cdot Z_{X_2} \cdot Z_{X_3} \cdot Z_{X_4} \cdot Z_{X_5}}{Z_{X_1}^2 \cdot Z_{X_2}^2 \cdot Z_{X_3}^3 \cdot Z_{X_4}^1 \cdot Z_{X_5}^1}$$

Partition Function



$$Z = \frac{Z_{f_A} \cdot Z_{f_B} \cdot Z_{f_C} \cdot Z_{f_D} \cdot Z_{f_E} \cdot Z_{X_1} \cdot Z_{X_2} \cdot Z_{X_3} \cdot Z_{X_4} \cdot Z_{X_5}}{Z_{X_1}^2 \cdot Z_{X_2}^2 \cdot Z_{X_3}^3 \cdot Z_{X_4}^1 \cdot Z_{X_5}^1}$$

Partition Function



$$Z = \frac{Z_{f_{\mathrm{A}}} \cdot Z_{f_{\mathrm{B}}} \cdot Z_{f_{\mathrm{C}}} \cdot Z_{f_{\mathrm{D}}} \cdot Z_{f_{\mathrm{E}}} \cdot Z_{X_1} \cdot Z_{X_2} \cdot Z_{X_3} \cdot Z_{X_4} \cdot Z_{X_5}}{Z_{X_1}^2 \cdot Z_{X_2}^2 \cdot Z_{X_3}^3 \cdot Z_{X_4}^1 \cdot Z_{X_5}^1}$$

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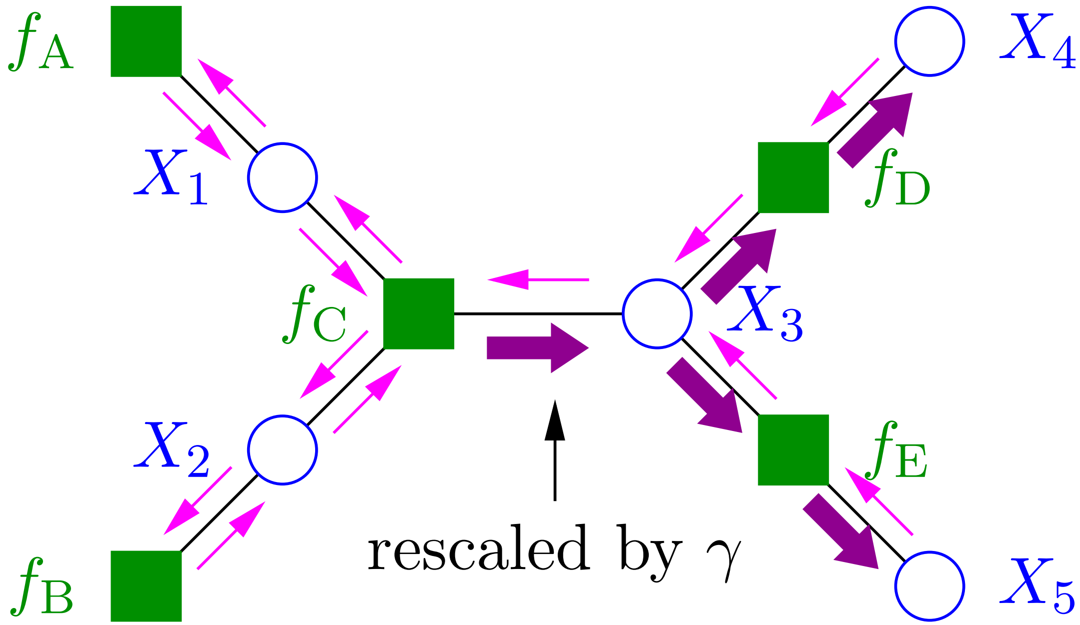

$$Z = \frac{Z_{f_{\mathrm{A}}} \cdot Z_{f_{\mathrm{B}}} \cdot Z_{f_{\mathrm{C}}} \cdot Z_{f_{\mathrm{D}}} \cdot Z_{f_{\mathrm{E}}} \cdot Z_{X_1} \cdot Z_{X_2} \cdot Z_{X_3} \cdot Z_{X_4} \cdot Z_{X_5}}{Z_{X_1}^2 \cdot Z_{X_2}^2 \cdot Z_{X_3}^3 \cdot Z_{X_4}^1 \cdot Z_{X_5}^1}$$

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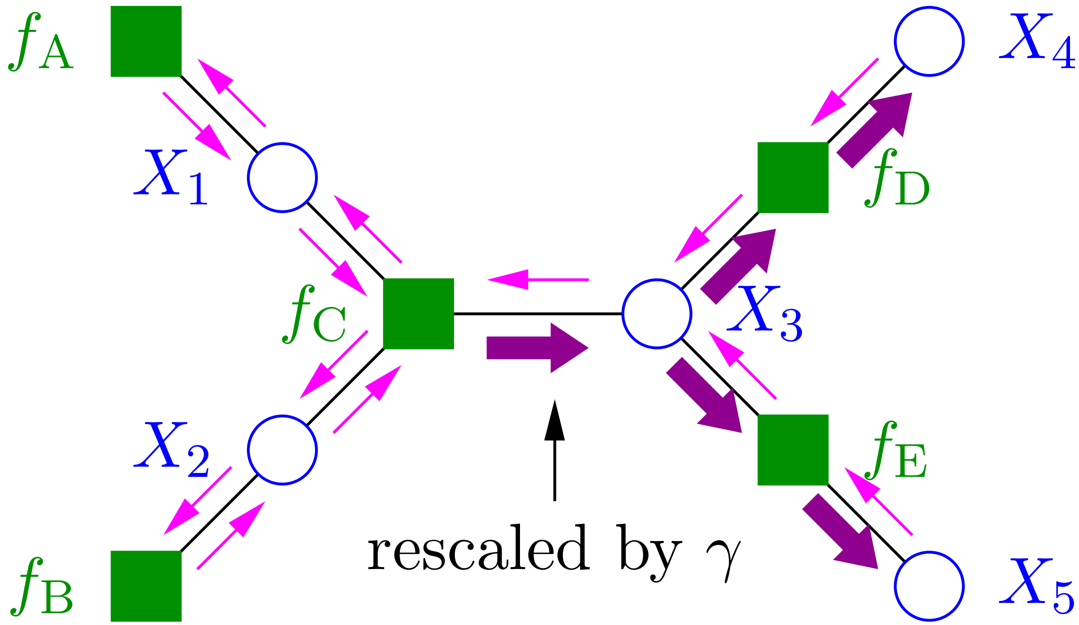
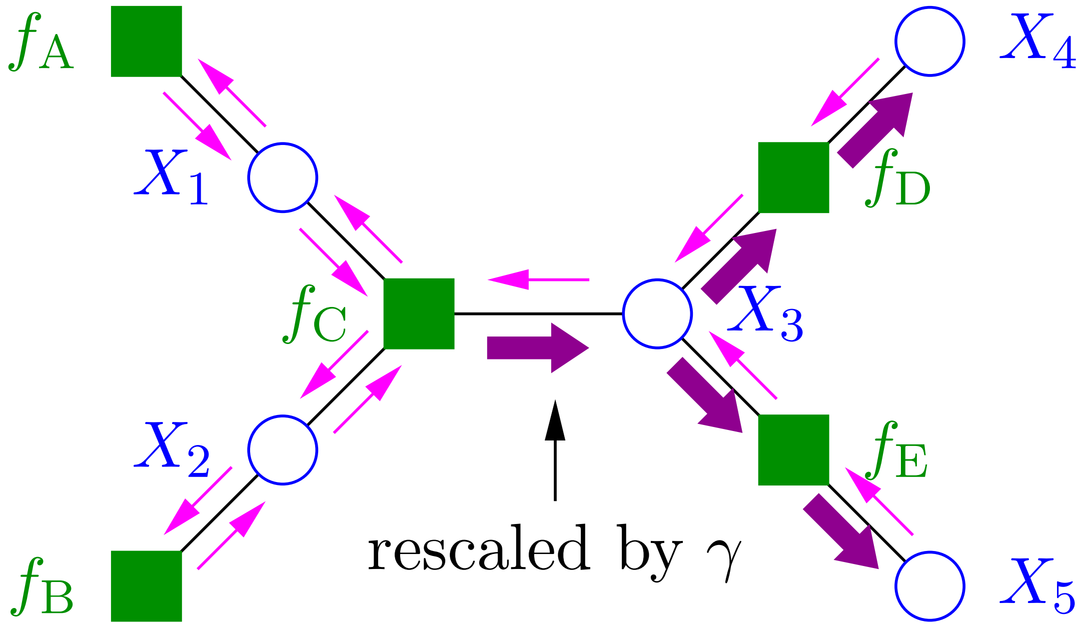


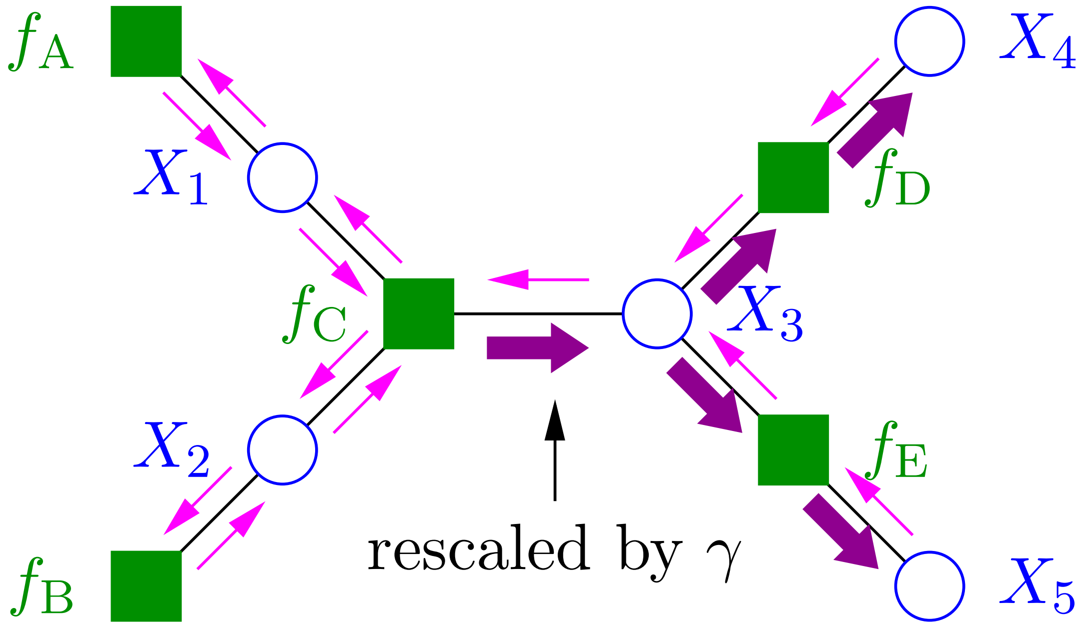
$$Z = \frac{Z_{f_{\mathrm{A}}} \cdot Z_{f_{\mathrm{B}}} \cdot Z_{f_{\mathrm{C}}} \cdot \boxed{\hat{Z}_{f_{\mathrm{D}}}} \cdot \boxed{\hat{Z}_{f_{\mathrm{E}}}} \cdot Z_{X_1} \cdot Z_{X_2} \cdot \boxed{\hat{Z}_{X_3}} \cdot \boxed{\hat{Z}_{X_4}} \cdot \boxed{\hat{Z}_{X_5}}}{Z_{X_1}^2 \cdot Z_{X_2}^2 \cdot \boxed{\hat{Z}_{X_3}^3} \cdot \boxed{\hat{Z}_{X_4}^1} \cdot \boxed{\hat{Z}_{X_5}^1}}$$

Partition Function



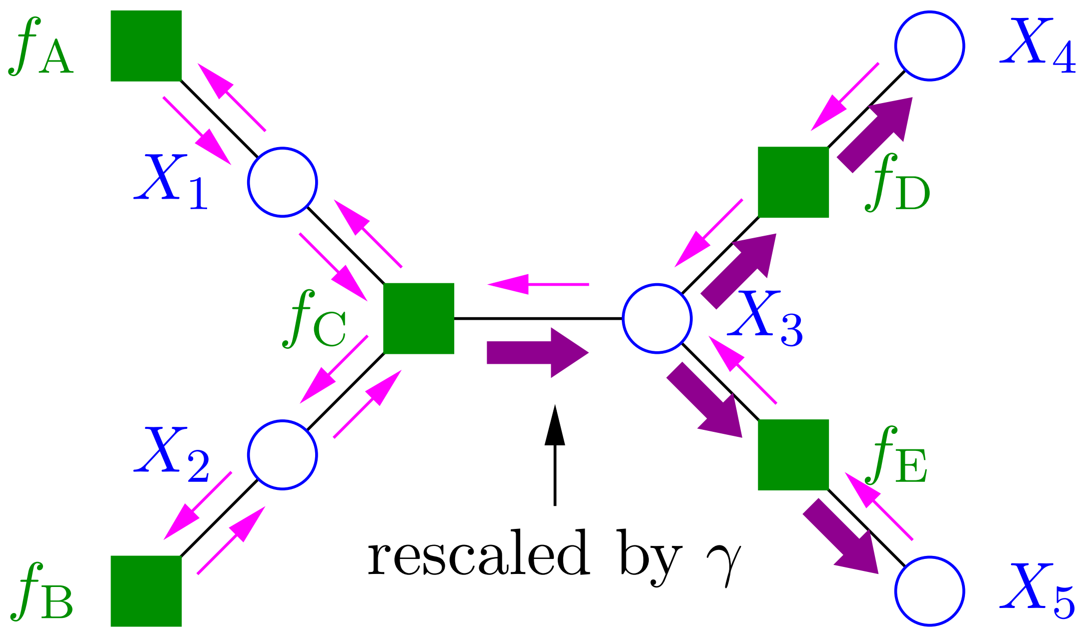
$$Z = \frac{Z_{f_{\mathrm{A}}} \cdot Z_{f_{\mathrm{B}}} \cdot Z_{f_{\mathrm{C}}} \cdot \boxed{\gamma Z_{f_{\mathrm{D}}}} \cdot \boxed{\gamma Z_{f_{\mathrm{E}}}} \cdot Z_{X_1} \cdot Z_{X_2} \cdot \boxed{\gamma Z_{X_3}} \cdot \boxed{\gamma Z_{X_4}} \cdot \boxed{\gamma Z_{X_5}}}{Z_{X_1}^2 \cdot Z_{X_2}^2 \cdot \boxed{\gamma^3 Z_{X_3}^3} \cdot \boxed{\gamma^1 Z_{X_4}^1} \cdot \boxed{\gamma^1 Z_{X_5}^1}}$$

Partition Function



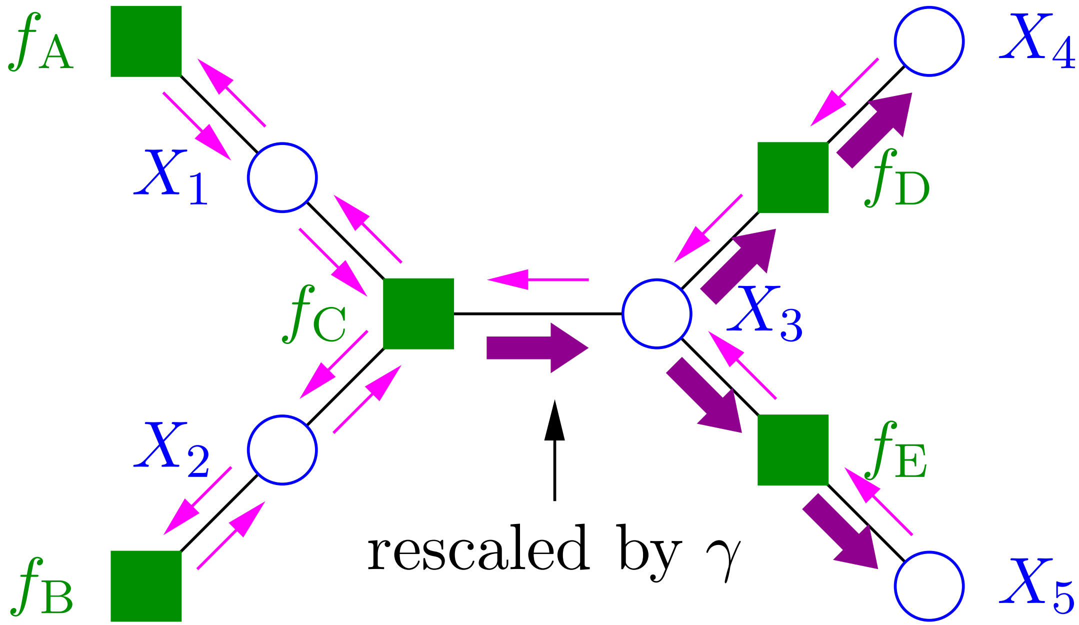
$$\color{red}{Z} = \frac{\gamma^5}{\gamma^5} \cdot \frac{Z_{f_A} \cdot Z_{f_B} \cdot Z_{f_C} \cdot Z_{f_D} \cdot Z_{f_E} \cdot Z_{X_1} \cdot Z_{X_2} \cdot Z_{X_3} \cdot Z_{X_4} \cdot Z_{X_5}}{Z_{X_1}^2 \cdot Z_{X_2}^2 \cdot Z_{X_3}^3 \cdot Z_{X_4}^1 \cdot Z_{X_5}^1}$$

Partition Function



$$Z = \frac{Z_{f_{\mathrm{A}}} \cdot Z_{f_{\mathrm{B}}} \cdot Z_{f_{\mathrm{C}}} \cdot Z_{f_{\mathrm{D}}} \cdot Z_{f_{\mathrm{E}}} \cdot Z_{X_1} \cdot Z_{X_2} \cdot Z_{X_3} \cdot Z_{X_4} \cdot Z_{X_5}}{Z_{X_1}^2 \cdot Z_{X_2}^2 \cdot Z_{X_3}^3 \cdot Z_{X_4}^1 \cdot Z_{X_5}^1}$$

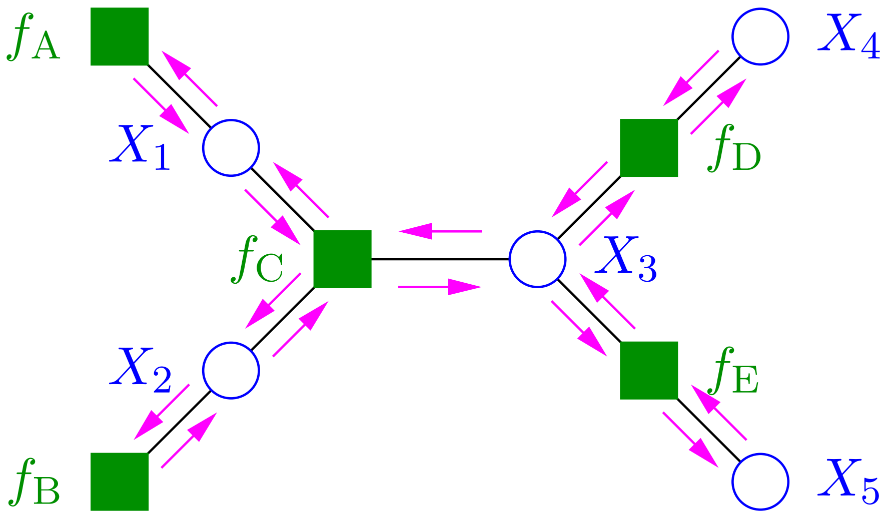
Partition Function



$$Z = \frac{Z_{f_A} \cdot Z_{f_B} \cdot Z_{f_C} \cdot Z_{f_D} \cdot Z_{f_E} \cdot Z_{X_1} \cdot Z_{X_2} \cdot Z_{X_3} \cdot Z_{X_4} \cdot Z_{X_5}}{Z_{X_1}^2 \cdot Z_{X_2}^2 \cdot Z_{X_3}^3 \cdot Z_{X_4}^1 \cdot Z_{X_5}^1}$$

Remarkable: this expression is invariant to rescaling of *function-node-to-variable-node* messages!

Partition Function



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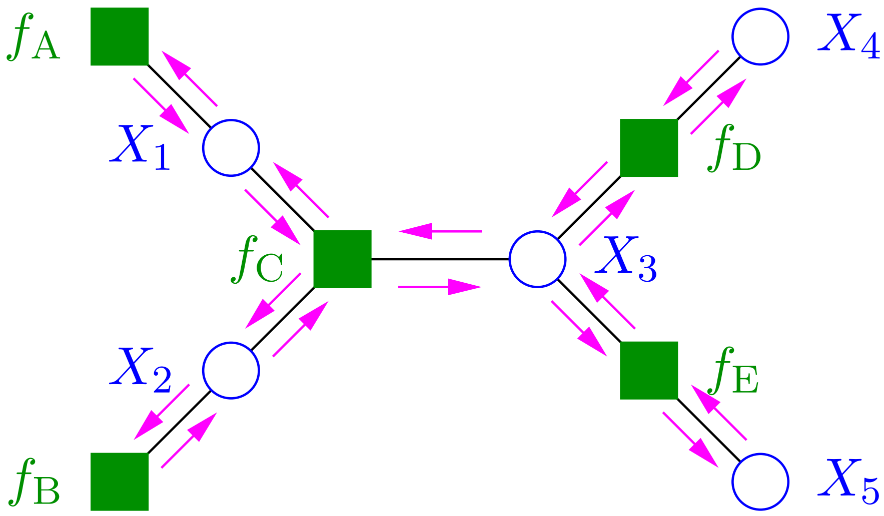
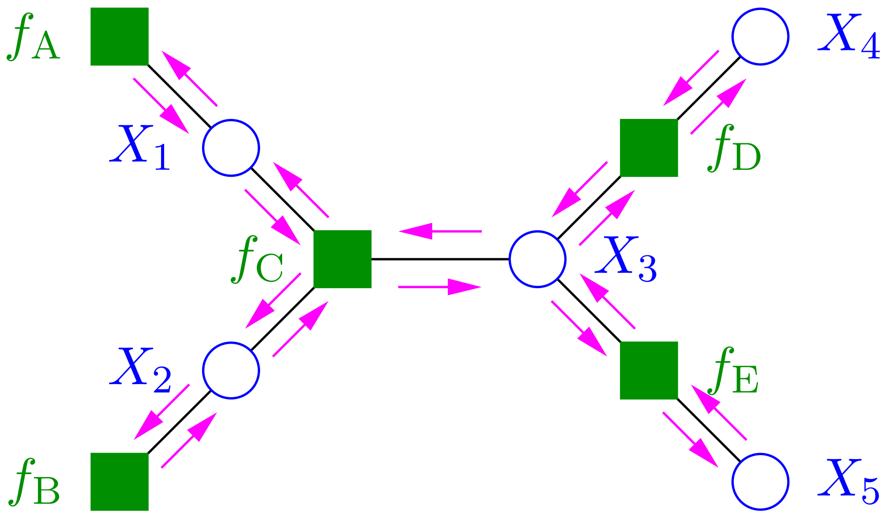
Partition Function



$$Z = \frac{\prod_f Z_f \cdot \prod_X Z_X}{\prod_X Z_X^{\deg(\mathrm{X})}}$$

Partition Function

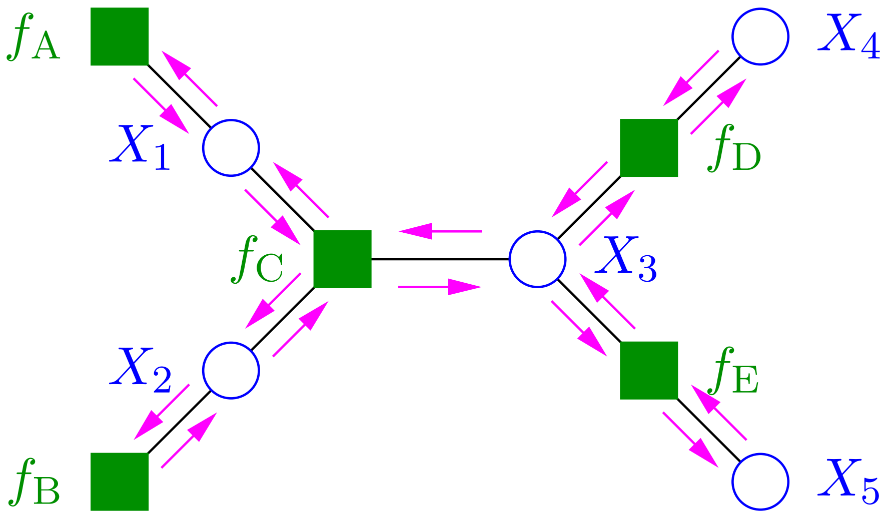


$$\color{red}{Z} = \frac{\prod_f Z_f \cdot \prod_X Z_X}{\prod_X Z_X^{\deg(X)}}$$

Bethe approximation:

Use the above type of expression also when factor graph has cycles.

Partition Function



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$$\rightarrow Z'_{\text{Bethe}}$$

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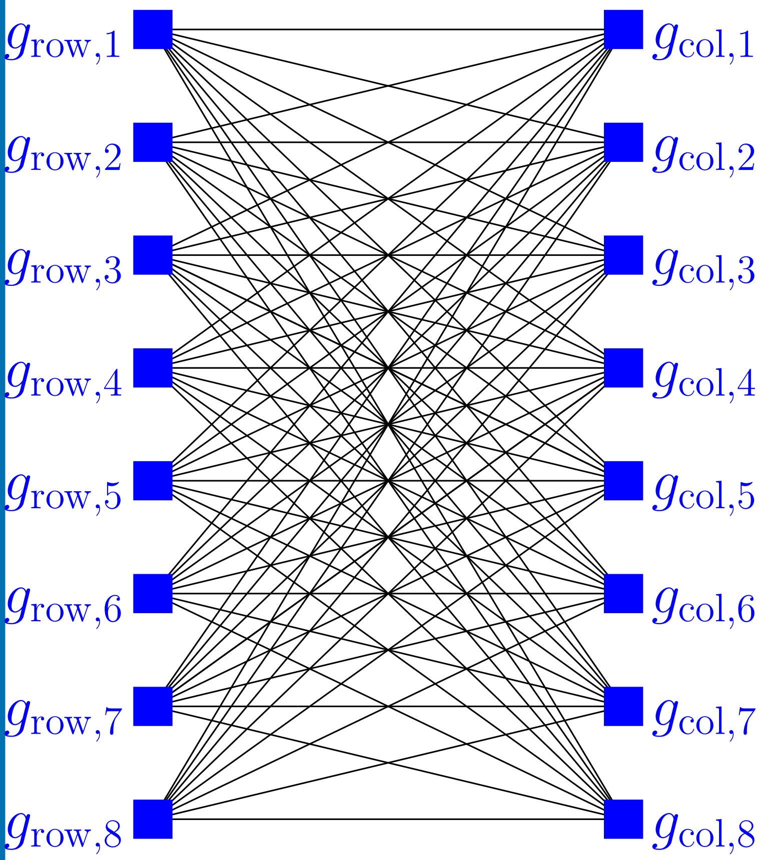
  Therefore, we call Z'_{Bethe} a (local) Bethe partition function only if we are at a fixed point of the SPA.

- Factor graph with cycles: the SPA can have multiple fixed points. We define the Bethe partition function to be

$$Z_{\mathrm{Bethe}} \triangleq \max_{\text{fixed points of SPA}} Z'_{\mathrm{Bethe}}.$$

Graphical Model for Permanent



(function nodes are suitably defined based on $\boldsymbol{\theta}$)

(variable nodes have been omitted)

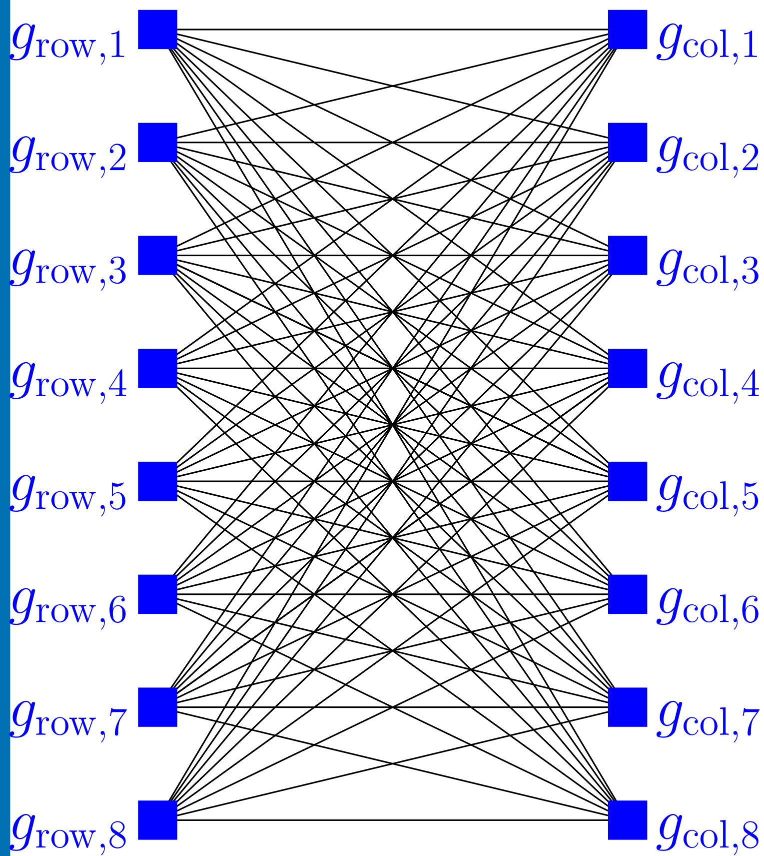
Global function:

$$g(a_{1,1}, \ldots, a_{8,8}) = \prod_j g_{\text{col},j}(a_{1,j}, \ldots, a_{8,j}) \times \prod_i g_{\text{row},i}(a_{i,1}, \ldots, x_{i,8})$$

Permanent:

$$\text{perm}(\boldsymbol{\theta}) = Z = \sum_{a_{1,1},\ldots,a_{8,8}} g(a_{1,1}, \ldots, a_{8,8})$$

Graphical Model for Permanent



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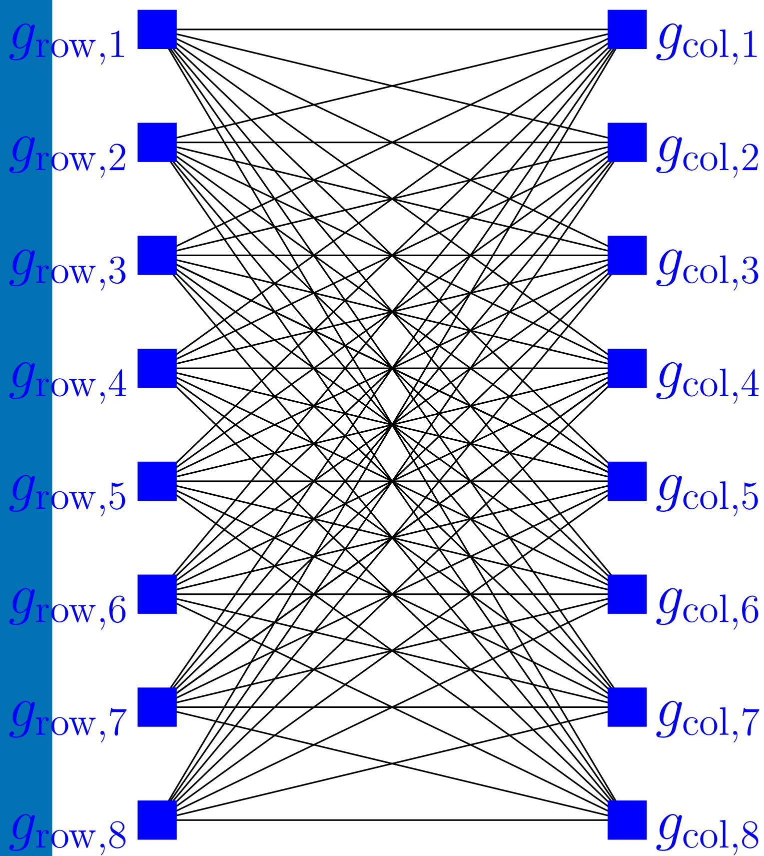
Bethe Permanent:

$$\text{perm}_{\text{B}}(\boldsymbol{\theta}) \triangleq Z_{\text{Bethe}}$$

Graphical Model for Permanent



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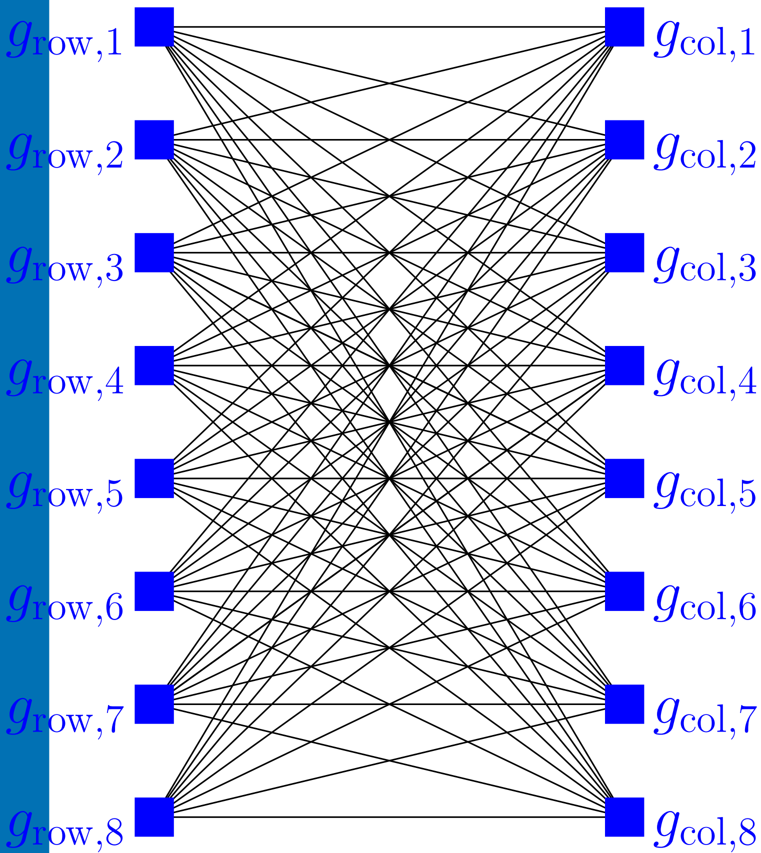
However, the SPA is a locally operating algorithm and so has its limitations in the conclusions that it can reach.

Graphical Model for Permanent



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Bethe Permanent:

$$\mathrm{perm}_{\mathrm{B}}(\boldsymbol{\theta}) \triangleq Z_{\mathrm{Bethe}}$$

This locality of the SPA turns out to be well-captured by so-called finite graph covers, especially at fixed points of the SPA.

A combinatorial interpretation of the Bethe permanent

Combinatorial Characterization of the Permanent

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Consider the matrix

$$\boldsymbol{\theta} = \begin{pmatrix} \theta_{1,1} & \theta_{1,2} \\ \theta_{2,1} & \theta_{2,2} \end{pmatrix} \qquad \text{with} \qquad \operatorname{perm}(\boldsymbol{\theta}) = \theta_{1,1}\theta_{2,2} + \theta_{2,1}\theta_{1,2}.$$

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In particular,

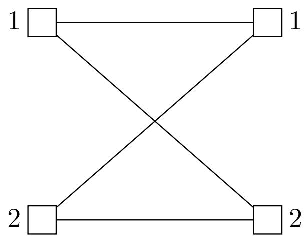
$$\boldsymbol{\theta} = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \qquad \text{with} \qquad \mathrm{perm}(\boldsymbol{\theta}) = 1 \cdot 1 + 1 \cdot 1 = 2.$$

Combinatorial Characterization of the Permanent

Recall that the permanent of a zero/one matrix like

$$\boldsymbol{\theta} = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$$

equals the number of perfect matchings in the following bipartite graph:

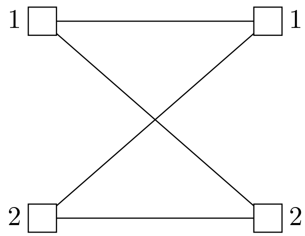


Combinatorial Characterization of the Permanent

Recall that the permanent of a zero/one matrix like

$$\boldsymbol{\theta} = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$$

equals the number of perfect matchings in the following bipartite graph:



Namely,

A Combinatorial Interpretation of the Bethe Permanent

- Consider the non-negative matrix $\boldsymbol{\theta}$ of size $n \times n$.
- Let $\mathcal{P}_{M \times M}$ be the set of all permutation matrices of size $M \times M$.
- For every positive integer M, we define Ψ_M be the set

$$\Psi_M \triangleq \left\{ \mathbf{P} = \left\{ \mathbf{P}^{(i,j)} \right\}_{(i,j) \in [n]^2} \,\middle|\, \mathbf{P}^{(i,j)} \in \mathcal{P}_{M \times M} \right\}.$$

- For $\mathbf{P} \in \Psi_M$ we define the $\mathbf{P}$-lifting of $\boldsymbol{\theta}$ to be the following $(nM) \times (nM)$ matrix

$$\boldsymbol{\theta} = \begin{pmatrix} \theta_{1,1} & \cdots & \theta_{1,n} \\ \vdots & & \vdots \\ \theta_{n,1} & \cdots & \theta_{n,n} \end{pmatrix} \quad \underset{\text{of } \boldsymbol{\theta}}{\xrightarrow{\mathbf{P}\text{-lifting}}} \quad \boldsymbol{\theta}^{\uparrow \mathbf{P}} \triangleq \begin{pmatrix} \theta_{1,1}\mathbf{P}^{(1,1)} & \cdots & \theta_{1,n}\mathbf{P}^{(1,n)} \\ \vdots & & \vdots \\ \theta_{n,1}\mathbf{P}^{(n,1)} & \cdots & \theta_{n,n}\mathbf{P}^{(n,n)} \end{pmatrix}.$$

Degree-M Bethe Permanent

Definition: For any positive integer M, we define the degree-M Bethe permanent of $\boldsymbol{\theta}$ to be

$$\mathrm{perm}_{\mathrm{B},M}(\boldsymbol{\theta}) \triangleq \sqrt[M]{\left\langle \mathrm{perm}\left(\boldsymbol{\theta}^{\uparrow \mathbf{P}}\right) \right\rangle_{\mathbf{P} \in \Psi_M}}.$$

Theorem:

$$\mathrm{perm}_{\mathrm{B}}(\boldsymbol{\theta}) = \limsup_{M \to \infty} \ \mathrm{perm}_{\mathrm{B},M}(\boldsymbol{\theta}).$$

Special Case: Deg.-M Bethe Permanent for $n = 2$

We want to obtain some appreciation why the Bethe permanent of $\boldsymbol{\theta}$ is close to the permanent of $\boldsymbol{\theta}$, and where the differences are.

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As before, consider the matrix

$$\boldsymbol{\theta} = \begin{pmatrix} \theta_{1,1} & \theta_{1,2} \\ \theta_{2,1} & \theta_{2,2} \end{pmatrix} \qquad \text{with} \qquad \mathrm{perm}(\boldsymbol{\theta}) = \theta_{1,1}\theta_{2,2} + \theta_{2,1}\theta_{1,2}.$$

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In particular,

$$\boldsymbol{\theta} = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \qquad \text{with} \qquad \operatorname{perm}(\boldsymbol{\theta}) = 1 \cdot 1 + 1 \cdot 1 = 2.$$

Special Case: Deg.-M Bethe Permanent for $n = 2$

For this $\boldsymbol{\theta}$, a $\mathbf{P}$-lifting looks like

$$\boldsymbol{\theta}^{\uparrow \mathbf{P}} = \begin{pmatrix} 1 \cdot \mathbf{P}_{1,1} & 1 \cdot \mathbf{P}_{1,2} \\ 1 \cdot \mathbf{P}_{2,1} & 1 \cdot \mathbf{P}_{2,2} \end{pmatrix} = \begin{pmatrix} \mathbf{P}_{1,1} & \mathbf{P}_{1,2} \\ \mathbf{P}_{2,1} & \mathbf{P}_{2,2} \end{pmatrix}.$$

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Applying some row and column permutations, we obtain

$$\operatorname{perm}\left(\boldsymbol{\theta}^{\uparrow \mathbf{P}}\right) = \operatorname{perm}\begin{pmatrix} \mathbf{I} & \mathbf{I} \\ \mathbf{I} & \mathbf{P}_{2,1}^{-1}\mathbf{P}_{2,2}\mathbf{P}_{1,2}^{-1}\mathbf{P}_{1,1} \end{pmatrix}.$$

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Therefore,

$$\mathrm{perm}_{\mathrm{B},M}(\boldsymbol{\theta}) \triangleq \sqrt[M]{\left\langle \mathrm{perm}\begin{pmatrix} \mathbf{I} & \mathbf{I} \\ \mathbf{I} & \mathbf{P}'_{2,2} \end{pmatrix} \right\rangle_{\mathbf{P}'_{2,2} \in \mathcal{P}_{M \times M}}}.$$

Special Case: Degree-2 Bethe Permanent for $n = 2$

For $M = 2$ we have

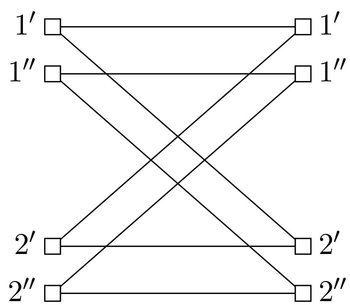
$$\mathrm{perm}_{\mathrm{B},2}(\boldsymbol{\theta}) \triangleq \sqrt[2]{\left\langle \mathrm{perm}\begin{pmatrix} \mathbf{I} & \mathbf{I} \\ \mathbf{I} & \mathbf{P}'_{2,2} \end{pmatrix} \right\rangle_{\mathbf{P}'_{2,2} \in \mathcal{P}_{2\times 2}}}$$

Special Case: Degree-2 Bethe Permanent for $n = 2$

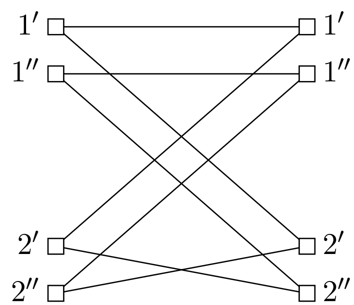
For $M = 2$ we have

$$\mathrm{perm}_{\mathrm{B},2}(\boldsymbol{\theta}) \triangleq \sqrt[2]{\left\langle \mathrm{perm}\begin{pmatrix} \mathbf{I} & \mathbf{I} \\ \mathbf{I} & \mathbf{P}'_{2,2} \end{pmatrix} \right\rangle_{\mathbf{P}'_{2,2} \in \mathcal{P}_{2\times 2}}}$$

corresponds to computing the average number of perfect matchings in the following 2-covers (and taking the 2nd root):



4



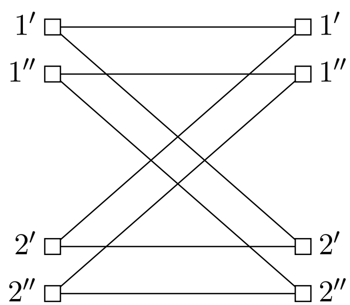
2

Special Case:
Degree-2 Bethe Permanent for $n = 2$

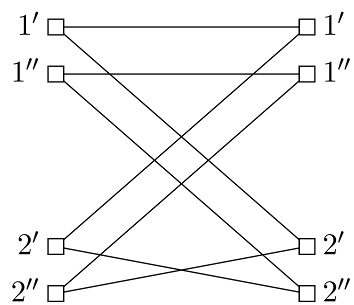
For $M = 2$ we have

$$\mathrm{perm}_{\mathrm{B},2}(\boldsymbol{\theta}) = \sqrt[2]{\frac{1}{2!} \cdot (4 + 2)}$$

corresponds to computing the average number of perfect matchings in the following 2-covers (and taking the 2nd root):



4



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Special Case:
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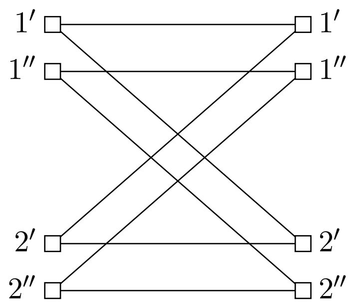
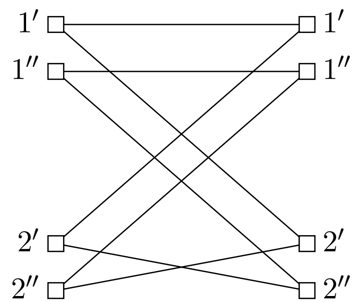
For $M = 2$ we have

$$
\begin{aligned}
\mathrm{perm}_{\mathrm{B},2}(\boldsymbol{\theta}) &= \sqrt[2]{\frac{1}{2!} \cdot (4+2)} \\
&= \sqrt[3]{\frac{1}{2!} \cdot 6} = \sqrt[2]{3} \approx 1.732
\end{aligned}
$$

corresponds to computing the average number of perfect matchings in the following 2-covers (and taking the 2nd root):



4



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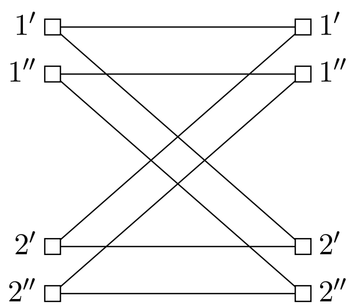
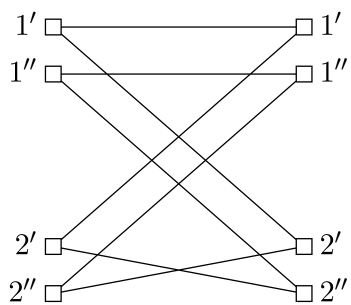


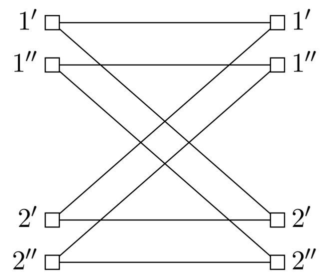
4



2

Special Case: Degree-2 Bethe Permanent for $n = 2$

Let us have a closer look at the perfect matchings in the graph



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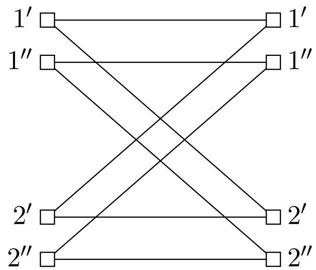
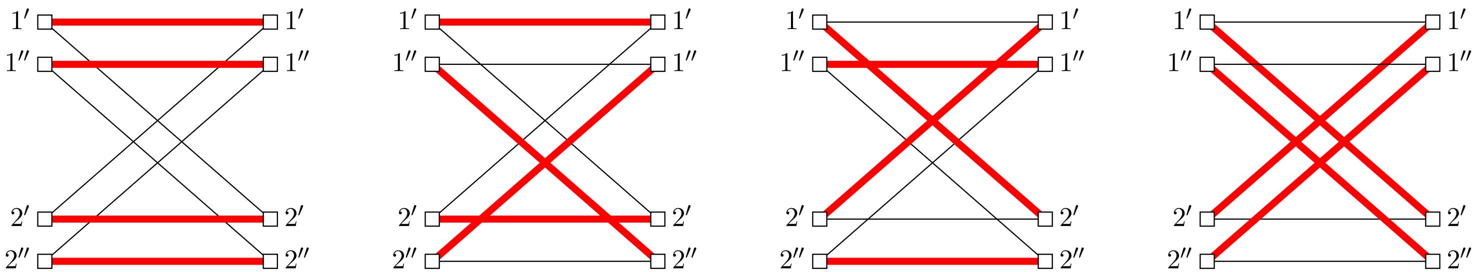


For this graph, the perfect matchings are



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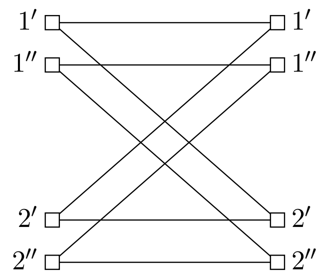

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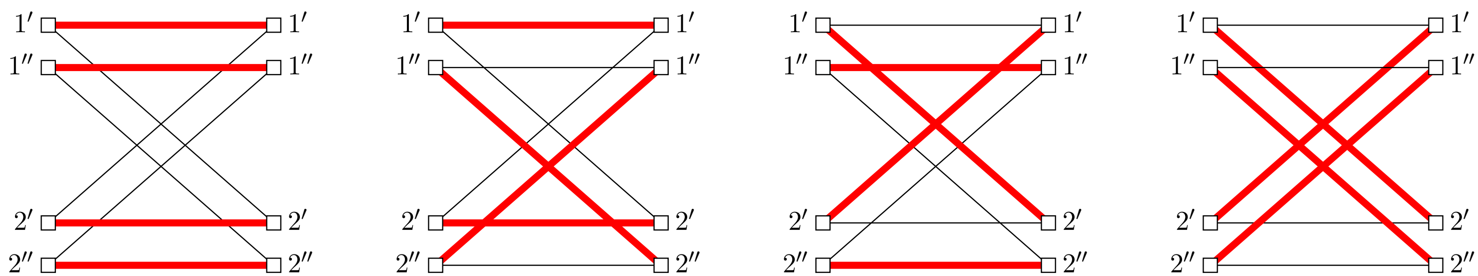

Because this double cover consists of two ***independent copies*** *of the base graph, the number of perfect matchings is* $2^2 = 4$.

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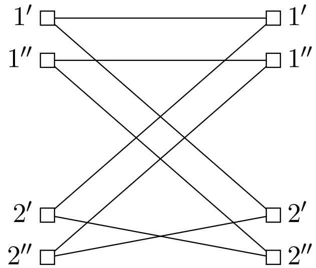

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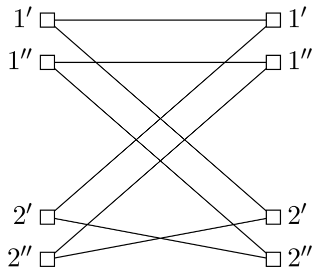

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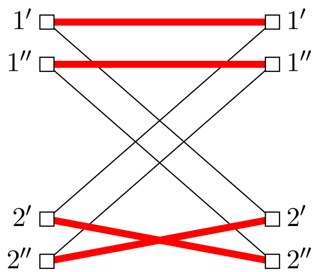

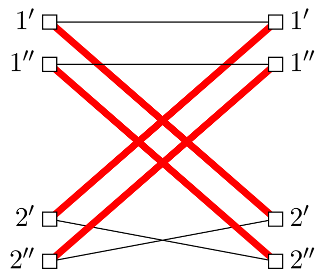

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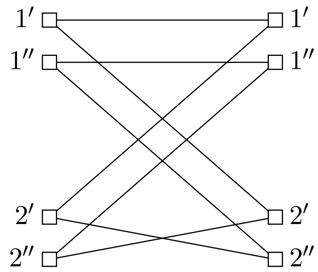

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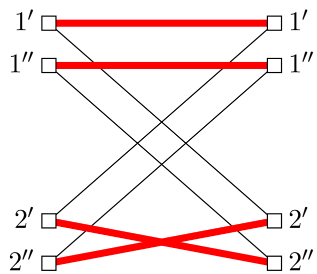
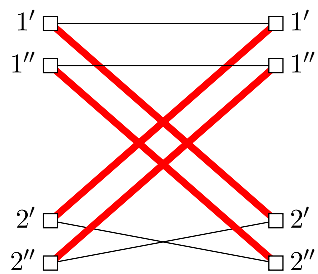




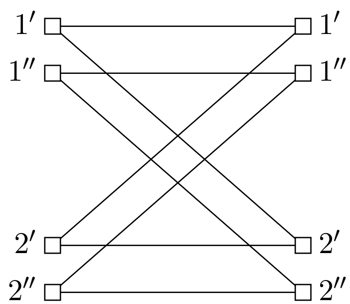
The ***coupling of the cycles*** *causes this graph to have fewer than 2^2 perfect matchings!*

Special Case:
Degree-2 Bethe Permanent for $n = 2$

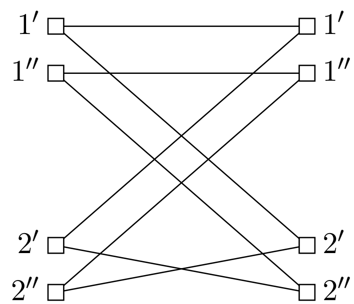
On the other hand, for $M = 2$ we have

$$\begin{aligned}\mathrm{perm}_{\mathrm{B},2}(\boldsymbol{\theta}) &= \sqrt[2]{\frac{1}{2!} \cdot (4+2)} \\ &= \sqrt[3]{\frac{1}{2!} \cdot 6} = \sqrt[2]{3} \approx 1.732 \; < \; \sqrt[2]{4} = 2 = \mathrm{perm}(\boldsymbol{\theta})\end{aligned}$$

corresponds to computing the average number of perfect matchings in the following 2-covers (and taking the 2nd root):



4



2

Special Case: Degree-3 Bethe Permanent for $n = 2$

On the other hand, for $M = 3$ we have

$$\mathrm{perm}_{\mathrm{B},3}(\boldsymbol{\theta}) \triangleq \sqrt[3]{\left\langle \mathrm{perm}\begin{pmatrix} \mathbf{I} & \mathbf{I} \\ \mathbf{I} & \mathbf{P}'_{2,2} \end{pmatrix} \right\rangle_{\mathbf{P}'_{2,2} \in \mathcal{P}_{3\times 3}}}$$

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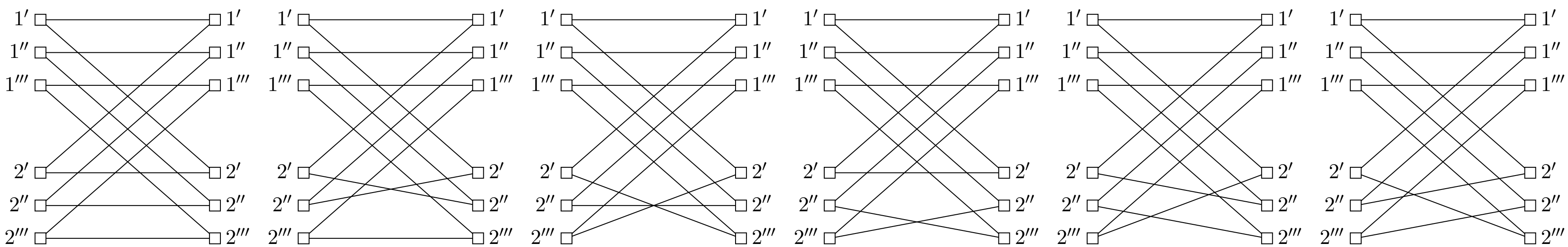
corresponds to computing the average number of perfect matchings in the following 3-covers (and taking the 3rd root):



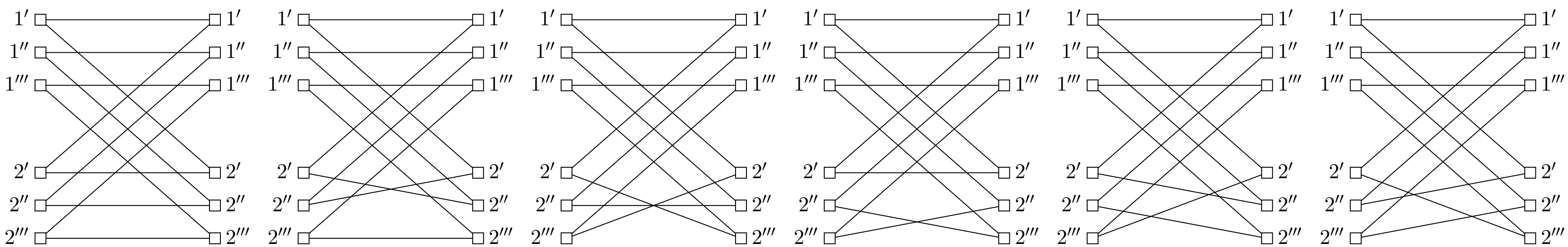
8 4 4 4 2 2

Special Case:
Degree-3 Bethe Permanent for $n = 2$

On the other hand, for $M = 3$ we have

$$\mathrm{perm}_{\mathrm{B},3}(\boldsymbol{\theta}) = \sqrt[3]{\frac{1}{3!} \cdot (8 + 4 + 4 + 4 + 2 + 2)}$$

corresponds to computing the average number of perfect matchings in the following 3-covers (and taking the 3rd root):



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On the other hand, for $M = 3$ we have

$$\begin{aligned}\mathrm{perm}_{\mathrm{B},3}(\boldsymbol{\theta}) &= \sqrt[3]{\frac{1}{3!} \cdot (8+4+4+4+2+2)} \\ &= \sqrt[3]{\frac{1}{3!} \cdot 24} = \sqrt[3]{4} \approx 1.587\end{aligned}$$

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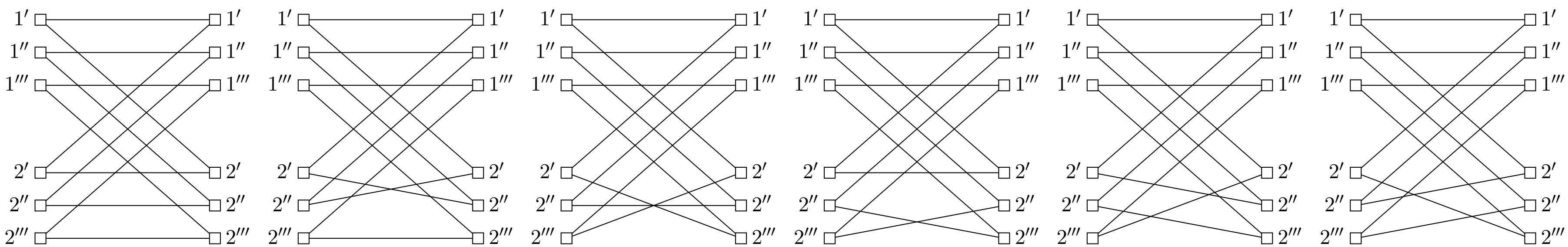

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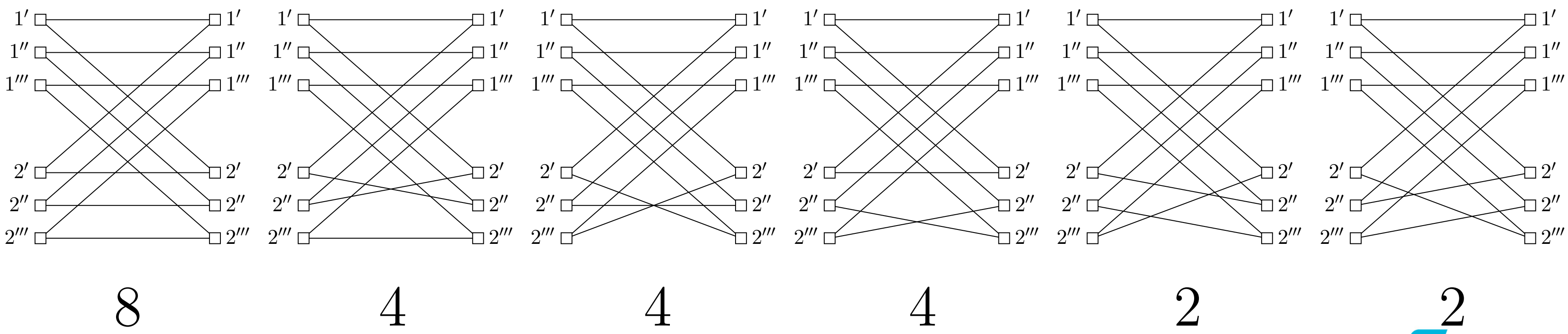
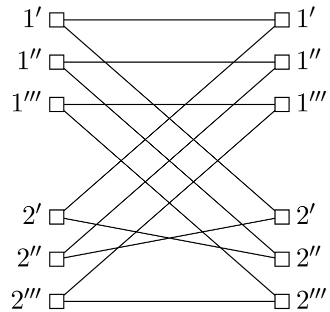


Special Case: Degree-3 Bethe Permanent for $n = 2$

Let us have a closer look at the perfect matchings in the graph



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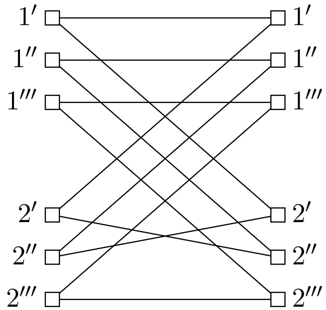
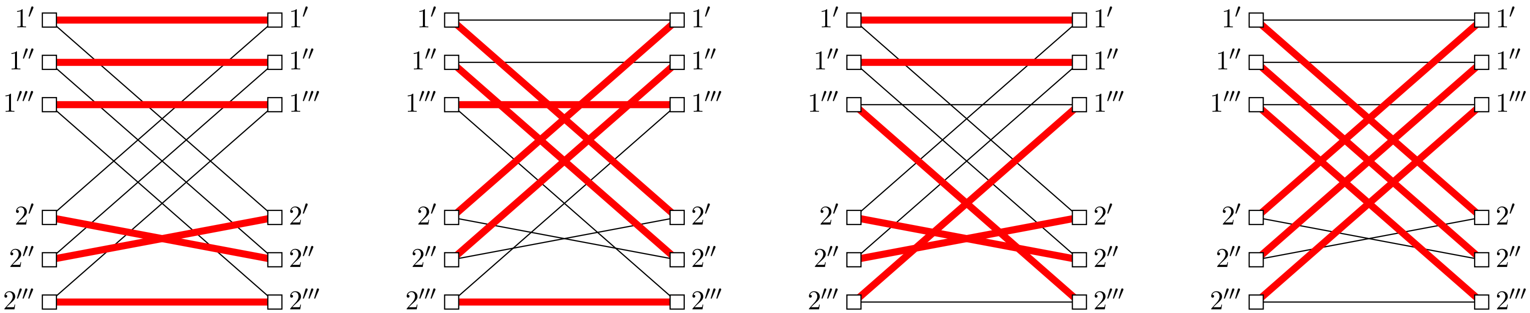


For this graph, the perfect matchings are



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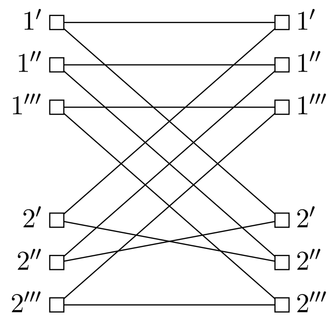

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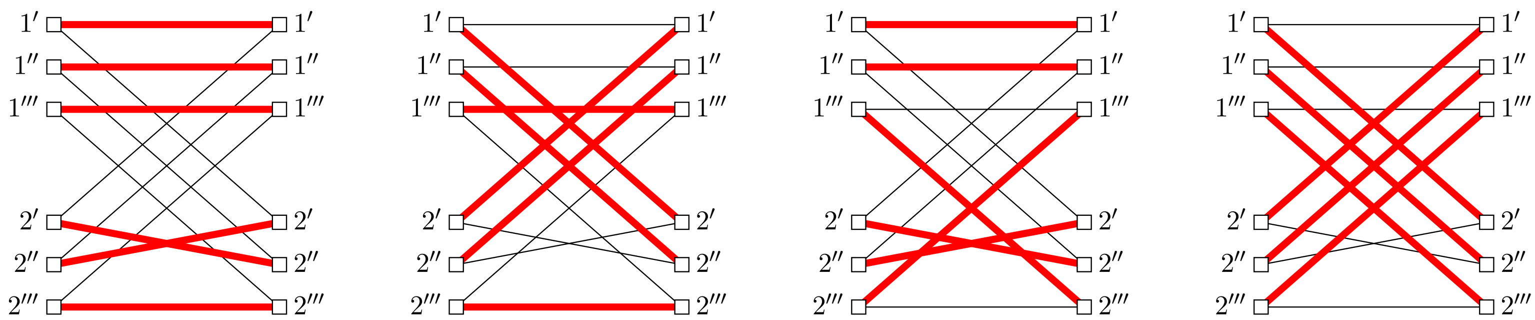


The ***coupling of the cycles*** *causes this graph to have fewer than* 2^3 *perfect matchings!*

Special Case: Deg.-M Bethe Permanent for $n = 2$

For general M we obtain

$$\mathrm{perm}_{\mathrm{B},M}(\boldsymbol{\theta}) = \sqrt[M]{\zeta_{S_M}} = \sqrt[M]{M+1},$$

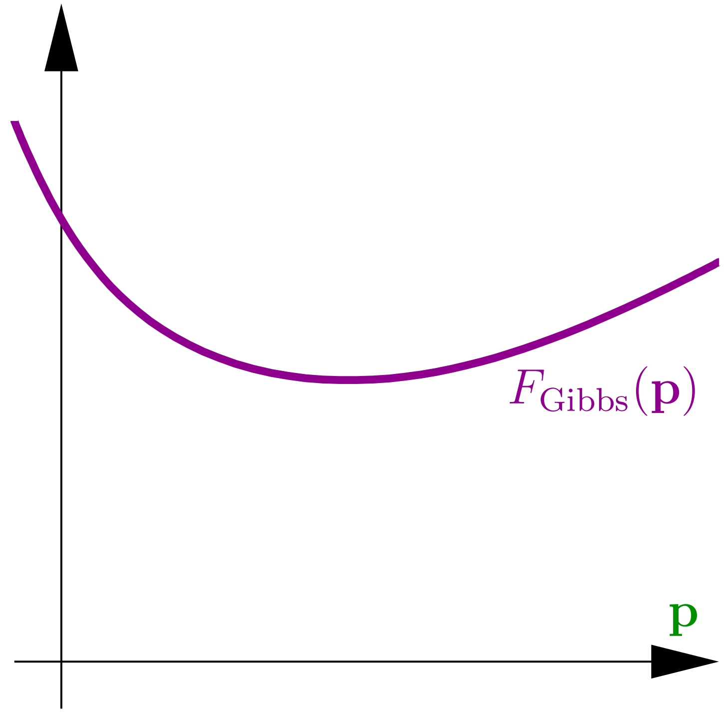
ζ_{S_M}: cycle index of the symmetric group over M elements.

The Gibbs free energy function

Gibbs Free Energy Function

The Gibbs free energy function

$$F_{\mathrm{Gibbs}}(\mathbf{p}) \triangleq -\sum_{\mathbf{a}} p_{\mathbf{a}} \cdot \log\big(g(\mathbf{a})\big) + \sum_{\mathbf{a}} p_{\mathbf{a}} \cdot \log(p_{\mathbf{a}}).$$



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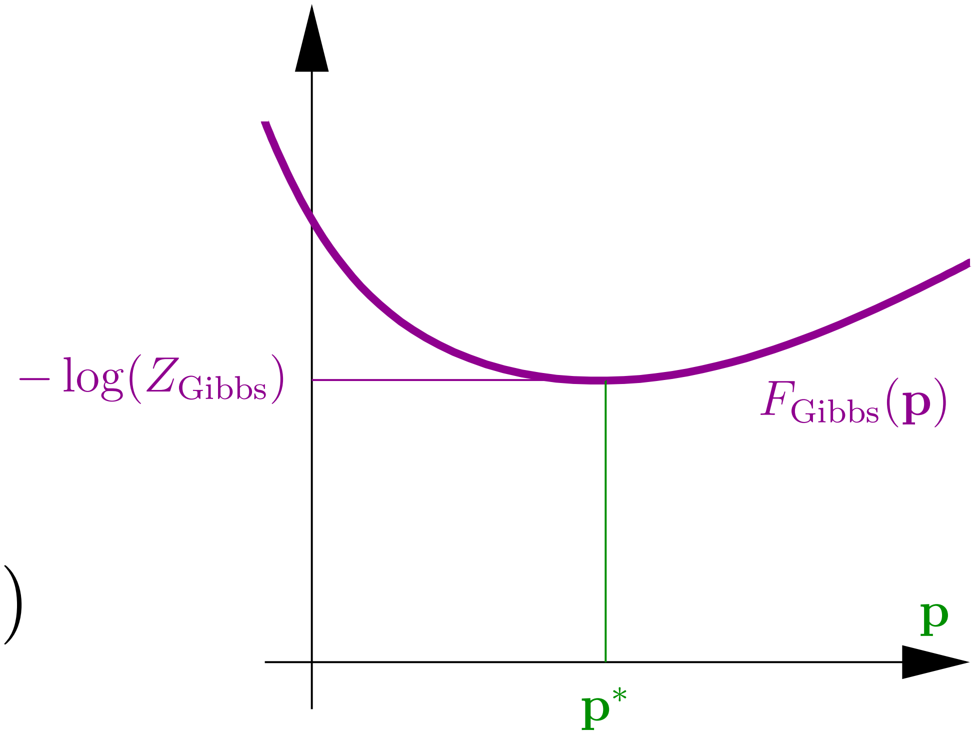

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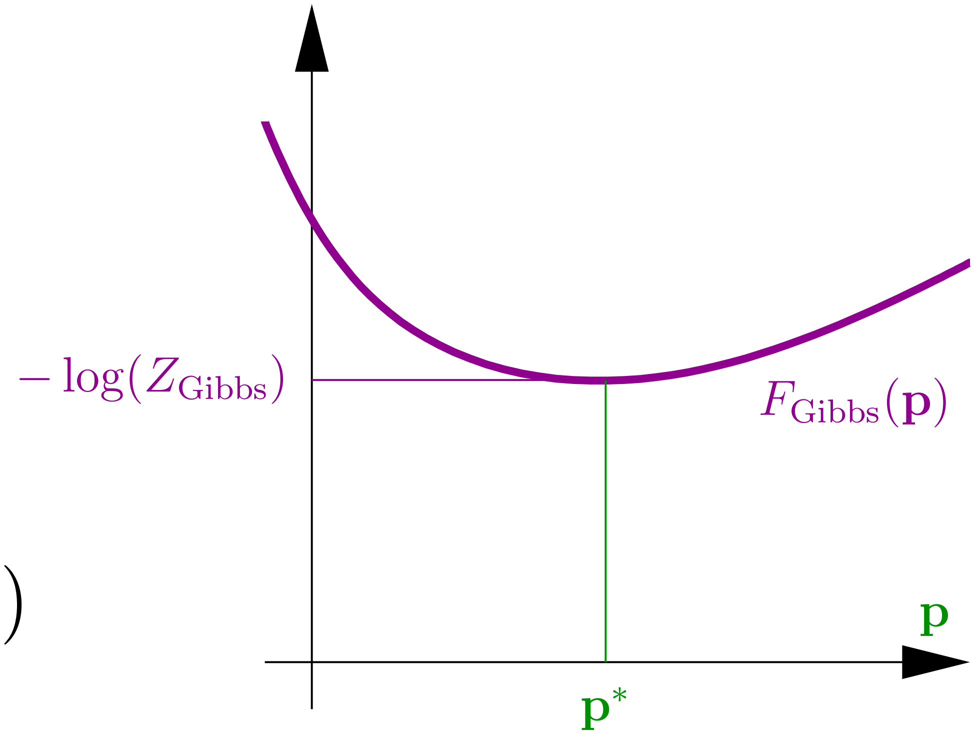

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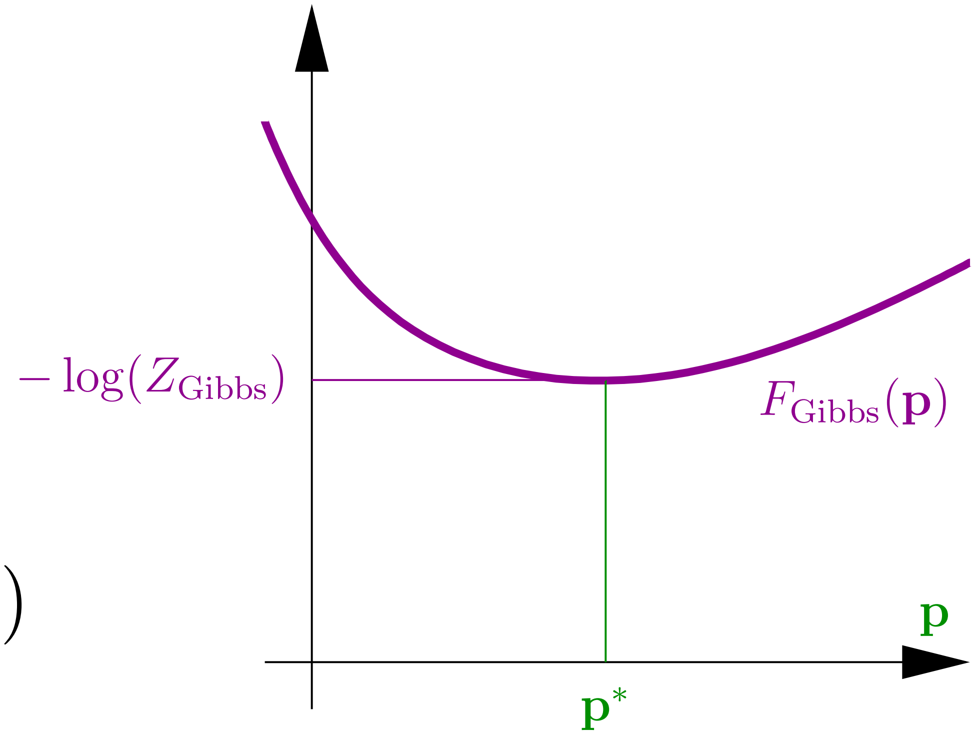
is defined such that its minimal value is related to the partition function:

$$Z = \exp\left(-\min_{\mathbf{p}} \ F_{\mathrm{Gibbs}}(\mathbf{p})\right).$$

Gibbs Free Energy Function



The Gibbs free energy function

$$F_{\mathrm{Gibbs}}(\mathbf{p}) \triangleq -\sum_{\mathbf{a}} p_{\mathbf{a}} \cdot \log\left(g(\mathbf{a})\right) + \sum_{\mathbf{a}} p_{\mathbf{a}} \cdot \log(p_{\mathbf{a}}).$$

is defined such that its minimal value is related to the partition function:

$$\mathrm{perm}(\boldsymbol{\theta}) = Z = \exp\left(-\min_{\mathbf{p}} \; F_{\mathrm{Gibbs}}(\mathbf{p})\right).$$

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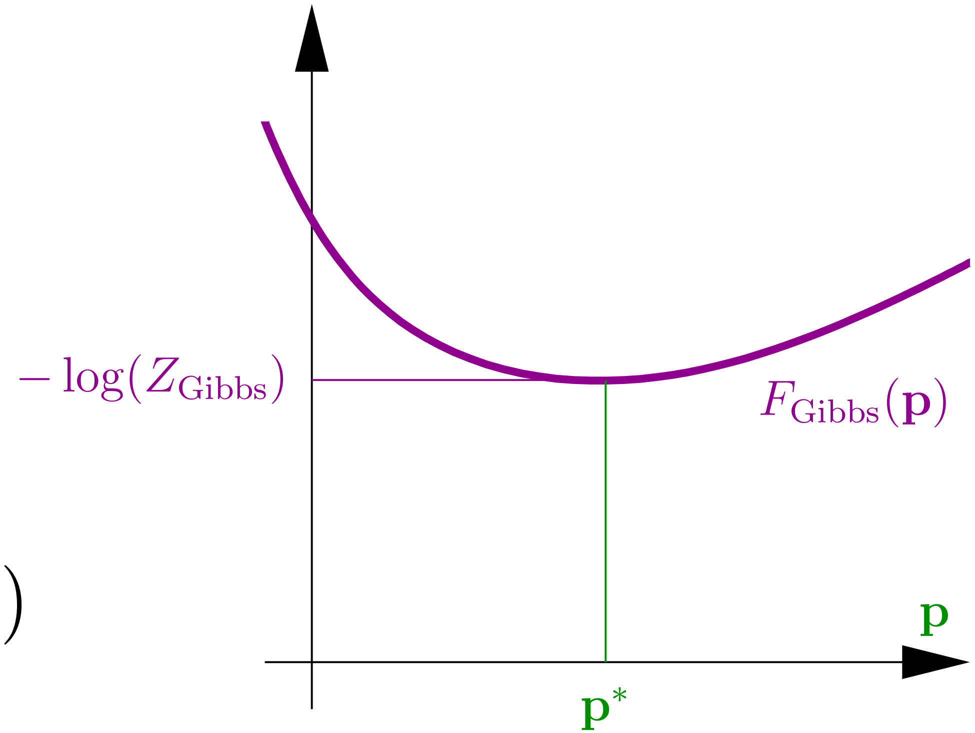

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Nice, but it does not yield any computational savings by itself.

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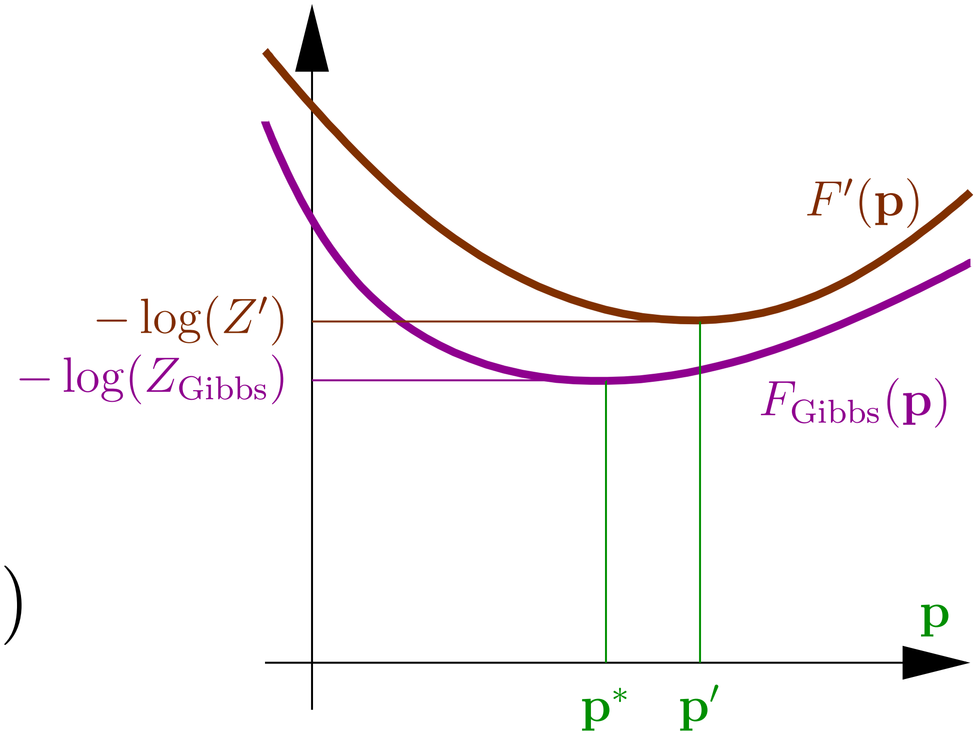

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But it suggests other optimization schemes.

The Bethe approximation

Bethe Approximation

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This approximation is interesting because of the following theorem:

Theorem (Yedidia/Freeman/Weiss, 2000):
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Fixed points of the sum-product algorithm (SPA) correspond to stationary points of the Bethe free energy function.

Definition: We define the Bethe permanent of $\boldsymbol{\theta}$ to be

$$\mathrm{perm}_{\mathrm{B}}(\boldsymbol{\theta}) = Z_{\mathrm{Bethe}} = \exp\left(- \min_{\boldsymbol{\beta}} \; F_{\mathrm{Bethe}}(\boldsymbol{\beta}) \right) .$$

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- The Bethe free energy function might have multiple local minima.
- It is unclear how close the (global) minimum of the Bethe free energy is to the minimum of the Gibbs free energy.
- It is unclear if the sum-product algorithm converges (even to a local minimum of the Bethe free energy).

Bethe Approximation

Luckily, in the case of the permanent approximation problem, the above-mentioned normal factor graph $\mathsf{N}(\boldsymbol{\theta})$ is such that the Bethe free energy function is very well behaved. In particular, one can show that:

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- The minimum of the Bethe free energy is quite close to the minimum of the Gibbs free energy. *(More details later.)*

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- The minimum of the Bethe free energy is quite close to the minimum of the Gibbs free energy. *(More details later.)*
- The sum-product algorithm converges to the minimum of the Bethe free energy. *(More details later.)*

Relationship between Permanent and Bethe Permanent

Theorem (Gurvits, 2011) $\quad$ **Conjecture (Gurvits, 2011)**

$$\mathrm{perm}_{\mathrm{B}}(\boldsymbol{\theta}) \;\underset{\text{Theorem (Gurvits, 2011)}}{\leq}\; \mathrm{perm}(\boldsymbol{\theta}) \;\underset{\text{Conjecture (Gurvits, 2011)}}{\leq}\; \sqrt{2}^{\,n} \cdot \mathrm{perm}_{\mathrm{B}}(\boldsymbol{\theta})$$

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This can be rewritten as follows:

Theorem **Conjecture**

$$\frac{1}{n} \log \mathrm{perm}_{\mathrm{B}}(\boldsymbol{\theta}) \;\overset{\downarrow}{\leq}\; \frac{1}{n} \log \mathrm{perm}(\boldsymbol{\theta}) \;\overset{\downarrow}{\leq}\; \frac{1}{n} \log \mathrm{perm}_{\mathrm{B}}(\boldsymbol{\theta}) + \log(\sqrt{2})$$

Relationship between Permanent and Bethe Permanent

Problem: find large classes of random matrices such that w.h.p.

Theorem (Gurvits, 2011)

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$$\frac{1}{n} \log \mathrm{perm}_{\mathrm{B}}(\boldsymbol{\theta}) \overset{\downarrow}{\leq} \frac{1}{n} \log \mathrm{perm}(\boldsymbol{\theta}) \leq \frac{1}{n} \log \mathrm{perm}_{\mathrm{B}}(\boldsymbol{\theta}) + O\left(\frac{1}{n} \log(n)\right)$$

Sum-Product Algorithm Convergence

Theorem: Modulo some minor technical conditions on the initial messages, the sum-product algorithm converges to the (global) minimum of the Bethe free energy function [V., 2010, 2011].

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Comment: the first part of the proof of the above theorem is very similar to the SPA convergence proof in

Bayati and Nair, "*A rigorous proof of the cavity method for counting matchings*," Allerton 2006.

Note that they consider matchings, not perfect matchings. (Although the perfect matching case can be seen as a limiting case of the matching setup, the convergence proof of the SPA is incomplete for that case.)

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- Inspired by the approaches mentioned in this talk, Ryuhei Mori recently showed that many replica method computations can be simplified and made quite a bit more intuitive.


Thank you!

References

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LABS hp