Kernel-Size Lower Bounds: The Evidence from Complexity Theory

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IAS

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Part 2/3

Note

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Outline

1. Introduction
2. OR/AND-conjectures and their use
3. Evidence for the conjectures

Outline

2. OR/AND-conjectures and their use

To be proved

Evidence for the OR, AND conjectures:

Theorem

Assume $\mathsf{NP} \not\subseteq \mathsf{coNP}/\mathsf{poly}$. *If* L *is* NP*-complete,* $t(k) \leq \mathrm{poly}(k)$,

1. *[Fortnow-Santhanam'08] No deterministic poly-time reduction* R *from* $\mathrm{OR}_{=}(L)^{t(\cdot)}$ *to any problem can have output size*

$$|R(\overline{x})| \leq O(t \log t) .$$

2. *[D.'12] No probabilistic poly-time reduction* R *from*

$$\mathrm{OR}_{=}(L)^{t(\cdot)} , \; \mathrm{AND}_{=}(L)^{t(\cdot)}$$

to any problem, with $\Pr[\text{success}] \geq .99]$, *can achieve*

$$|R(\overline{x})| \leq t .$$

Background

- Let's back up and discuss:
- What does $\mathsf{NP} \not\subseteq \mathsf{coNP}/\mathsf{poly}$ mean?
- Why believe it?

Background: circuits

- We use ordinary model of Boolean circuits: $\wedge, \vee, \neg$ gates, bounded fanin.

- Say that decision problem L has poly-size circuits, and write $L \in \mathsf{P}/\mathsf{poly}$, if

$$\exists \ \{ \ C_n : \{0,1\}^n \rightarrow \{0,1\} \ \}_{n>0} \ :$$

$$\text{size}(C_n) \ \leq \ \mathsf{poly}(n), \quad C_n(x) \ \equiv \ L(x) \ .$$

- Non-uniform complexity class: def'n of C_n may depend uncomputably on n!

- Example: if $L \subseteq 1^*$, then $L \in \mathsf{P}/\mathsf{poly}$. Also, $\mathsf{BPP} \subset \mathsf{P}/\mathsf{poly}$.

Background: nondeterminism

Recall: decision problem L is in NP if:

$\exists$ poly-time algorithm $A(x, y)$ on $n + \text{poly}(n)$ input bits :

$$x \in L \iff \exists\, y : \; A(x, y) \; = 1 \, .$$

Background: nondeterminism

Say that decision problem L is in $\mathsf{NP}/\mathsf{poly}$ if:

$\exists$ poly-sized ckts $\{C_n(x, y)\}_n$ on $n + \mathrm{poly}(n)$ input bits :

$$x \in L_n \iff \exists\, y : \; C_n(x, y) = 1 \, .$$

- “Non-uniform NP”

Background: nondeterminism

- Recall that $\mathsf{coNP} = \{L : \overline{L} \in \mathsf{NP}\}$.
- Complete problem: $\mathsf{UNSAT} = \{\psi : \psi \text{ is unsatisfiable}\}$.
- $\mathsf{coNP/poly} = \{L : \overline{L} \in \mathsf{NP/poly}\}$.

Uniform and nonuniform complexity

- Connect questions about non-uniform computation to uniform questions?
- Yes!
- Need a broader view of nondeterminism...

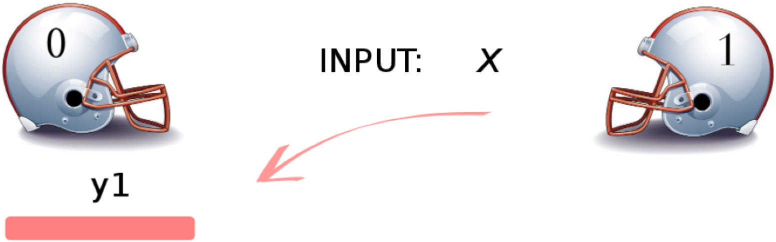
Games and computation

- Given a circuit $C(y^1, y^2, \ldots, y^k)$ with k input blocks, consider 2-player game where Player 1 wants $C \to 1$, P0 wants $C \to 0$.
- Take turns setting $y^1, \ldots, y^k$.

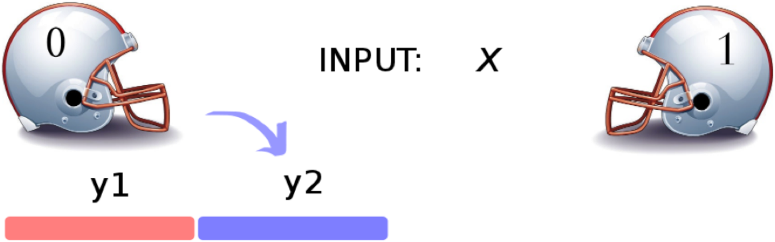
Games and computation

INPUT: x

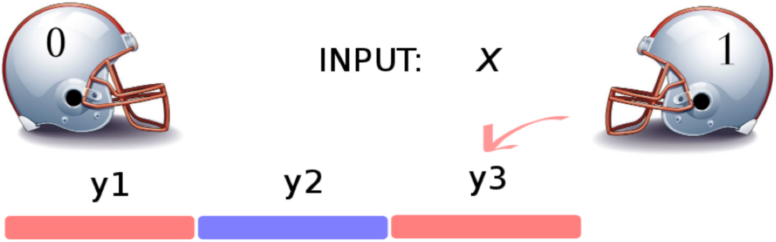
Games and computation



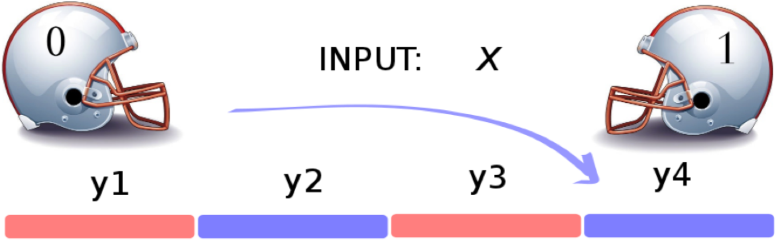
Games and computation



Games and computation



Games and computation



Games and computation

Define d-ROUND GAME ($\exists$) as:

- Input: a d-block circuit $C(y^1, \ldots, y^d)$.
- Decide: on 2-player game where P1 goes first, can P1 force a win? ($C = 1$)

Define complexity class

$$\Sigma_d^p$$

as set of languages poly-time (Karp)-reducible to d-ROUND GAME ($\exists$).

"d^{th} level of Polynomial Hierarchy"

Games and computation

- **Facts:** $\mathrm{NP} = \Sigma_1^p$ ("solitaire"); $\Sigma_d^p \subseteq \Sigma_{d+1}^p$.
- **Common conjecture:** for all $d > 0$, $\Sigma_d^p \neq \Sigma_{d+1}^p$.
- Otherwise we could efficiently reduce a $(d+1)$-round game to an equivalent d-round one, and

how the heck do you do that??

Games and computation

- Games allow us to connect uniform and non-uniform complexity questions:

Theorem (Karp-Lipton '82)

Suppose NP *is in* P/poly.
Then, for all $d > 2$,

$$\Sigma_d^p = \Sigma_2^p .$$

Games and computation

- Games allow us to connect uniform and non-uniform complexity questions:

Theorem (Yap '83)

Suppose NP *is in* coNP/poly.
Then, for all $d > 3$,

$$\Sigma_d^p = \Sigma_3^p .$$

So: the assumption

$$\mathrm{NP} \not\subseteq \mathrm{coNP/poly}$$

can be based on an (easy-to-state, likely) assumption:

"One cannot efficiently reduce a 100-round game
to an equivalent 3-round game!"

The minimax theorem

- An extremely useful tool.
- Many applications in complexity theory, beginning with [Yao'77].
- Gives alternate (but similar) proof of [Fortnow-Santhanam'08] result;
- seems crucial for best results in [D.'12].

The minimax theorem

- **Setting:** a 2-player, simultaneous-move, zero-sum game.
- Players 1, 2 have finite sets X, Y. **("possible moves")**
- **"Payoff function"** $\mathrm{Val}(x, y) : X \times Y \to [0, 1]$.
- $\mathrm{Val}(x, y)$ defines **"payoff from P1 to P2,"** given moves (x, y).
- (P1 trying to minimize $\mathrm{Val}(x, y)$, P2 trying to maximize)

The minimax theorem

- **Mixed strategy for P1:** A distribution $\mathcal{D}_X$ over X.
- (Mixed strategies can be useful...)
- Minimax thm says: for P1 to do well against **all** P2 strategies...

  it's enough if P1 can do well against **any fixed** mixed strategy.

The minimax theorem

Theorem (Minimax—Von Neumann)

Suppose that for every mixed strategy $\mathcal{D}_Y$ *for P2, there is a P1 move* $x \in X$ *such that*

$$\mathbb{E}_{\mathbf{y} \sim \mathcal{D}_Y} \left[\mathrm{Val}(x, \mathbf{y}) \right] \ \leq \ \alpha \ .$$

Then, there is a P1 mixed strategy $\mathcal{D}_X^*$ *such that, for all P2 moves* y*,*

$$\mathbb{E}_{\mathbf{x} \sim \mathcal{D}_X^*} \left[\mathrm{Val}(\mathbf{x}, y) \right] \ \leq \ \alpha \ .$$

- Follows from LP duality theorem.

Back to business

- Time to apply these tools.
- Let's restate the [Fortnow-Santhanam'08] result.
- Will switch from k's to n's...

Back to business

Theorem (FS'08, restated)

Let L be an NP-complete language, L' another language, and $t(n) \leq \mathrm{poly}(n)$.
Suppose there is a poly-time reduction

$$R(\overline{x}) \;=\; R(x^1, \ldots, x^{t(n)})$$

taking $t(n)$ inputs of length n, and producing output such that

$$R(\overline{x}) \;\in L' \quad \Longleftrightarrow \quad \bigvee_j \left[x^j \in L\right] .$$

Suppose too we have the output-size bound

$$|R(\overline{x})| \;\leq\; O(t(n) \log t(n)) .$$

Then, $\mathsf{NP} \subseteq \mathsf{coNP}/\mathrm{poly}$.

Simplifying

To ease discussion:

- Assume $L' = L$;
- Fix $t(n) = n^{10}$;
- Assume
$$\left| R\left(x^1, \ldots, x^{n^{10}}\right) \right| \equiv n^3 .$$

(No more ideas needed for general case!)

Proof strategy

- Recall L is NP-complete. To prove theorem, enough to show that

$$L \in \mathsf{coNP/poly}\ , \qquad \text{i.e.,} \qquad \overline{L} \in \mathsf{NP/poly}\ .$$

- Thus, want to use R to build a non-uniform proof system witnessing membership in $\overline{L}$.

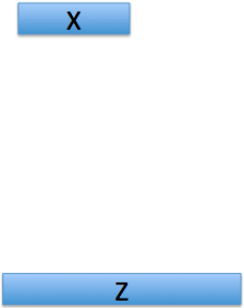
Shadows

- For $x \in \{0,1\}^n$, say that $\overline{x} = (x^1, \ldots, x^{n^{10}})$ contains x if x occurs as one of the x^j's.

- Define the shadow of $x \in \{0,1\}^n$ by

$$\text{shadow}(x) \;:=\; \{z = R(\overline{x}) : \overline{x} \;\text{ contains }\; x\} \qquad \subseteq \; \{0,1\}^{n^3} \;.$$

Shadows



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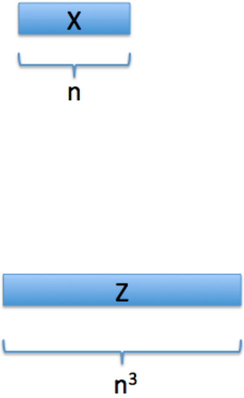

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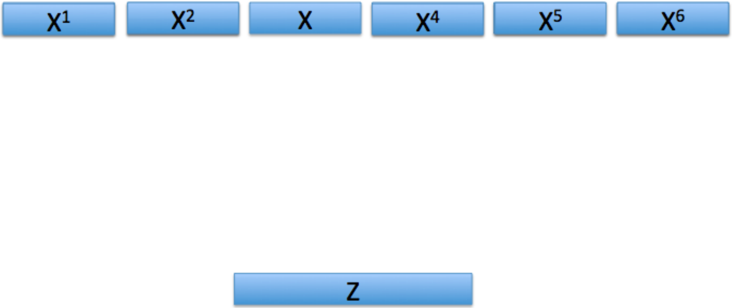

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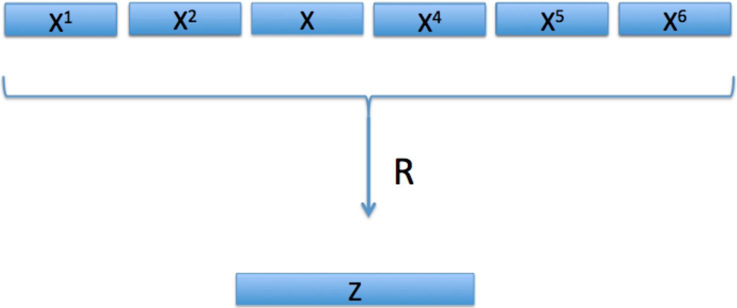

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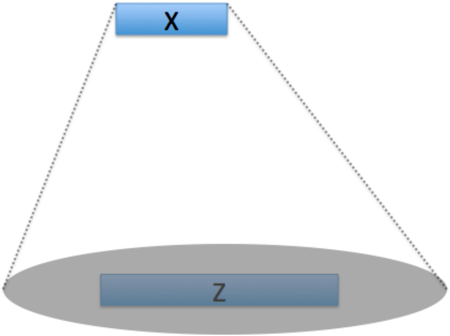

Shadows

- **Fact:** if some $z \notin L$ is in the shadow of x, then $x \notin L$.

  (by OR-property of R...)

- This is our **basic form of evidence** for membership in $\overline{L}$!

  (z will be non-uniform advice...)

The main claim

Claim (FS '08)

There exists a set $Z \subseteq \overline{L}_{n^3}$*, with*

$$|Z| \leq \text{poly}(n) ,$$

such that for every $x \in \overline{L}_n$,

$$\text{shadow}(x) \cap Z \neq \emptyset .$$

Intuition: the massive compression by $R \implies$ some z is the image of many sequences $\overline{x}$, hence is in many shadows. Can collect these "popular" z's to hit all shadows (of $\overline{L}_n$).

Claim easily implies $\overline{L} \in \text{NP}/\text{poly}$... take Z as non-uniform advice.

Shadows

To prove $x \in \overline{L}$...

Shadows

To prove $x \in \overline{L}$... nondeterministically choose $\overline{x} \supset x$ and $z \in Z$, and check:

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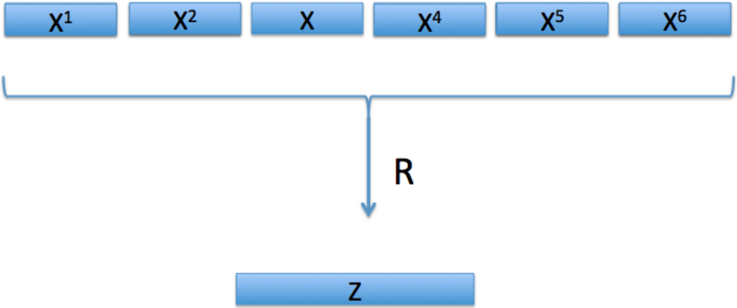

Conclusion: $x \in \overline{L}$.

The main claim

Claim (FS '08)

There exists a set $Z \subseteq \overline{L}_{n^3}$*, with*

$$|Z| \leq \text{poly}(n)\ ,$$

such that for every $x \in \overline{L}_n$,

$$\text{shadow}(x) \cap Z \neq \emptyset\ .$$

Proof by game

To prove Claim, consider the following simul-move game between P1 ("Maker") and P2 ("Breaker"):

Game

- **P1:** *chooses* $z \in \overline{L}_{n^3}$;
- **P2:** *chooses* $x \in \overline{L}_n$;
- **Payoff to P2:** $\mathrm{Val}(x, z) = 1$ *if* $z \notin \mathrm{shadow}(x)$, *otherwise* 0.

Lemma

There is a P1 strategy (dist'n $\mathcal{D}^*$ over $\overline{L}_{n^3}$) such that for any x,

$$\mathbb{E}_{\mathbf{z} \sim \mathcal{D}^*}[\mathrm{Val}(\mathbf{z}, x)] \ \leq \ o(1) \ .$$

Our Claim follows easily, with $|Z| = O(n)$. (Repeated sampling!)

Proof by game

Lemma

There is a P1 strategy (dist'n D^* over $\overline{L}_{n^3}$) such that for any x,

$$\mathbb{E}_{\mathbf{z}\sim\mathcal{D}^*}[\mathrm{Val}(\mathbf{z}, x)] \;\leq\; o(1)\ .$$

- By minimax theorem, it's enough to beat any **fixed** P2 mixed strategy

$$\mathcal{D}_n \qquad \text{(dist'n over } \overline{L}_n \text{)} \quad .$$

- **Idea:** use P1 strategy induced by outputs of R on inputs from $\mathcal{D}_n$...

Proof by game

- Say that $z \in \overline{L}_{n^3}$ is bad, if

$$\Pr_{\mathbf{x} \sim \mathcal{D}_n} [z \in \text{shadow}(\mathbf{x})] \leq 1 - 1/n \ .$$

- We've beaten strategy $\mathcal{D}_n$ if some z is not bad.

Proof by game

- Let $\mathcal{D}_n^{\otimes t}$ denote t ind. copies of $\mathcal{D}_n$.
- Define dist'n $\mathcal{R}$ by

$$\mathcal{R} = R\left(\mathcal{D}_n^{\otimes n^{10}}\right) .$$

- We claim that

$$\Pr_{\mathbf{z}\sim\mathcal{R}} [\mathbf{z} \text{ is bad }] = o(1) .$$

Proof by game

- Let $\overline{x} = (\mathbf{x}^1, \ldots, \mathbf{x}^{n^{10}}) \sim \mathcal{D}_n^{\otimes n^{10}}$, and
$$\mathbf{z} \;=\; R(\overline{x}) \;.$$

- Consider any bad z. For $[\mathbf{z} = z]$, we must have
$$\mathbf{x}^j \;\in\; \text{shadow}(z) \qquad \forall j,$$
with happens with probability
$$\leq\; (1 - 1/n)^{n^{10}} \;< 2^{-n^9}.$$

- Union bound over all bad z completes proof:
$$\Pr[\mathbf{z} \text{ is bad}] \;\leq\; \frac{2^{n^3}}{2^{n^9}} \;= o(1) \;.$$

$\square$

Summary

- If
$$R(\overline{x}) : \{0,1\}^{n \times n^{10}} \rightarrow \{0,1\}^{n^3}$$
is a compressive mapping with the "OR-property" for L, then there are $z \in \overline{L}_{n^3}$ lying in the "shadow" of many $x \in \overline{L}_n$.

- We collect a small set Z of these non-uniformly, use it to prove membership in $\overline{L}_n$.
- Note: nondeterminism still required to verify membership in $\overline{L}_n$: we have to guess extensions $x \rightarrow \overline{x}$ which map to z!

- Minimax theorem made our job easier.

Summary

- Fortnow-Santhanam technique applies to randomized algorithms avoiding false negatives. Also to co-nondeterministic alg's (observed in [DvM '10]).
- [FS '08] also used their tools to rule out "succinct PCPs" for NP...
- Left open: evidence again two-sided error OR-compression; any strong evidence against AND-compression.
- poly(k)-kernelizability of problems like k-Treewidth left open...

Side note: a mystery

- Fortnow-Santhanam prove we cannot efficiently compress $\mathrm{OR}_{=}(L)_n^{t(n)}$ instances to size $O(t(n)\log t(n))$.
- Input size is $t(n) \cdot n$.
- There is still a big gap here; consequences of compression to size $O(t(n) \cdot \sqrt{n})$? If, e.g., $L = \mathrm{SAT}$?
- Same issue with [D'12] bounds...