DDD: Density Distribution Determination

VISION FLASH 36

by

B.K.P. Horn

Massachusetts Institute of Technology

Artificial Intelligence Laboratory

Robotics Section

8 MAR 1973

Abstract

This paper presents a solution to the problem of determining the distribution of an absorbing substance inside a non-opaque non-scattering body from images or ray samplings. It simultaneously solves the problem of determining the distribution of emitting substance in a transparent non-scattering medium. The relation to more common vision problems is discussed.

This is largely a cleaned up version of a solution found sometime ago when two other related problems were of interest. The one is the special situation when the density can have only two values, which has been solved for special cases by J. Kloustad. The other is the problem of shape determination from silhouettes, that is when the density is infinite in a simple region.

Work reported herein was conducted at the Artificial Intelligence Laboratory, a Massachusetts Institute of Technology research program supported in part by the Advanced Research Projects Agency of the department of Defense and monitored by the Office of Naval Research under Contract Number N00014-70-A-0362-0003.

TABLE OF CONTENTS:

APPENDICES:

OUR NORMAL VISION WORLD:

On the usual scale the visual world consists largely of opaque, non-emissive, cohesive objects. At times they may even be smooth or convex. The opaqueness causes surfaces to have special significance and many descriptors we have for objects relate to their surfaces only. Such properties as color, matt reflectivity and tactile texture require a surface layer of finite thickness. The details of this layer are beyond our resolution however and we summarize its behavior using colour, texture and translucency descriptors.

Given that the visual world consists of separate objects or that objects are non-convex and have protrusions we turn our attention to edges. Most will be occlusion edges. Some, particularly on man-made objects will correspond to discontinuities of the surface derivatives. Edges of course, need not be straight. Where edges meet we have vertices. Most of these will be T-joints with a few X-joints where a shadow edge crosses some other edge. Interior edges on some objects will produce Y-joints. This establishes the importance of edges and vertices in understanding the visual world.

We can even say some things about the most common lighting situations and surface properties. In many cases we have a point source and an additional more uniform illumination. The ratio of these two intensities may vary widely. Surfaces usually have both a specular reflectivity and some matte reflectivity. Again either may be the predominant feature in a given situation.

All of this knowledge and particularly the opaqueness of most objects makes it reasonable for an organism to attempt to build an internal model of the world using mere two-dimensional images. A dozen or so depth-cues belong in its arsenal of image processing heuristics to help it recover the three-dimensional information. The study of these heuristics is the subject matter of much of the research on vision.

SOME NOT QUITE SO NATURAL VISUAL WORLDS:

As was mentioned, what we consider surface layer depends on the resolution of the image. The visual quality of objects then will change with magnification. At high magnifications for example, we will be seeing what would normally be called the surface layer. Since most non-metallic objects become almost transparent at high magnification we are now faced by a quite different image analysis problem and find it hard to interpret what we see.

Another example can be found in planetary astronomy. When looking at the moon or one of the planets the surface detail which is too fine to be resolved may be of the order of large rocks or even mountains. This then constitutes the surface layer giving rise to such properties as reflectivity. We cannot then expect the reflectivity function to be similar to that applying to pebbles. And indeed the moon, for example, has a most unusual property which causes it to reflect uniformly over the whole face when illuminated straight on and so appear flat to organisms more accustomed to pebbles.

Finally, there is an example of a non-optical device which produces images that we find easy to interpret. This is the scanning electron microscope which samples a very thin surface layer with a beam of electrons. Typical objects are much more opaque to electrons than photons so that we can go to much higher magnification before the disturbing effects of translucency and transparency set in. This device produces images which have no shadows, corresponding to pictures taken with the light source at the image sensing device. The shading is dramatic and provides the needed depth-cue. It is for this reason that this device has become more popular than the transmission electron microscope even though the latter provides higher magnification.

IMAGES IN A NON-OPAQUE DOMAIN:

Some things of course are not opaque. Such objects present new problems. We can deal with special cases as mere variants of opaque objects. This goes for transparent objects and those with translucent surface layers, people for example. In other cases we consider almost opaque objects embedded in an almost transparent medium. Looking at the blood corpuscles in the capillaries of a small fish falls into this category. Our normal visual world is not very different (since we are embedded in a near-transparent medium). Such heuristics not only solve the perceptual problem but also that of internal representation.

Harder problems arise if we are interested in the distribution of some absorbing substance in a non-opaque object. Since we are no longer interested in mere two-dimensional surfaces but three-dimensional distributions we can no longer hope to extract all the information required using only one or two two-dimensional images. In addition, we now lack adequate representation techniques. Perhaps the simplest approach to this problem is to slice the object and take an image of each section. The series of images constitutes the lacking extra dimension. This is a technique often used with the transmission electron microscope and optical microscopes at high magnification. Of course, this technique may not always be acceptable for the object may be a galactic nebula or your liver.

HOW SUCH IMAGES ARISE:

The problem of determining the distribution of some substance inside a non-opaque object arises in many imaging devices used in research. We have already mentioned high magnification optical microscopes and transmission electron microscopes. Other examples are found in X-ray images and X-ray diffraction patterns. Say for example you wanted to determine the electron density in some giant molecule to prove that it has a helical structure.

It has to be said that in many practical cases the full reconstruction of absorbing substance density distribution is not undertaken because of the paucity of processing techniques and the need for many images if such techniques are to be employed. If one can assume that the density has only a few discrete values adequate guesses can be made from one or two images. Bone and tissue for example, have vastly different X-ray absorbing properties and a single X-ray can give some useful information.

So far we have been assuming a varying density of absorbing substance inside an object being sampled by rays of some kind, not necessarily electro-magnetic. A different problem turns out to have the same solution. This is the case of a variable density of emitting substance, such as a radio-active isotope. In the one case we can determine the total amount of absorbing substance that a single ray passes through when traversing the object. In the other case, a point on the image tells us the total amount of emitting substance along a particular line through the object. An example of the latter is found in isotope scans made to determine which tissues absorb a given substance. More familiar examples are a flame, emission nebulae and the solar corona in Mg^{X} light.

THE GENERAL RECONSTRUCTION PROBLEM:

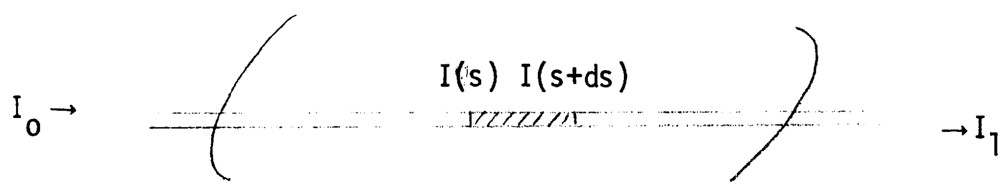
The problem we are interested in here is a special case of the problem of reconstructing the internal details of an object from information obtained from some process that allows us to sample its interior. The sampling energy may be electrical , electro-magnetic, acoustic or seismic for example. The information obtained is usually some complex function of the physical processes involved and the internal details. In the simplest cases it may be a convolution of the quantity we are interested in. In our normal visual world we are fortunate that the rays of light reflected from the surfaces of objects are relatively unmolested on their way to our eyes and a given point in the image corresponds to a definite point on some object. In this more general problem each measurement reflects the effect of many internal details. If we are lucky only those along a well defined ray are involved. If scattering is present the image analysis problem is very hard and even for the simple case of thin layers has only approximate solutions.

In our special case we assume that no scattering takes place and can in fact show that the information at one image point corresponds to internal details along a ray only. A simple transformation takes this raw data into a form which is a convolution of the density we are looking for and a spreading function. This spreading function has nothing to do with limited resolution, but instead is determined by the geometry and physics of absorption. The appearance of the word convolution immediately tells us that problems of this form are amenable to solution using the bag of tricks in linear systems theory provided we extend it to more than one dimension. We will need to know about fourier transforms and deconvolutions to do this.

We will not here touch on the problem of representing the information extracted in other than numerical terms.

A SINGLE RAY PASSING THROUGH THE OBJECT:

Our problem is to determine the distribution of absorbing substance in an object. We have available collimated rays of suitable radiation and a device for measuring their intensity. First consider a single ray traversing the object. Let the distance along the ray be s and let f(s) be the density of absorbing substance. The rate of decrease in intensity along the beam at any point is proportional to the intensity and the density of absorbing substance. You can see this by considering an infinitesimal segment along the ray.



$$I'(s) = - I(s) f(s)$$

$$\frac{I'(s)}{I(s)} = - f(s)$$

Suppose the ray enters the object at $s=s_0$ with intensity I_0 and leaves it at $s=s_1$ with intensity I_1. Then we can integrate the above equation.

$$\log I_0 - \log I_1 = \int_{s_0}^{s_1} f(s)\, ds$$

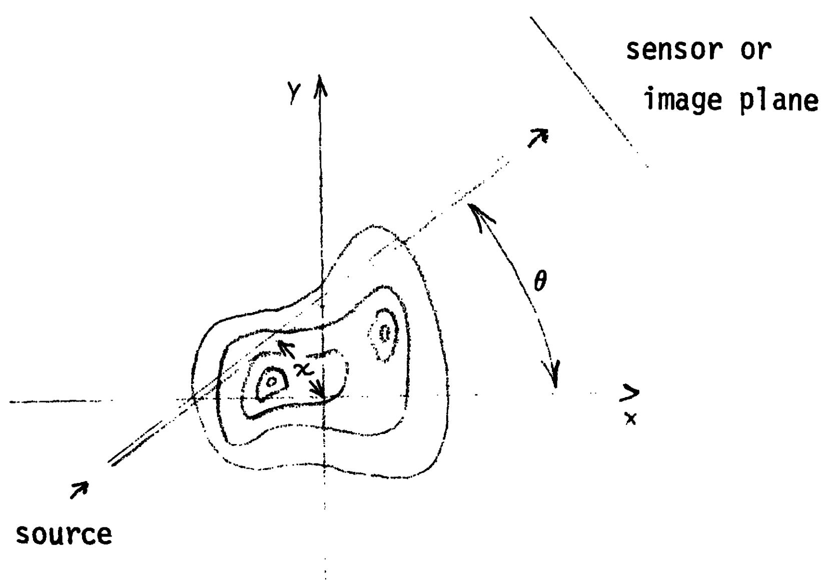
So by taking logs of the ratio of intensities entering and leaving, we determine the total amount of absorbing substance traversed by the ray. Note that we are in trouble if f(s) is infinite anywhere, that is if part of the object is truly opaque.

A similar argument can be made for the case of emitting substance except that we do not take logs, but obtain the total amount of emitting substance along a ray directly.

THIN SLICES, THE TWO-DIMENSIONAL PROBLEM:

We'll next consider thin slices through the object to get an idea of the reconstruction operations involved. We do not however look down on the slice and obtain the density distribution directly, as we would if we were using the usual transmission electron microscope technique. Instead we look along the slice. This may seem counter-productive but it is done for a good reason. Firstly the slice may be only a conceptual device for thinking about a part of an object. In other words if we can understand a slice by looking along it, we can understand complete objects by thinking of them as made up of such slices. Secondly solving the two-dimensional case first gives us insight into the intuitively less obvious three-dimensional case.

We now consider all possible ways of sampling the slice without leaving the plane in which it lies. We have two degrees of freedom in choosing the sampling rays. You may think of this as taking an image for each possible angular orientation of the slice. Each such image is a line of intensities. We define in this way a function $h(\chi,\theta)$ which is the total absorbing density along a ray at angle θ and distance χ from the origin of some rectangular coordinate system.



COLLECTING ALL THE DATA RELATING TO ONE POINT:

Our raw data certainly has the right number of dimensions, but what does it tell us about the density at a particular point? We may be tempted at this stage to look at all rays passing through this point. And indeed if we sum all rays passing through one point we get an interesting result. First we will augment our rectangular coordinate system with a polar one centered on the point of interest (x_0,y_0).

$$h(\chi,\theta) \quad = \quad \int_{-\infty}^{\infty} f(r,\theta)\, dr \qquad\qquad (\chi \; = \; -(x_0 \sin\theta + y_0 \cos\theta)\,)$$

Integrating over all angles from 0 to π:

$$\begin{aligned} g(x_0,y_0) \quad &= \quad \int_0^{\pi} h(\chi,\theta)\, d\theta \\ &= \quad \int_0^{\pi}\int_{-\infty}^{\infty} f(r,\theta)\, dr\, d\theta \\ &= \quad \int_{-\pi}^{\pi}\int_0^{\infty} f(r,\theta)\, \frac{1}{r}\, r\, dr\, d\theta \end{aligned}$$

We now change back to a rectangular coordinate system:

$$g(x_0,y_0) \quad = \quad \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x-x_0,y-y_0)\, \frac{1}{r}\, dx\, dy$$

So the function $g(x,y)$ we constructed is the convolution of the density function we are looking for and the function $(1/r)$. It is thus a good first approximation to $f(x,y)$, being spread, smeared or smoothed out. We will call $(1/r)$ the spreading function in this case. If the singularity at the origin worries you consider the effect of finite resolution in the imaging device. Also note that the apparently infinite integrals only really extend to the edge of the object which we assume is of finite extend.

RECONSTRUCTING THE DENSITY IN A THIN SLICE:

We are now ready to apply linear function theory. Clearly one thing to do will be to take the fourier transform, which will change the convolution into a product. We can then divide out the term due to the spreading function and transform back to obtain f(x,y).

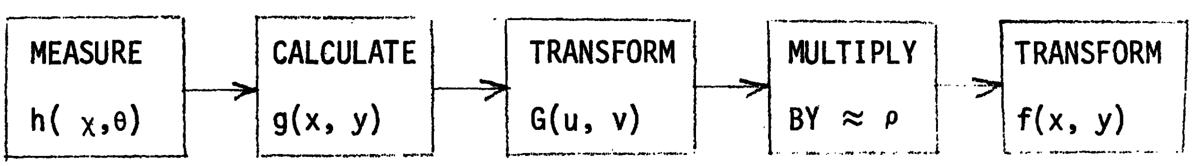
$$g = f \circledast (1/r) \qquad\qquad G = F * FT(1/r)$$

Here F and G are the two-dimensional fourier transform of f and g respectively. The function (1/r) is rotationally symmetric and fairly well behaved and its two-dimensional fourier transform turns out to be $(1/\rho)$ (see Appendix). Here ρ is the radius in the polar coordinate systems of the transform space.

$$G = F / \rho \qquad \text{and so} \qquad F = G \rho$$

In other words to find f(x,y) we transform g(x,y), multiply this by ρ and transform back.

At this stage we have to make some practical comments. Since G will be rather anemic in high frequency components and strong in the lows we have the usual worries about noise when we perform this operation. This will limit the resolution attainable. If we know the statistical properties of the signal and the noise as well as the transfer characteristic of the imaging device, we can apply Wiener-style least-square filtering. Typically we will end up multiplying G not by ρ, but some function which behaves like ρ for low frequencies and then tends to zero for high frequencies.



USING DE-CONVOLUTION INSTEAD:

Can we perform this recovery of the density function without using fourier transforms? Yes, by using another convolution. Suppose we have a function d(r) which when convolved with our spreading function (1/r) is zero everywhere except at the origin where we obtain a unit pulse. Then because of the associativity of convolutions we get:

$$g = [f \circledast (1/r)] \quad \text{so } g \circledast d(r) = f \circledast [(1/r) \circledast d(r)] = f \text{ !}$$

Note that while we so avoid the use of fourier transforms this method may be vastly more inefficient computationally. It does on the other hand provide valuable insights since the function d(r) will have more intuitive appeal to us than strange operations in the transform domain.

What form does d(r) take? Unfortunately we have to go into the transform domain to find this.

$$(1/r) \circledast d(r) = \delta(r) \qquad (1/\rho)*D = 1 \qquad D = \rho$$

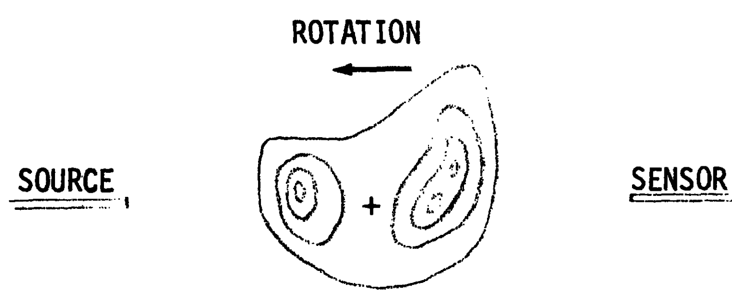
Here $\delta(r)$ is the unit pulse at the origin which transforms into 1, and D is the two-dimensional fourier transform of d(r). All that remains to be done is to transform D back again. Now while ρ is rotationally symmetric it certainly is not well behaved and we have to use convergence factors to inverse transform it (see Appendix). Anyway we get a central pulse surrounded by a region in which d(r) is proportional to $(-1/r^3)$. Note that all along we assumed d(r) would be rotationally symmetrical, which we have now demonstrated since it is the transform of a rotationally symmetric function.

If we have to use Wiener-style filtering, d(r) won't look quite so pathological since the central pulse will be spread into a positive blob.

SOME MATTERS OF IMPLEMENTATION:

We have already mentioned the problem of noise and how to deal with it. We then have to deal with the quantisation of the data. Clearly we will not be able to sample for all possible positions of the ray, or if we are using an imaging technique we will have limited resolution. In any case if the processing is to be done in a digital computer rather than some coherent optical set-up we will find the data quantised in both its dimensions. This matches nicely our capability to perform efficiently discrete fourier transforms in a digital computer using the fast-fourier-transform algorithm.

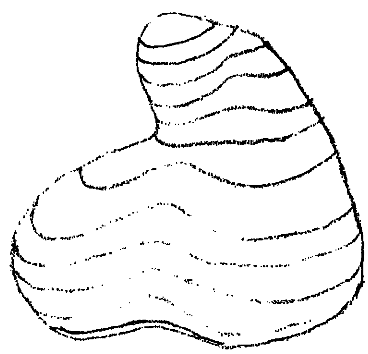
Next we address the question of practical ways of determining $g(x,y)$. We have so far made little distinction between imaging techniques and probing with a single ray. If the medium is at all likely to scatter the latter method is preferable even though more tedious. Then we may ask if we can obtain $g(x,y)$ directly instead of calculating it from $h(\chi,\theta)$. A simple method consists of a rotating mount for the slice, the center of rotation being adjustable to fall anywhere inside the slice. A ray passes through this center of rotation. The average over one revolution of the log of recorded intensity is then proportional to $g(x,y)$. It is amusing to compare this to the heuristics used by people for examining the interior of partially absorbing objects.



Other schemes might involve many source/sensor pairs at different angles all pointing at one point in the object. Summing the log of the outputs of the sensors would again produce a result proportional to $g(x,y)$. The object can then be moved to explore all values of x and y.

FINALLY: THREE-DIMENSIONAL OBJECTS:

As mentioned before the simplest way to deal with three-dimensional objects is to consider them made up of slices, since we know how to deal with those. Since for each slice the data is two-dimensional the totality of data will now be three dimensional and can be gathered by taking images of the object, each image differing from the one before it by a slight rotation of the object about an axis fixed in the object. Preferably this axis should be through the longest line through the object so as to help ensure that the sampling rays traverse as short a distance in the object as possible.



Intuitively we might feel that we are not sampling the object optimally if we only consider for each point the rays passing though it contained in one plane. We could better average out contributions of other parts of the body and obtain better accuracy if we used all possible rays through each point. This of course also adds another dimension to the raw data and forces us to learn about three-dimensional convolutions and fourier transforms.

Let us as before define a function g(x,y,z) which is the sum of logs of intensities for all rays through a point. This is now a double integral. It can be shown that this again is the convolution of f(x,y,z,), the density distribution, and a spreading function. Only now the spreading function is $(1/r^2)$, not $(1/r)$ (see Appendix).

RECONSTRUCTING THE DENSITY FROM THREE-DIMENSIONAL DATA:

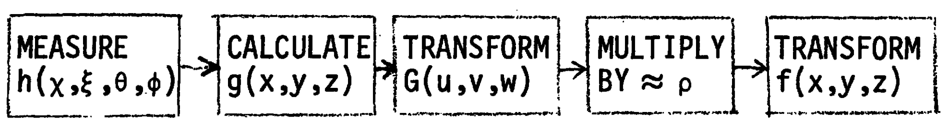
The form of the spreading function shows that we are averaging contributions over a large area and hence each contribution has a smaller effect. This may be one reason for using this method rather than the simpler slicing method. Again we fourier transform:

$$g = f \circledast (1/r^2) \qquad \text{so} \qquad G = F * FT(1/r^2)$$

Where F and G are the three-dimensional fourier transforms of f and g respectively. The function $(1/r^2)$ is rotationally symmetric and fairly well behaved and its three-dimensional fourier transform turns out to be $(1/\rho)$ (see Appendix). Here ρ is the radius in the spherical coordinate system of the transform space.

$$G = F/\rho \qquad \text{and so} \qquad F = G\rho$$

In other words to find f(x,y,z) we transform g(x,y,z), multiply this by ρ and transform back. Similar comments apply as regards noise and quantization as for the two-dimensional case.



We may also ask again about a de-convolution function. We have to find the inverse transform of ρ. While ρ is rotationally symmetric it is not well behaved and we have to use convergence factors to inverse transform it (see Appendix). The result is a central pulse surrounded by a region in which the function is proportional to $(-1/r^4)$. This again demonstrates the better localization obtained with the three-dimensional method. That is a local spot will perturb the computation less.

MISCELLANEOUS RELATED MATTERS:

In practice it is sometimes possible to assume that the density can take on only two values, that is that we have an almost opaque object in an almost transparent medium. If the object has a reasonable shape with no holes it seems reasonable to assume that we can obtain its shape from data of lower dimensionality. In fact under certain conditions a rather ad hoc procedure developed by J. Kloustad while at the R.L.E. will solve for the shape given only two images. It should be noted that his use of fourier series is not related to the transforms here discussed, but only to least-squares fitting of points to a closed curve. It is interesting that he opted for a slice by slice approach. This procedure has been applied to the problem of determining the changing shape of the heart using two X-ray motion picture films. One of the problems he investigated was the effect of perspective which somewhat complicates the calculation of ray geometries.

A related problem is silhouette-reconstruction. Here the object is completely opaque and thus precludes any estimation of its depth at a given point. It should be clear that a series of images is enough to reconstruct such an object if it is convex. If it is not convex, but has no hidden concavities it can be reconstructed using images from all directions (a hidden concavity is a point at which the complement of the object is convex). Perhaps one could use this to automatically produce a bust from many photographs of a person.

The reader should not be astonished by the form of the de-convolution functions. The de-convolution function corresponding to any smoothing function will have the form of a positive central pulse surrounded by a negative region (a smoothing function is one which at least near the origin is positive, that is produces a local weighted average of the function it is convolved with). It is the precise shape of the negative skirt which varies from one de-convolution function to another. So the existence of central-on, peripheral-off cells in a retina is little help in comparing alternate theories of vision. Such a device could be useful in edge-detection, dynamic-range compression, color vision, increasing contrast, undoing defocusing and even handling density distribution reconstruction!

PURPOSE OF THE APPENDICES:

The main purpose of the appendices is to remove the boring details of the convolutions and fourier transforms from the text. The main result is the transform in k-dimensional space of the function r^n. Of particular interest are the cases ($n=-1$, $k=2$) and ($n=-2$, $k=3$) which are easy to deal with and ($n=+1$, $k=2$) and ($n=+1$, $k=3$) which do not converge in the normal sense.

In order to be able to find these transforms the following is covered:

1. The form of the fourier transform in 1, 2, 3 ... k dimensions.

2. Collapse of the multiple integral into $\int_0^\infty f(r)\, h(r, \rho, k)\, dr$ for rotationally symmetric functions $f(r)$.

3. The fact that the inverse transform has the same form with r and ρ exchanged.

4. The transform of r^n is $\rho^{-n-k} ** F_k(n)$ for $-k < n < (1-k)/2$

5. The method of extending the validity of the transform to any $n > -k$ using convergence factors.

6. The details of $h(r,\rho,k) = \rho^{1-k/2}\, r^{k/2}\, J_{k/2-1}(r\rho)$

7. The details of $F_k(n) = 2^{n+k/2}\, \Gamma((n+k)/2) \,/\, \Gamma(-n/2)$

ONE DIMENSIONAL FOURIER TRANSFORM:

We define $g(u) = (2\pi)^{-1/2} \int_{-\infty}^{\infty} f(x)\, e^{-ixu}\, dx$

At times $f(x) = (2\pi)^{-1/2} \int_{-\infty}^{\infty} g(u)\, e^{ixu}\, du$

Now if f(x) is symmetrical (Let $r=x$, $\rho=u$):

$$g(\rho) = \int_0^{\infty} f(r) \quad (2/\pi)^{1/2} \cos(r\rho)\, dr$$

The inverse has the same form with r and ρ exchanged.

Now suppose we want to find the transform of r^n:

$$(2/\pi)^{1/2} \int_0^{\infty} r^n \cos(r\rho)\, dr$$

This diverges for $\rho=0$, otherwise if the integral is defined:

$$\rho^{-n-1} (2/\pi)^{1/2} \int_0^{\infty} x^n \cos x\, dx \qquad \text{(Where } x=r\rho\text{)}$$

So $FT_1(r^n) = \rho^{-n-1} * F_1(n)$ Where $F_1(n) = (2/\pi)^{1/2} \int_0^{\infty} x^n \cos x\, dx$

This integral is defined for some n, for example for n=-1/2 we have:

$$\int_0^{\infty} \cos x / \sqrt{x}\, dx = (\pi/2)^{1/2} \quad \text{so} \quad F_1(-1/2) = 1 \quad \text{[pg 313, 2]}$$

Which makes sense since $r^{-1/2}$ maps into $\rho^{-1/2}$. In fact it can be shown that:

$$F_1(n) = -(2/\pi)^{1/2}\, \Gamma(n+1) \sin n\pi/2 \qquad -1 < n < 0$$

The same form also applies for $n \geq 0$ if we use convergence factors (see later).

TWO-DIMENSIONAL FOURIER TRANSFORM:

We define $g(u,v) = (2\pi)^{-1}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x,y)\ e^{-i(xu+vy)}\ dx\ dy$

At times $f(x,y) = (2\pi)^{-1}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} g(u,v)\ e^{i(xu+vy)}\ du\ dv$

If we introduce polar coordinates we get:

$$x = r\cos\theta,\ y = r\sin\theta \quad \text{and} \quad u = \rho\cos\alpha,\ v = \rho\sin\alpha$$

$$g(\rho,\alpha) = \frac{1}{2\pi}\int_0^{\infty}\int_0^{2\pi} f(r,\theta)\ e^{ir\ (\cos\theta\cos\alpha + \sin\theta\sin\alpha)}\ r\ d\theta\ d\rho$$

Now if f(x,y) is rotationally symmetric and we note that:

$$\cos\theta\cos\alpha + \sin\theta\sin\alpha = \cos(\theta-\alpha)$$

$$(1/\pi)\int_0^{\pi} e^{ix\cos\theta}d\theta = J_0(x) \qquad \text{[pg 57, 3]}$$

then $$g(\rho) = \int_0^{\infty} f(r)\ r\ J_0(r\rho)\ dr$$

The inverse has the same form with r and ρ exchanged.

Now suppose we want to find the transform of r^n:

$$\int_0^{\infty} r^{n+1}\ J_0(r\rho)\ dr$$

This diverges for $\rho = 0$, otherwise if the integral is defined:

$$\rho^{-n-2}\int_0^{\infty} x^{n+1}\ J_0(x)\ dx \qquad \text{(Where } x = r\rho\text{)}$$

So $FT_2(r^n) = \rho^{-n-2} * F_2(n)$ Where $F_2(n) = \int_0^{\infty} x^{n+1}\ J_0(x)\ dx$

This integral is defined for some n, for example for n=-1 we have:

$$\int_0^\infty J_0(x)\,dx = 1 \qquad \text{so } F_2(-1) = 1 \qquad \text{[pg 58, 3]}$$

Which again makes sense since r^{-1} maps into ρ^{-1}. In fact it can be shown:

$$F_2(n) = 2^{n+1}\,\Gamma(1+n/2)\,/\,\Gamma(-n/2) = -(2^{n+1}/\pi)(\Gamma(1+n/2))^2 \sin \pi n/2$$

for $-2 < n < -1/2$

The same form also applies for $n \geq -1/2$ if we use convergence factors(see later).

THREE DIMENSIONAL FOURIER TRANSFORM:

We define $g(u,v,w) = (2\pi)^{-3/2} \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x,y,z)\, e^{-i(xu+vy+wz)}\, dx\, dy\, dz$

At times $f(x,y,z) = (2\pi)^{-3/2} \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} g(u,v,w)\, e^{i(xu+vy+wz)}\, du\, dv\, dw$

If we introduce spherical coordinates we get:

$$x=r\cos\theta\cos\phi,\ y=r\sin\theta\cos\phi,\ z=r\sin\phi$$
$$u=\rho\cos\alpha\cos\beta,\ v=\rho\sin\alpha\cos\beta,\ w=\rho\sin\beta$$

Let $A = r\rho(\cos\phi\cos\theta\cos\beta\cos\alpha + \cos\phi\sin\theta\cos\beta\sin\alpha + \sin\phi\sin\beta)$
$= r\rho\cos\phi\cos\beta\cos(\theta-\alpha) + r\rho\sin\phi\sin\beta$

So $g(\rho,\alpha,\beta) = (2\pi)^{-3/2} \int_0^{\infty}\int_{-\pi/2}^{\pi/2}\int_0^{2\pi} f(r,\theta,\phi)\, e^{iA}\, r^2\cos\phi\, d\theta\, d\phi\, dr$

Now if f(x,y,z) is rotationally symmetric and we note that:

$$\int_0^{2\pi} e^{i r\rho\cos\phi\cos\beta\cos(\theta-\alpha)}\, d\theta = 2\pi J_0(r\rho\cos\theta\cos\beta) \text{ (as before)}$$

Then we have to find $\int_{-\pi/2}^{\pi/2} e^{ix\sin\phi\sin\beta} J_0(x\cos\phi\cos\beta)\cos\phi\, d\phi$

Which also is $2\int_0^{\pi/2} \cos(x\sin\phi\sin\beta)\, J_0(x\cos\phi\cos\beta)\cos\phi\, d\phi$

This hairy thing needs to be approached with caution. Lets try $\beta=\pi/2$ first:

$$(2/x)\int_0^{\pi/2} \cos(x\sin\phi)\, x\cos\phi\, d\phi$$

Let $A=x\sin\phi$, $dA=x\cos\phi\, d\phi$, then:

$$(2/x)\int_0^{x} \cos A\, dA \quad = 2(\sin x)/x$$

Next we will try another special case: $\beta=0$:

$$2\int_0^{\pi/2} J_0(x\cos\phi)\cos\phi\, d\phi$$

$$=2\int_0^{\pi/2} J_0(x\sin\phi)\sin\phi\, d\phi \quad = 2\,(\sin x)/x \qquad \text{[pg 99, 3]}$$

You can substitute A = x sin ϕ or x cos ϕ to see this. Anyway the integral is in fact independent of the parameter β . This makes sense since we expect the transform of a rotationally symmetric function to also be rotationally symmetric. So we get:

$$g(\rho) = \int_0^{\infty} f(r)\quad (2/\pi)^{1/2}(r/\rho)\,\sin(r\rho)\, dr$$

The inverse has the same form with r and ρ exchanged.

Now consider the transform of r^n:

$$\int_0^{\infty} (r^{n+1}/\rho)\,\sin r\rho\quad (2/\pi)^{1/2}\, dr$$

This diverges for $\rho=0$, otherwise if the integral is defined:

$$\rho^{-n-3}\,(2/\pi)^{1/2}\int_0^{\infty} x^{n+1}\sin x\, dx \qquad (\text{Where } x=r\rho\ )$$

So $FT_3(r^n) = \rho^{-n-3} * F_3(n)$ Where $F_3(n) = (2/\pi)^{1/2}\int_0^{\infty} x^{n+1}\sin x\, dx$

This integral is defined for some n, for example for n=-3/2 we have:

$$\int_0^{\infty} (\sin x)/\sqrt{x}\, dx = (\pi/2)^{1/2} \quad \text{so} \quad F_3(-3/2)=1 \qquad \text{[pg 313, 2]}$$

Which makes sense since $r^{-3/2}$ maps into $\rho^{-3/2}$. In fact it can be shown:

$$F_3(n) = -(2/\pi)^{1/2}\,\Gamma(n+2)\sin n\pi/2 \qquad -3<n<-1$$

The same form also applies for $n\geqslant -1$ if we allow convergence factors(see later) k DIMENSIONAL FOURIER TRANSFORM:

We define $g(\underline{u}) = (2\pi)^{-k/2} \int_V f(\underline{x})\, e^{-i\, \underline{x}\cdot\underline{u}}\, dV$

At times $f(\underline{x}) = (2\pi)^{-k/2} \int_V g(\underline{u})\, e^{i\, \underline{x}\cdot\underline{u}}\, dV$

Where $\underline{x}$ is a k dimensional vector in the source domain and $\underline{u}$ is a k dimensional vector in the transform domain.

Now if $f(\underline{x})$ is rotationally symmetric we get:

$$g(\rho) = \int_0^\infty f(r)\, \rho^{1-k/2}\, r^{k/2}\, J_{k/2-1}(r\rho)\, dr$$

The inverse has the same form with r and ρ exchanged.

Now consider the transform of r^n: This diverges for $\rho=0$, but otherwise:

$$g(\rho) = \rho^{-k-n} \int_0^\infty x^{n+k/2}\, J_{k/2-1}(x)\, dx \qquad (\text{Let } x = r\rho)$$

So $FT_k(r^n) = \rho^{-n-k} * F_k(n)$ Where $F_k(n) = \int_0^\infty x^{n+k/2}\, J_{k/2-1}(x)\, dx$

This integral is defined for some n, for example for n=-k/2 we have:

$$\int_0^\infty J_{k/2-1}(x)\, dx = 1 \quad \text{so } F_k(-k/2) = 1 \qquad [\text{pg } 96,\ 3]$$

Which is as it should be since $r^{-k/2}$ maps into $\rho^{-k/2}$ and visa versa.
In fact it can be shown (see later):

$$F_k(n) = 2^{n+k/2}\, \Gamma((n+k)/2) \,/\, \Gamma(-n/2) \quad \text{for } -k < n < (1-k)/2$$

The same formula also applies for $n \geq (1-k)/2$ if we permit convergence factors.

THE SO-CALLED CONVERGENCE FACTORS:

In dealing with fourier transforms we often find integrals which fail to converge because of an oscillating integrand which does not tend to zero. Since these integrals often occur in consort with functions that do tend to zero at infinity it seems reasonable to assign a value to them in any case. A way of dealing with this problem is the use of convergence factors. These are functions of a parameter a, with the feature that the functions are constant for a = 0.

$$c(0) = 1, \; c(ax) \approx 1 \text{ for small } ax, \; \approx 0 \text{ for large } ax$$

If $A = \int_0^\infty f(x)\, dx$ exists it will be equal to $B = \lim_{a \to 0} \int_0^\infty f(x)\, c(ax)\, dx$

This will be true for any convergence factor. Now if instead we have an integrand f(x) that does not tend to zero as x tends to infinity, but instead oscillates, A will be undefined, by B may have a meaningful value. Similar techniques are used for dealing with series whose sums oscillate. Common functions are:

$$c(ax) = e^{-ax} \qquad \text{or} \qquad c(ax) = e^{-(ax)^2}$$

A closely related technique is that of avoiding poles in calculating Laplace transforms:

$$f(s) = \int_0^\infty e^{-st}\, F(t)\, dt \qquad\qquad F(t) = \frac{1}{2\pi i} \int_{a+\infty i}^{a+\infty i} e^{st}\, f(s)\, ds$$

Where a is chosen so that the path of integration lies to the right of any poles of f(s). In this way one finds that the transform of t^n is:

$$s^{-1-n}\, \Gamma(n+1) \qquad \text{for } n>-1$$

whereas without this technique the transform is only defined for $-1<n<0$.

THE INTEGRAL $\int_0^\infty x^q \cos x \, dx$:

$$\int_0^\infty x^q \cos x \, dx = -\Gamma(q+1) \sin q\pi/2 \qquad -1 < q < 0$$

from $\int_0^\infty x^{q-1} \cos mx \, dx$ [pg 223, 1]

For $q \leq -1$ we have convergence problems at the origin. For $q \geq 0$ the integrand does not vanish at infinity, but instead oscillates about zero. So we can try convergence factors.

$$\int_0^\infty e^{-ax} x^q \cos x \, dx = \Gamma(q+1) / r^{q+1} \cos(q+1)\theta \qquad -1 < q, \ a > 0$$

Where $r = \sqrt{1+a^2}$ $\quad \sin\theta = 1/r, \cos\theta = a/r$

from $\int_0^\infty x^{p-1} e^{-ax} \cos mx \, dx$ [pg 235, 1]

$$\lim_{a \to 0} \int_0^\infty e^{-ax} x^q \cos x \, dx = -\Gamma(q+1) \sin q\pi/2 \qquad -1 < q$$

Which is the same form as above.

THE INTEGRAL $\int_0^\infty x^q \, J_0(x) \, dx$:

$$\int_0^\infty x^q \, J_0(x) \, dx = 2^q \, \Gamma((1+q)/2) \, / \, \Gamma((1-q)/2) \qquad -1 < q < 1/2$$

from $\int_0^\infty t^{\mu} J_{\nu}(t) \, dt$ [pg 486, 4]

For $q < -1$ we have convergence problems at the origin. For $q > 1/2$ the integrand does not vanish at infinity, but instead oscillates about zero. So we can try convergence factors.

$$\int_0^\infty e^{-(ax)^2} \, x^q \, J_0(x) \, dx = \Gamma((q+1)/2)/(2a^{q+1}) \; M((q+1)/2,\, 1,\, -1/(4a^2))$$

for $-1 < q$ and $a^2 > 0$ from $\int_0^\infty e^{-a^2 t^2} \, t^{\mu-1} J_{\nu}(bt) \, dt$ [pg 486, 4]

Now we have for the asymptotic expansion of the confluent hypergeometric function:

$$M(\alpha, \beta, z) = \Gamma(\beta)/\Gamma(\alpha) \;\; e^z \, z^{\alpha-\beta} \, (1 + o(|z|)^{-1}) \qquad z > 0$$

$$M(\alpha, \beta, z) = \Gamma(\beta)/\Gamma(\beta-\alpha) \; (-z)^{-\alpha} \, (1 + o(|z|)^{-1}) \qquad z < 0$$

[pg 504, 4]

$$\therefore M((q+1)/2,\, 1,\, -1/(4a^2)) = \left[\Gamma(1)/\Gamma((1-q)/2)\right] (4a^2)^{(q+1)/2} \, (1 + o(|z|)^{-1}) \qquad z < 0$$

$$= (2a)^{q+1} / \Gamma((1-q)/2) \; (1 + o(|z|)^{-1}) \qquad z < 0$$

$$\int_0^\infty e^{-(ax)^2} \, x^q \, J_0(x) \, dx = \left[\Gamma((1+q)/2) \, / \, \Gamma((1-q)/2)\right] 2^q (1 + o(|z|)^{-1})$$

$$\lim_{a \to 0} \int_0^\infty e^{-(ax)^2} \, x^q \, J_0(x) \, dx = 2^q \;\; \Gamma((1+q)/2) \, / \, \Gamma((1-q)/2) \qquad -1 < q$$

Which is the same form as above.

THE INTEGRAL $\int_0^\infty x^q \sin x\, dx$:

$$\int_0^\infty x^q \sin x\, dx = \Gamma(q+1) \cos q\pi/2 \qquad -2 < q < -1, \quad -1 < q < 0$$

$$= \pi/2 \qquad q=-1$$

from $\int_0^\infty x^{q-1} \sin mx\, dx$ [pg 223, 1]

For q $\leq$-2 we have convergence problems at the origin. For q$\geq$0 the integrand does not vanish at infinity, but instead oscillates about zero. So we can try convergence factors.

$$\int_0^\infty e^{-ax} x^q \sin x\, dx = \Gamma(q+1) / r^{q+1} * \sin(q+1)\theta \qquad -1 < q,\ a > 0$$

Where $r = \sqrt{1+a^2}$ $\sin\theta = 1/r$, $\cos\theta = a/r$

from $\int_0^\infty x^{p-1} e^{-ax} \sin mx\, dx$ [pg 234, 1]

$$\lim_{a\to 0} \int_0^\infty e^{-ax} x^q \sin x\, dx = \Gamma(q+1) \cos q\pi/2 \qquad -1 < q$$

Which is the same form as above.

THE INTEGRAL $\int_0^\infty x^q J_p(x)\, dx$:

$$\int_0^\infty x^q J_p(x)\, dx = 2^q \Gamma((p+q+1)/2)/\Gamma((p-q+1)/2)$$

for $-1-p < q < 1/2$ from $\int_0^\infty t^\mu J_\nu(t)\, dt$ [pg 486, 4]

For $q \leq -1-p$ we have convergence problems at the origin. For $q \geq 1/2$ the integrand does not vanish at infinity, but instead oscillates about zero. So we can try convergence factors.

$$\int_0^\infty e^{-(ax)^2} x^q J_p(x)\, dx = \Gamma((p+q+1)/2)/(2^{p+1} a^{p+q+1} \Gamma(p+1))$$

$$* \; M((p+q+1)/2,\ p+1,\ -1/(4a^2))$$

for $-1-p < q$, $a^2 > 0$ from $\int_0^\infty e^{-a^2t^2} t^{\mu-1} J_\nu(bt)\, dt$ [pg 486,4]

We now use the asymptotic expansion of the confluent hyper-geometric function M for large negative values of the third argument:

$$M(\alpha,\beta,z) = \Gamma(\beta)/\Gamma(\beta-\alpha)\ (-z)^{-\alpha}\ (1 + o(|z|)^{-1}) \qquad z < 0$$

[pg 504, 4]

$$M((p+q+1)/2,\ p+1,\ -1/(4a^2)) = \left[\Gamma(p+1)/\Gamma((p-q+1)/2)\right](2a)^{p+q+1}(1+o(|z|)^{-1})$$

So $$\int_0^\infty e^{-(ax)^2} x^q J_p(x)\, dx = \left[\Gamma((p+q+1)/2)/\Gamma((p-q+1)/2)\right] 2^q\ (1 + o(|z|)^{-1})$$

$$\lim_{a\to 0}\int_0^\infty e^{-(ax)^2} x^q J_p(x)\, dx = 2^q \Gamma((p+q+1)/2)/\Gamma((p-q+1)/2) \qquad -1-p < q$$

Which is the same form as above.

THE FORM $h(r,\rho,k) = \rho^{1-k/2}\, r^{k/2}\, J_{k/2-1}(r\rho)$:

This is the kernel of the single integral into which the fourier transform collapses when applied to a rotationally symmetric function.

For k=1 we have: $\rho^{1/2}\, r^{1/2}\, J_{-1/2}(r\rho)$

Now $J_{-1/2}(x) = (2/(\pi x))^{1/2} \cos x$ [pg 194, 1]

so $h(r,\rho,1) = (2/\pi)^{1/2} \cos(r\rho)$

For k=2 we have:

$$h(r,\rho,2) = r\, J_0(r\rho)$$

For k=3 we have: $\rho^{-1/2}\, r^{3/2}\, J_{1/2}(r\rho)$

Now $J_{1/2}(x) = (2/(\pi x))^{1/2} \sin x$ [pg 193, 1]

so $h(r,\rho,3) = (2/\pi)^{1/2} (r/\rho) \sin(r\rho)$

THE FORM $F_k(n) = 2^{n+k/2}\,\Gamma((n+k)/2)\,/\,\Gamma(-n/2) \qquad -k < n < (1-k)/2$

The above form can be written in other ways to allow simplification for particular k. First we note that:

$$\Gamma(z)\,\Gamma(1-z) = \pi/\sin\pi z \quad \text{so } 1/\Gamma(-z) = -z\,\Gamma(z)(\sin\pi z)/\pi$$

Using $z\,\Gamma(z) = \Gamma(z+1)$ [pg 256, 4]

So $F_k(n) = 2^{n+k/2}\,(-n/2)\,\Gamma(n/2)\,\Gamma((n+k)/2)\ \sin\pi n/2\ /\pi$

This form is useful for k=2 for example. Next we note that:

$$\Gamma(2z) = (2\pi)^{-1/2}\,2^{2z-1/2}\,\Gamma(z)\,\Gamma(z+1/2) \qquad \text{[pg 256, 4]}$$

So $\Gamma(n/2)\,\Gamma((n+1)/2) = \Gamma(n)\sqrt{\pi}\,2^{1-n}$

That gives $F_k(n) = -2^{k/2}/\sqrt{\pi}\left[\Gamma((n+k)/2)/\Gamma((n+1)/2)\right]\Gamma(n+1)\ \sin\pi n/2$

This is useful for k=1,3. So we can use the above to get:

For k=1: $F_1(n) = -(2/\pi)^{1/2}\,\Gamma(n+1)\ \sin\pi n/2 \qquad F_1(-1/2)=1$

For k=2: $F_2(n) = -2^{n+1}/\pi\ \left(\Gamma(n/2+1)\right)^2\ \sin\pi n/2 \qquad F_2(-1) = 1$

For k=3: $F_3(n) = -(2/\pi)^{1/2}\,\Gamma(n+2)\ \sin\pi n/2 \qquad F_3(-3/2)=1$

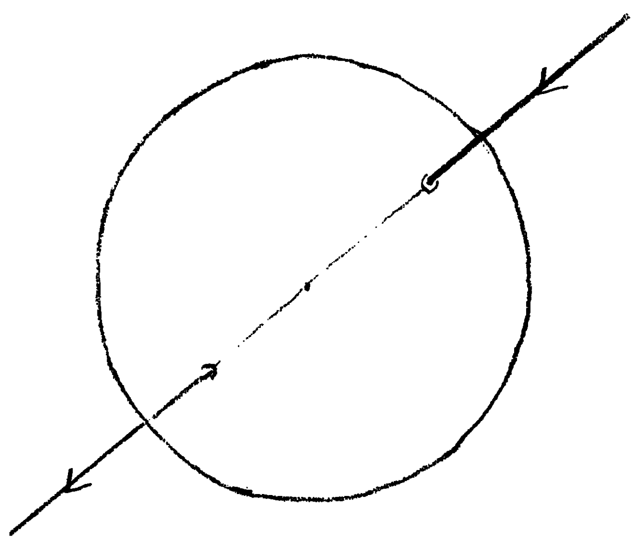
THE SPREADING FUNCTION IN k DIMENSIONS:

Consider a spherical shell of thickness dr, absorbing density D/2 and radius r. Each ray through the centre of this shell will be attenuated by D dr and so the integral over all ray directions will be the total solid angle times D dr and is independent of the radius of the shell. This integral is also equal to the area of the shell times the spreading function.

$$A_k(r) * s_k(r)\ D\ dr = A_k(1) * D\ dr$$

Where $A_k(r) = \pi(2r)^{k-1}$ is the surface area of the shell. So we get:

$$s_k(r) = r^{1-k}$$



TRANSFORM OF THE SPREADING FUNCTION IN k DIMENSIONS:

Since the form of the spreading function is r^{1-k} we have:

$$g(\underline{x}) = f(\underline{x}) \bullet r^{1-k}$$

This transforms into:

$$FT_k(r^{1-k}) = \rho^{-(1-k)-k} * F_k(1-k) = (1/\rho) * F_k(1-k)$$

For the de-convolution function we want the algebraic inverse of this inverse transformed:

$$FT_k(\rho) = r^{-1-k} * F_k(1)$$

Now $F_k(1-k) = 2^{1-k/2}\,\Gamma(1/2) / \Gamma((k-1)/2)$

$= 2^{1-k/2}\sqrt{\pi} / \Gamma((k-1)/2)$ Which is positive

$F_k(1) = 2^{1+k/2}\,\Gamma((1+k)/2) / \Gamma(-1/2)$

$= -2^{k/2} / \sqrt{\pi}\,\Gamma((1+k)/2)$ Which is negative k DIMENSIONAL TRANSFORM OF LOW PASS FILTER:

Consider $f(r) = 1$ for $r \leq R$ and $f(r) = 0$ for $r > R$

Now $g(\rho) = f(r)\ \rho^{1-k/2}\ r^{k/2}\ J_{k/2-1}(r\rho)\ dr$

$$g(\rho) = \rho^{-k} \int_0^{R\rho} x^{k/2}\ J_{k/2-1}(x)\ dx \qquad (\text{let } x = r\rho)$$

$$= \rho^{-k}\ (R\rho)^{k/2}\ J_{k/2}(R\rho) \qquad \text{for } k > 0$$

from $\int_0^{\infty} t^{\nu}\ J_{\nu-1}(t)\ dt$ [pg 484, 4]

$$g(\rho) = R^k\ J_{k/2}(R\rho)\ /\ (R\rho)^{k/2}$$

For k=1: $R\ J_{1/2}(R\rho)/(R\rho)^{1/2} = R\ (2/\pi)^{1/2} \sin(R\rho)/(R\rho)$

For k=2: $R^2\ J_1(R\rho)/(R\rho)$

For k=3: $R^3\ J_{3/2}(R\rho)/\ (R\rho)^{3/2} = R^3(2/\pi)^{1/2}(\ \sin(R\rho)-(R\rho)\cos(R\rho))/(R\rho)^3$

Since $J_{3/2}(x) = (2/(\pi x))^{1/2}((\sin x)/x - \cos x)$ [pg 95, 3]

REFERENCES:

1. Dwight H.B. "Tables of integrals and other mathematical data" MacMillan, 1964
2. Hodgman C.D. "C.R.C. Standard Mathematical Tables" Chemical Rubber Publishing, 1962
3. Bowman F. "Introduction to Bessel functions" Dover, 1958
4. Abramowits M & Segun I.A. "Handbook of Mathematical Functions" Dover, 1968
5. Gradshteyn I.S. & Ryzhik I.M. "Table of integrals, series and products" Academic Press, 1965
6. Wiener N. "Generalized Harmonic Analysis and Tauberian Theorems" M.I.T. Press, 1966
7. Wiener N. "Time Series" M.I.T. Press, 1966
8. Horn B.K.P. "The Application of Fourier Transform Methods to Image Processing" S.M. Thesis, M.I.T. E.E. Department, June 1968
9. Horn B.K.P. "VISMEM: A bag of robotics formulae" Vision Flash 34, Artificial Intelligence Laboratory, M.I.T., December 1972