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The Position of the Sun

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Abstract. The appearance of a surface depends dramatically on how it is illuminated. In order to interpret properly satellite and aerial imagery, it is necessary to know the position of the sun in the sky. This is particularly important if this interpretation is to be done in an automated fashion. Techniques using relatively straightforward methods are presented here for calculating the position of the sun with more than enough accuracy.

Caution: Do not use this technique for navigational purposes. Correction terms have been omitted; as a result, the ephemeris data calculated may be in error by about one minute of arc, an amount which is of no significance for the application of this data in image analysis.

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1. Geographical Position of the Sub-solar Point.

In order to calculate the position of the sun in the sky above a point at longitude θ and latitude ϕ, it is convenient to determine the longitude θ' and latitude ϕ' of the sub-solar point. This is the point where an observer would find the sun in the zenith. Longitude is measured from the prime meridian, increasing in an easterly direction. Latitude is measured from the equator, increasing in a northerly direction. Figure 1a illustrates relative position of the points of interest.

The elevation, e, and azimuth, a, of the sun can now be calculated by concentrating on the relevant spherical triangle extracted from Figure 1a and shown in Figure 1b. The elevation is simply the angle between the sun and the horizon, while the azimuth is the direction of the sun measured clockwise from north, as shown in Figure 2. We can find the elevation by applying the law of cosines to the triangle in Figure 1b,

$$\cos(\frac{\pi}{2} - e) = \cos(\frac{\pi}{2} - \phi) \cos(\frac{\pi}{2} - \phi') + \sin(\frac{\pi}{2} - \phi) \sin(\frac{\pi}{2} - \phi') \cos(\theta' - \theta)$$

so,

$$\sin e = \sin \phi \sin \phi' + \cos \phi \cos \phi' \cos (\theta' - \theta) \qquad (1)$$

Since the elevation ranges between $-\pi/2$ and $+\pi/2$, there is no ambiguity in this solution (negative elevations have the obvious interpretation).

Applying the law of cosines in a second way,

$$\cos(\frac{\pi}{2} - \phi') = \cos(\frac{\pi}{2} - \phi) \cos(\frac{\pi}{2} - e) + \sin (\frac{\pi}{2} - \phi) \sin(\frac{\pi}{2} - e) \cos a$$

So,

$$\cos a = \frac{\sin \phi' - \sin \phi \sin e}{\cos \phi \cos e} \tag{2}$$

Applying the law of sines,

$$\frac{\sin a}{\sin (\frac{\pi}{2} - \phi')} = \frac{\sin (\theta' - \theta)}{\sin (\frac{\pi}{2} - e)}$$

or,

$$\sin a \cos e = \cos \phi' \sin (\theta' - \theta) \tag{3}$$

Using (1) and (3) to substitute for sin e and cos e in (2), one finds

$$\tan a = \frac{\cos \phi' \sin (\theta' - \theta)}{\cos \phi \sin \phi' - \sin \phi \cos \phi' \cos (\theta' - \theta)} \tag{4}$$

This form is more convenient than (2) because it is expressed directly in terms of the longitudes and latitudes and because the quadrant of the angle a is properly defined. That is, the numerator has the sign of sin a, while the denominator has the sign of cos a. [To use (2) one has to check the sign of $(\theta' - \theta)$ and use $2\pi - a$ instead of a if the sign is a negative.]

2. Date.

In order to calculate the position of heavenly bodies, the time that has elapsed since some fixed epoch must be known. This can be conveniently expressed in days. Because of our weird system of days, months, years, and leap years, this is not so easy to calculate. One can write a program which takes into account the different lengths of the months and the fact that there is a leap year every fourth year, except once a century, except once every four centuries. This program in all likelihood would contain a large number of conditional statements. Curiously, the same effect can be achieved by employing the truncation properties of integer-arithmetic! If we let Y, M and D be the year, month and day of the date in question, and let

$$F = -\left\lfloor \frac{14 - M}{12} \right\rfloor \tag{5}$$

$$L = F + Y + 4800 \tag{6}$$

then the Julian date can be expressed as

$$J = \left\lfloor \frac{367 * (M - 2 - F * 12)}{12} \right\rfloor - \left\lfloor \frac{3 * (1 + \lfloor L/100 \rfloor)}{4} \right\rfloor + \left\lfloor \frac{1461 * L}{4} \right\rfloor + D - 32075 \tag{7}$$

Purists should feel free to substitute (5) and (6) into (7) so as to obtain a single equation for the Julian date. In the above expressions, $\lfloor a/b \rfloor$ stands for truncated integer division. It is convenient to work relative to some recent epoch and since the Julian date of the first of January of 1975 is 2442412, one may use,

$$J' = J - 2442412 \tag{8}$$

the number of days since that point in time.

At times it is necessary to calculate the date in our usual notation from a Julian date. I have not come across an equally elegant method for doing this. If J is the Julian date, the following calculation will do:

$$A = J + 68569$$

$$B = \left\lfloor \frac{A * 4}{146097} \right\rfloor$$

$$C = A - \left\lfloor \frac{B * 146097 + 3}{4} \right\rfloor$$

$$\left.\begin{aligned} I &= \left\lfloor \frac{(C + 1) * 4000}{1461001} \right\rfloor \\ K &= C - \left\lfloor \frac{I * 1461}{4} \right\rfloor + 31 \\ J &= \left\lfloor \frac{K * 80}{2447} \right\rfloor \end{aligned}\right\} \tag{9}$$

$$D = K - \left\lfloor \frac{J * 2447}{80} \right\rfloor$$

$$E = \lfloor J/11 \rfloor$$

$$M = J - 12 * E + 2$$

$$Y = I + 100 * (B - 49) + E$$

where Y, M and D are the year, month and day. If you are wondering where some of these peculiar numbers come from, note that there are 4 * 365 + 1 = 1461 days in four years, 25 * 1461 - 1 = 36524 days in a century and 4 * 36524 + 1 = 146097 days in four centuries.

3. Time.

There are a number of different systems of time measurement, including

1. Ephemeris time, the independent time-argument of the ephemeris of the Sun, Moon and planets. It can be determined by observation of these bodies and departs from an ideal time scale only in so far as the theory of their motion is inadequate. It is by definition uniform.

2. Sidereal time, is directly related to the rotation of the earth. It can be determined from the diurnal motions of the fixed stars. Equal intervals of angular motion correspond to equal intervals of sidereal time. The small variations in the rate of rotation, which cannot be accurately predicted ensures that sidereal time cannot be related exactly to ephemeris time.

3. Universal time, is related to the diurnal motions of a "mean" sun. It is computed from sidereal time by a numerical formula and is related to the hour angle of a point moving with the mean speed of the Sun in its orbit by means of an empirical correction, which must be determined by observation.

One must be careful to distinguish the places in the calculation where ephemeris time is appropriate (as, for example, for orbital calculations) as opposed to those places where universal time is appropriate (as, for example, for earth rotation calculations). Universal and Ephemeris time were defined so that they coincided at 12^h January 0, 1900. By now, (downward) drifts in the angular rate of rotation of the earth has permitted universal time to drop behind ephemeris time by about 45 seconds.

International Atomic Time (TAI) is based on a second of 9 192 631 770 periods of radiation corresponding to a transition between two hyperfine levels of the ground state of the calsium atom 33. Ephemeris time is approximately equal to TAI plus 32.18 seconds.

4. The Orbit.

In order to determine the sub-solar point at a particular instant, it is necessary to calculate the position of the sun on the celestial sphere. By definition of the ecliptic as the intersection of the earth's orbital plane with the celestial sphere, it is clear that the sun will appear on the ecliptic when viewed from earth. To find out where on this great circle it is, one must determine the position of the earth in its orbit. Because the earth's orbit is not perfectly circular, one has to understand the motions of a planet on an elliptical orbit, a problem first addressed by Kepler.

It is convenient to introduce some auxilliary quantities in addition to the semi-major axis, a, and the eccentricity, e (see Figure 3). One such auxilliary variable is the eccentric anomaly, E, measured as shown in Figure 3 from the center of the ellipse, unlike the true anomaly, ν, measured from the sun, situated at one focus of the ellipse. Both angles are relative to the major axis of the ellipse.

The eccentricity, e, is related to the semi-major axis, b, by

$$b^2 = a^2 (1 - e^2) \tag{10}$$

From Figure 3, one finds

$$ae + r \cos \nu = a \cos E \tag{11}$$

$$r \sin \nu = b \sin E \tag{12}$$

Eliminating ν by adding $\sin^2 \nu$ and $\cos^2 \nu$ one finds

$$r = a\ (1 - e \cos E) \tag{13}$$

Substituting the value of cos E from (13) into (11) eliminates E

$$r = \frac{a\ (1 - e^2)}{(1 + e \cos \nu)} \tag{14}$$

Now one can eliminate r from (13) and (14) and solve for cos ν,

$$\cos \nu = \frac{\cos E - e}{1 - e \cos E} \tag{15}$$

Then,

$$\frac{1 - \cos \nu}{1 - \cos \nu} = \left(\frac{1 + e}{1 - e}\right) \frac{1 - \cos E}{1 + \cos E} \tag{16}$$

And consequently,

$$\tan \frac{\nu}{2} = \sqrt{\frac{1 + e}{1 - e}}\ \tan \frac{E}{2} \tag{17}$$

This then is the equation for the true anomaly, ν, in terms of the eccentric anomaly, E. Next one has to find a way of calculating the eccentric anomaly, E, from the mean anomaly, g. The mean anomaly is the angular position of a fictitious mean earth moving at a <u>uniform</u> angular rate such that it completes an orbit in the same time as the real earth moving at a non-uniform rate.

$$g = \frac{2\pi}{T} (t - t_0) \tag{18}$$

Here T is the orbital period, t the time of interest and t_0 the time of perihelion passage.

The celestial longitude of the earth as seen from the sun can be calculated by adding the true anomaly to the celestial longitude of perihelion of the earth. The celestial longitude of the sun, λ, as seen from the earth can be similarly calculated by adding the true anomaly, ν, to the celestial longitude of perigee, Γ, of the sun.

5. Kepler's Second Law.

At this point, we invoke Kepler's second law, stating that the radius vector sweeps out equal areas in equal times. The area swept out by the radius vector since perihelion passage is shown in Figure 4a to be decomposable into a triangle, A, and a section of the ellipse, B. The latter can be determined from the area of the sector, C, of the circle of radius a shown in Figure 4b. The area of the triangle, A, is

$$\frac{1}{2}(r \cos \nu)(r \sin \nu) = \frac{1}{2} ab \sin E (\cos E - e) \tag{19}$$

The area of the section of the ellipse B is (b/a) times the area of the sector C of the circle. So,

$$\frac{b}{a}\left[\frac{1}{2} a^2 E - \frac{1}{2} a^2 \sin E \cos E\right] = \frac{1}{2} ab (E - \sin E \cos E) \tag{20}$$

The total area swept out comes to

$$\frac{1}{2} ab (E - e \sin E) \tag{21}$$

This should be compared to the total area of the ellipse, πab. As a result, one may write

$$E - e \sin E = g \tag{22}$$

This is Kepler's famous equation. It is transcendental and has to be solved iteratively. Curiously, (17) a good first approximation when e is small is

$$\tan \frac{E}{2} \approx \sqrt{\frac{1 + e}{1 - e}} \tan \frac{g}{2} \tag{23}$$

The iteration can be based on

$$E_{n + 1} = g + e \sin E_n \tag{24}$$

It will be apparent to the reader that we are neglecting periodic components of the perturbations due to the other planets. These tend to contribute less than a minute of arc to the longitude and a fraction of a second of arc in celestial latitude.

6. Inclination of the Earth Axis of Rotation.

We now know where the sun is in a celestial coordinate system with the ecliptic as equator. In order to calculate the sub-solar point in geographical coordinates we have to convert to a different coordinate system with the projection of the earth's equator as the reference. In this system, commonly used for astronomical purposes, "latitude" is called declination and "longitude" is called right ascencion. Declination, like latitude increases northward and right ascension like longitude increases eastward, but is frequently given as an hour-angle, with 24 hours in a full circle. The intersection of the ecliptic and the equator is called the vernal equinox (see Figure 5a) and celestial longitude as well as right ascencion are measured from this point. Here the inclination of the axis of the earth is $\varepsilon \simeq 23\ 1/2$ degrees. The relationship between coordinates in the two system is apparent in Figure 5b.

To find the appropriate coordinate transformation, consider the spherical diagram in Figure 5c, where λ, β are celestial longitude and latitude, while α, δ are right ascension and declination. Using the cosine law

$$\sin\delta = \cos\varepsilon\sin\beta + \sin\varepsilon\cos\beta\sin\lambda \tag{25}$$

Again, applying the cosine law, one finds

$$\sin\beta = \cos\varepsilon\sin\delta - \sin\varepsilon\cos\delta\sin\alpha \tag{26}$$

Using the sine law,

$$\frac{\cos \alpha}{\cos \beta} = \frac{\cos \lambda}{\cos \delta} \tag{27}$$

Eliminating sin δ and cos δ in (26), using (25) and (27), leads to

$$\tan \alpha = \frac{\cos \varepsilon \cos \beta \sin \lambda - \sin \varepsilon \sin \beta}{\cos \beta \cos \lambda} \tag{28}$$

The quadrants of α are taken care of since the numerator has the sign of sin α and the denominator has the sign of cos α .

In the case of an object <u>on</u> the ecliptic, such as the sun, these equations become even simpler since then $\beta = 0$. So,

$$\sin \delta = \sin \varepsilon \sin \lambda \tag{29}$$

$$\tan \alpha = \cos \varepsilon \tan \lambda \tag{30}$$

7. Aberration.

Due to the finiteness of the speed of light, an object does not appear to an observer to lie in the direction defined geometrically by the true position at the time of observation. In the case of the sun as seen from earth, this can be taken care of by calculating the relative position at an earlier time. The time interval in question equals the time required for light to travel from the sun to the earth. The light time at unit distance is 499.012 seconds. Equations presented earlier can be used to calculate the true distance.

One can instead use an angular offset which depends on the ratio of the velocity of earth in its orbit to the velocity of light. At unit distance this comes to 20.496 seconds of arc. This has to be divided by the true distance derived from the orbital equations if high precision is required.

8. Precession of the Earth's Axis of Rotation.

The equatorial plane is unfortunately not fixed. The gravitational pull of the moon and the sun on the equatorial bulge of the earth result in a slow precession of the earth's axis of rotation. That is, the pole of the equator precesses about the pole of the ecliptic in about 26,000 years. The inclination of the axis of rotation remains constant but the equinox, the intersection of the equator and the ecliptic, precesses. Since the coordinate systems we employ use this intersection on a reference point, we have to take this movement into account. This effect is known as luni-solar precession and amounts to about 50.25 seconds of arc per year. The equations presented later for the geometric mean longitude of the sun, the mean longitude of the sun's perigee and the hour angle of the equinox include this effect.

9. Nutation.

The lunar orbit, inclined about 5° to the ecliptic precesses rather rapidly, with a period of about 18.6 years. As it does so, it produces a further small displacement of the earth's pole. The earth's pole describes a small circle of about 9.210 seconds of arc radius on the celestial sphere about its mean position. As a result the obliquity varies periodically, as does the position of the equinox, the intersection of celestial equator with the ecliptic. If the longitude of the ascending node of the lunar orbit is designated Ω, then the nutation in obliquity is found to be $9''.210 \cos \Omega$ and the nutation in celestial longitude (due to the movement of the equinox on the ecliptic) comes to $-17''.234 \sin \Omega$.

This is the largest component of nutation. The remaining components depend on the longitude of the sun and the longitude of the moon in its orbit and tend to be less than a second of arc.

10. Precession of the earth's orbital plane.

The earth's orbital plane is inclined slightly to the invariant plane of the solar system. Consequently, the gravitational pull of the other planets cause the pole of the orbit to precess at a very slow rate. This is termed planetary precession. The ecliptic plane is thus also in motion and the position of the equinox at the intersection of the ecliptic and the equator is further displaced. One important effect of planetary precession is a decrease in the obliquity, the angle between the ecliptic and the equator, of about 47 seconds of arc per century. Another effect in a movement of the equinox which amounts to about 12.5 seconds of arc per century.

11. Movement of perihelion.

Because of the perturbing effects of the planets, the major axis of the earth's orbit has a secular motion, too, and the perihelion, to which mean, eccentric and true anomaly are referred, is thus also in motion. All of these effects are taken into account in Simon Newcomb's theory of planetary motion [1, 7].

Newcomb's tables are based on the epoch 12^h E.T. on January 0, 1900. For convenience the equations presented here have been referred instead to 0^h E.T. on January 1, 1975. Higher order terms have been dropped as have the small periodic effects of planetary perturbation and short period terms of nutation.

Angles are given with 4 digits after the decimal, angular rates with 8 digits. Secular terms involving higher powers of time have been dropped. Eccentricity must be calculated with 2 more digits precision because of its use in Kepler's equation. The position of the moon's node can be calculated with 2 digits less since it only enters into the calculation of nutation.

12. Summary.

All angles are in degrees. ET is time in ephemeris days since 1975/01/01 00:00:00 E.T.

Geometric mean longitude of the sun, mean equinox of date,

$$L = 280.0271 + .98564734 * ET \qquad (31)$$

Mean longitude of sun's perigee, mean equinox of date,

$$\Gamma = 282.5105 + .00004709 * ET \qquad (32)$$

Mean anomaly, $L - \Gamma$

$$g = 357.5166 + .98560026 * ET \qquad (33)$$

Eccentricity of earth's orbit

$$e = .016720 - .000\ 000\ 0011 * ET \qquad (34)$$

Kepler's equation for eccentric anomaly [where $e' = (e/\pi) * 180$],

$$E - e'\sin(E) = g \qquad (35)$$

Equation for true anomaly, ν

$$\tan(\nu/2) = \sqrt{(1 + e)/(1 - e)}\ \tan(E/2) \qquad (36)$$

Longitude of sun, λ, is longitude of perigee plus true anomaly,

$$\lambda = \Gamma + \nu \tag{37}$$

True distance between sun and earth in astronomical units

$$R = 1 - e \cos (E) = (1 - e^2)/[1 + e \cos (\nu)] \tag{38}$$

Semi-diameter of sun as seen from earth, .2670/R.

Parallax, semi-diameter of earth as seen from sun, .0024/R.

Aberration, to be subtracted from sun's longitude, .0057/R.

Mean obliquity of ecliptic

$$\varepsilon = 23.4425 - .000\ 000\ 36 * ET \tag{39}$$

Longitude of the mean ascending node of the lunar orbit on the ecliptic, measured from mean equinox of date,

$$\Omega = 248.59 - .052954 * ET \tag{40}$$

Nutation in obliquity $\Delta\varepsilon = +.0026 \cos (\Omega)$

Nutation in longitude $\Delta\lambda = -.0048 \sin (\Omega)$

Conversion from true celestial longitude of the sun, λ, to declination and right ascension

$$\delta = \sin^{-1} (\sin \lambda \ \ \sin \varepsilon) \tag{41}$$

$$\alpha = \tan^{-1} (\tan \lambda \ \ \cos \varepsilon) \tag{42}$$

At this point we have to calculate the geographical position of the subsolar point. To do this we must calculate the rotational position of a reference meridian. Let UT be the time in universal days since 1975/01/01 00:00:00 UT. (Note that there is a difference, ΔT, between ephemeris time and universal time -- in 1975, it was around + 45.6 seconds). Let γ be the mean Greenwich hour angle of the equinox then,

$$\gamma = 100.0215 + 360.98564734 * UT \tag{43}$$

(This calculation is best done by splitting UT into an integer and a fractional part to avoid loss of precision). The true hour angle can be calculated by adding $\Delta\lambda * \cos \varepsilon$ to the mean hour angle for nutation.

The longitude, θ', of the subsolar point is then

$$\theta' = \alpha - \gamma \tag{44}$$

while the latitude, ϕ', equals the declination δ.

If the observer is at longitude θ and latitude ϕ, he will see the sun at elevation, e, and azimuth, a, where

$$e = \sin^{-1} [\sin \phi \sin \phi' + \cos \phi \cos \phi' \cos (\theta' - \theta)] \tag{45}$$

$$a = \tan^{-1} \left[\frac{\cos \phi' \sin (\theta' - \theta)}{\cos \phi \sin \phi' - \sin \phi \cos \phi' \cos (\theta - \theta')}\right] \tag{46}$$

13. Parallax and Refraction.

The topocentric position of the sun can be calculated from the geocentric position by taking into account the lateral displacement of the observer from the line connecting the centers of the earth and the sun. This is best accomplished by subtracting from the elevation the parallax times the cosine of the elevation. Parallax, as calculated in the previous section, is the semi-diameter of the earth as seen from the sun and amounts to only about 8.794 seconds of arc.

Finally, one has to take into account the refraction of light rays entering the atmosphere. If one ignores the curvature of the earth, it is easy to see that the elevation, e', of a celestial object when viewed through the atmosphere is related to the elevation, e, which the same object would have in the absence of the atmosphere by Snell's law. So

$$\cos e' = \cos e/\mu \tag{47}$$

where μ is the refractive index of air. The refractive index equals one plus a small constant times the density and is thus a function of the pressure and temperature. At normal pressure and temperature, the refractive index is approximately 1.0002824. Note that the apparent elevation, e', is greater than the geometric elevation, e, by an amount which increases with decreasing elevation. For an elevation of 45° the difference is about a minute of arc.

For small elevations, the curvature of the earth can no longer be neglected, and the above equation is not sufficiently accurate. The tables in the Nautical Almanac [4] can then be used. For elevations above 10° these are approximately equal to

$$e' - e = 0.016167 \tan (90^\circ - e') + .0000169 \tan^3 (90^\circ - e') \qquad (48)$$

For very small elevations even this is not accurate enough. For example, when the apparent elevation is zero, the geometric elevation is about minus 34 minutes of arc.

14. References.

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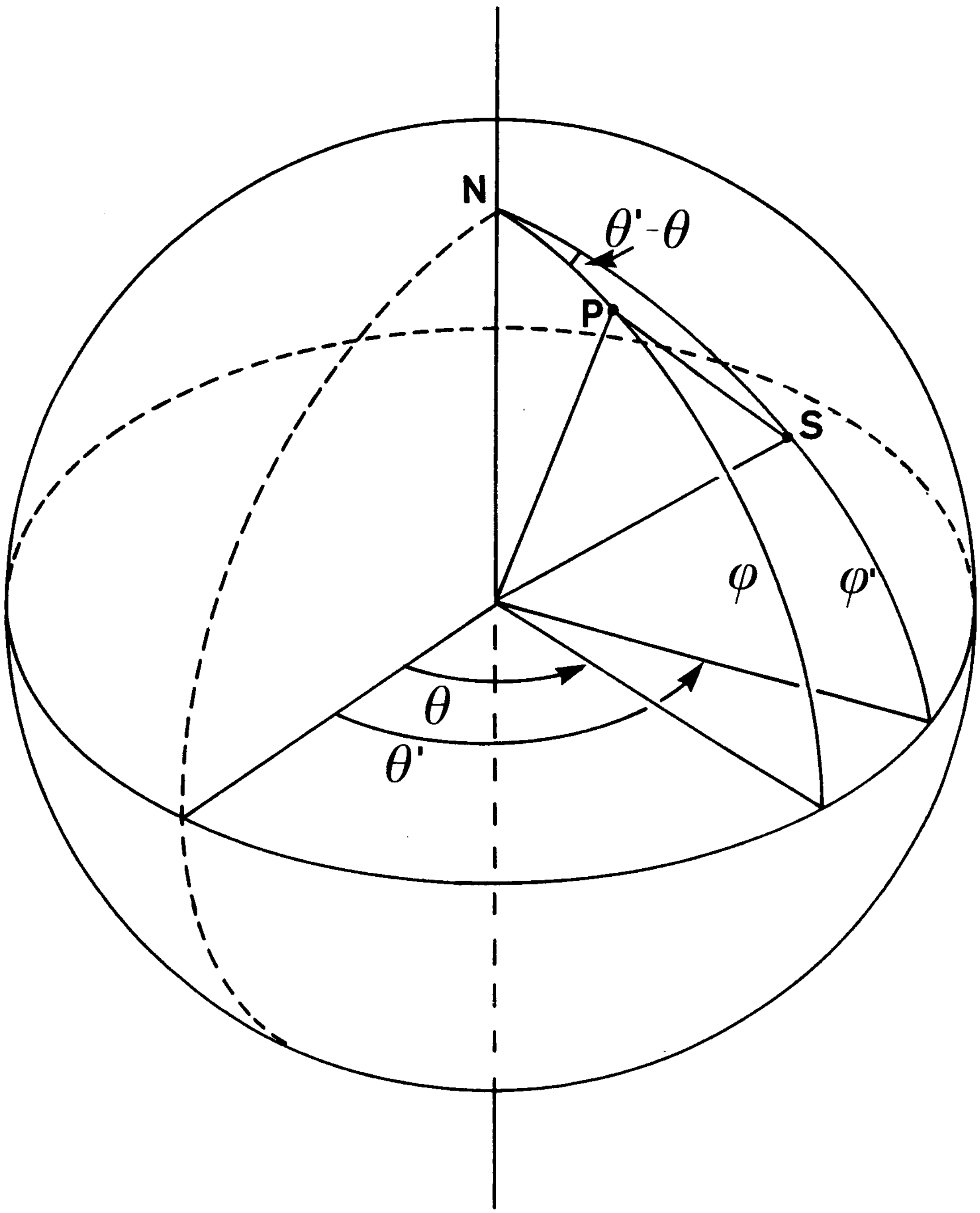


FIGURE 1a

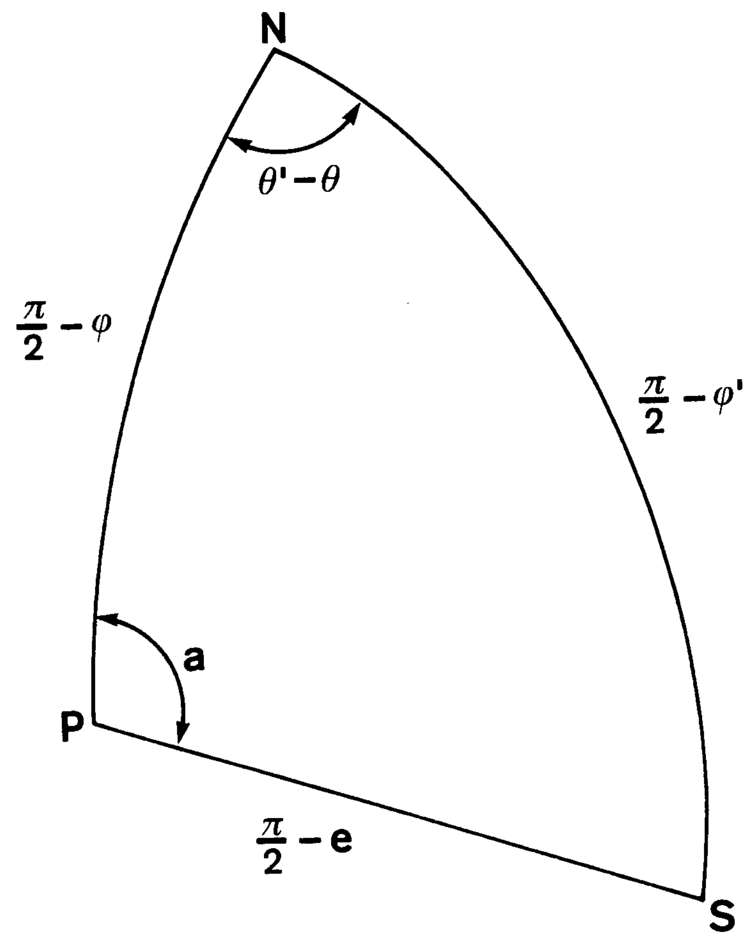


FIGURE 1b

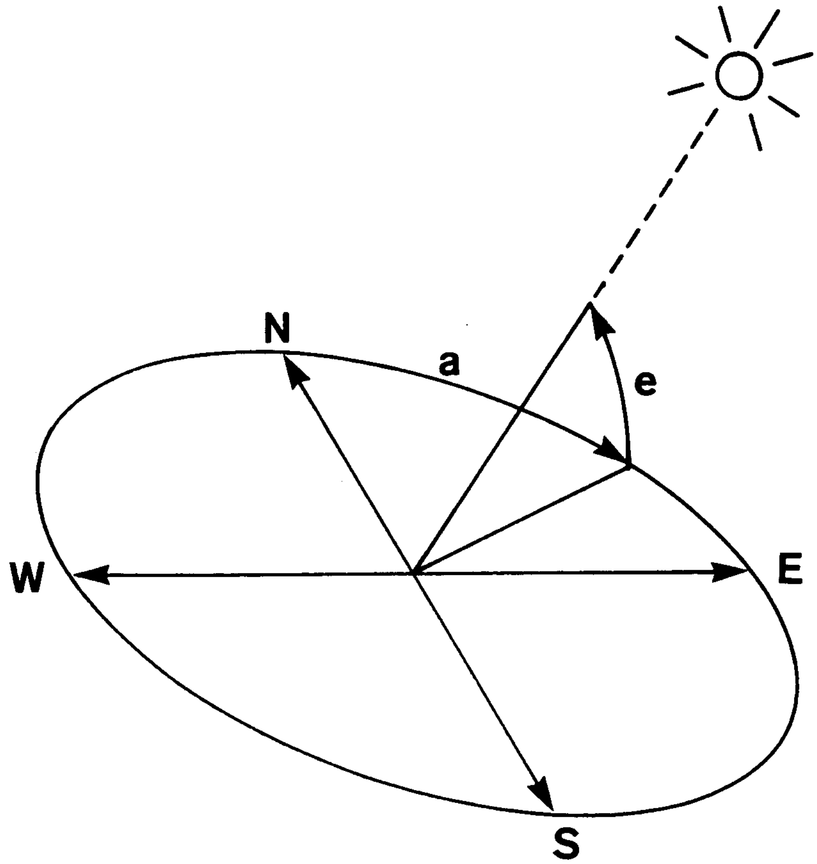


FIGURE 2

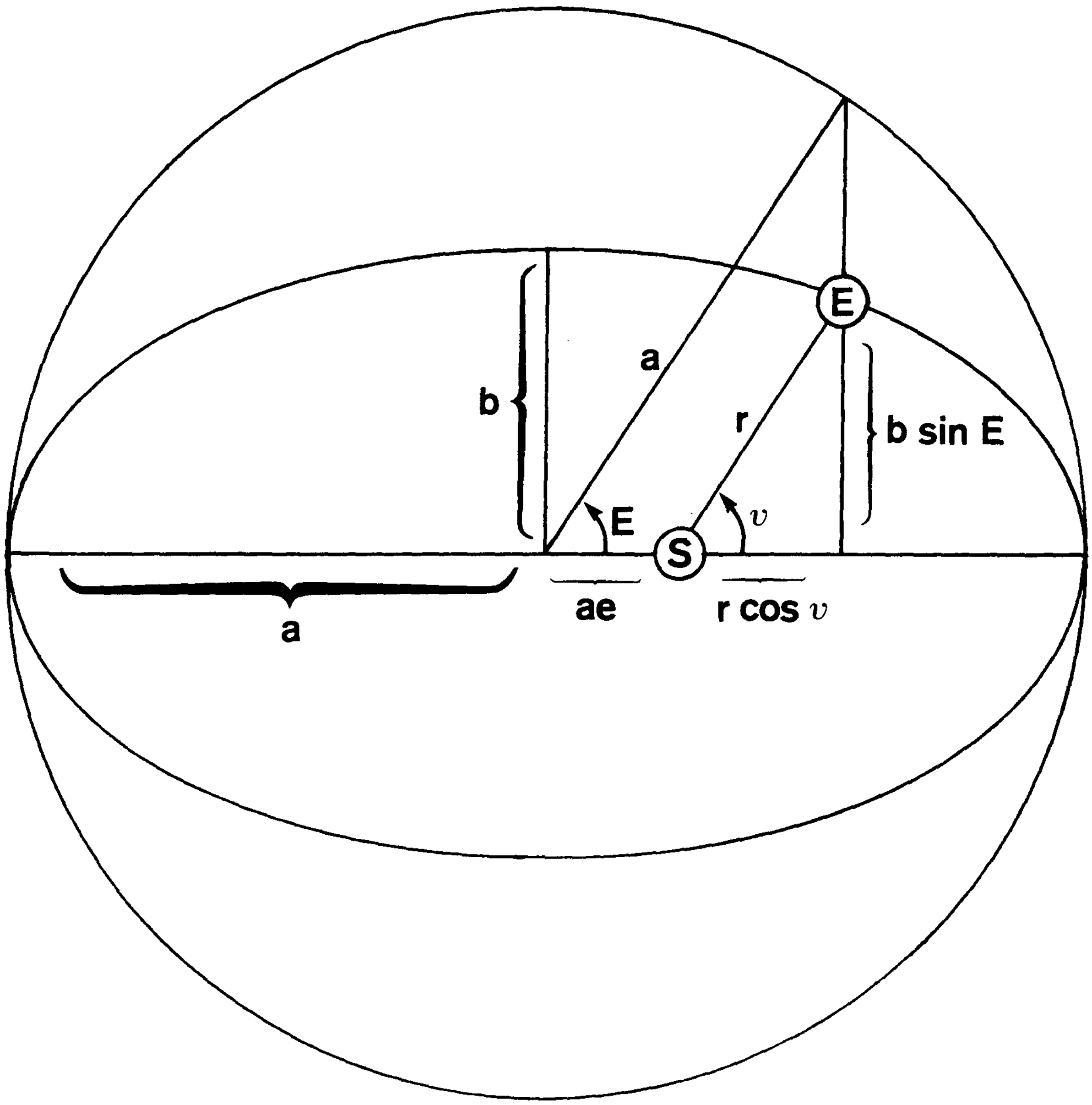


FIGURE 3

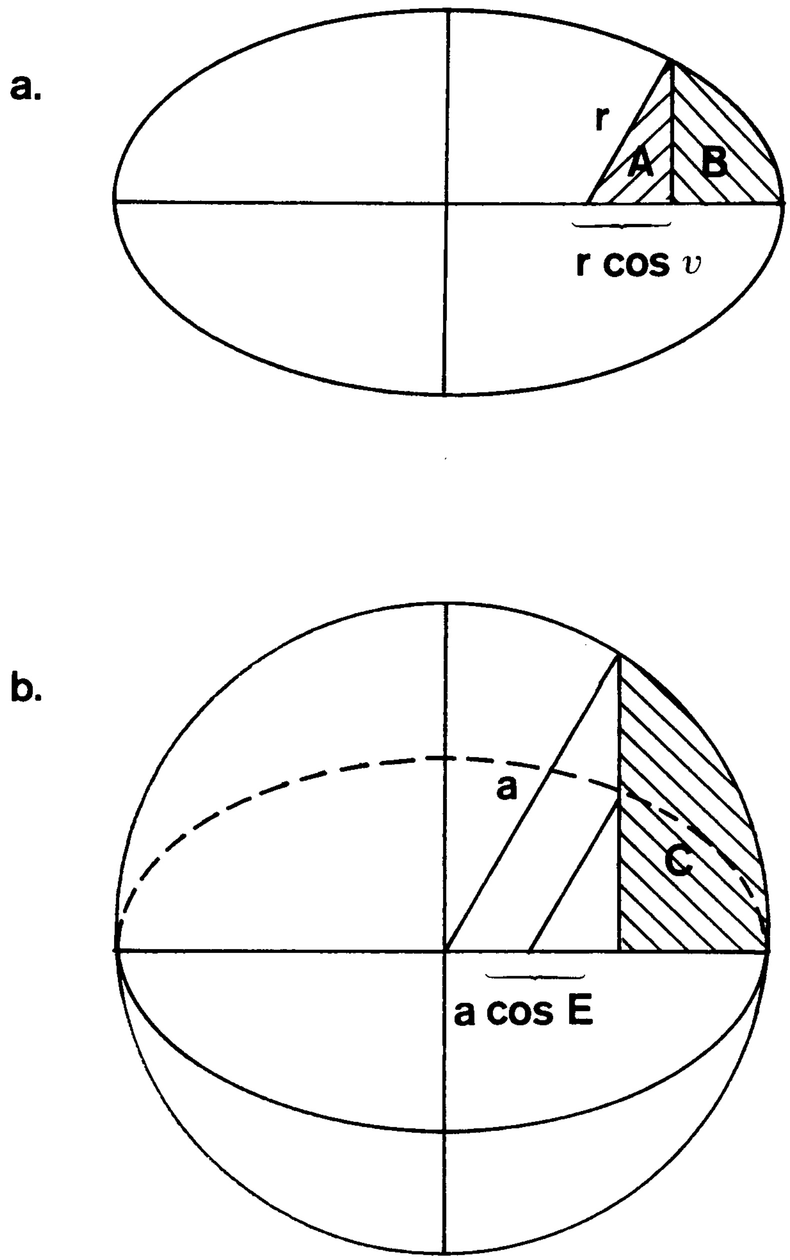


FIGURE 4

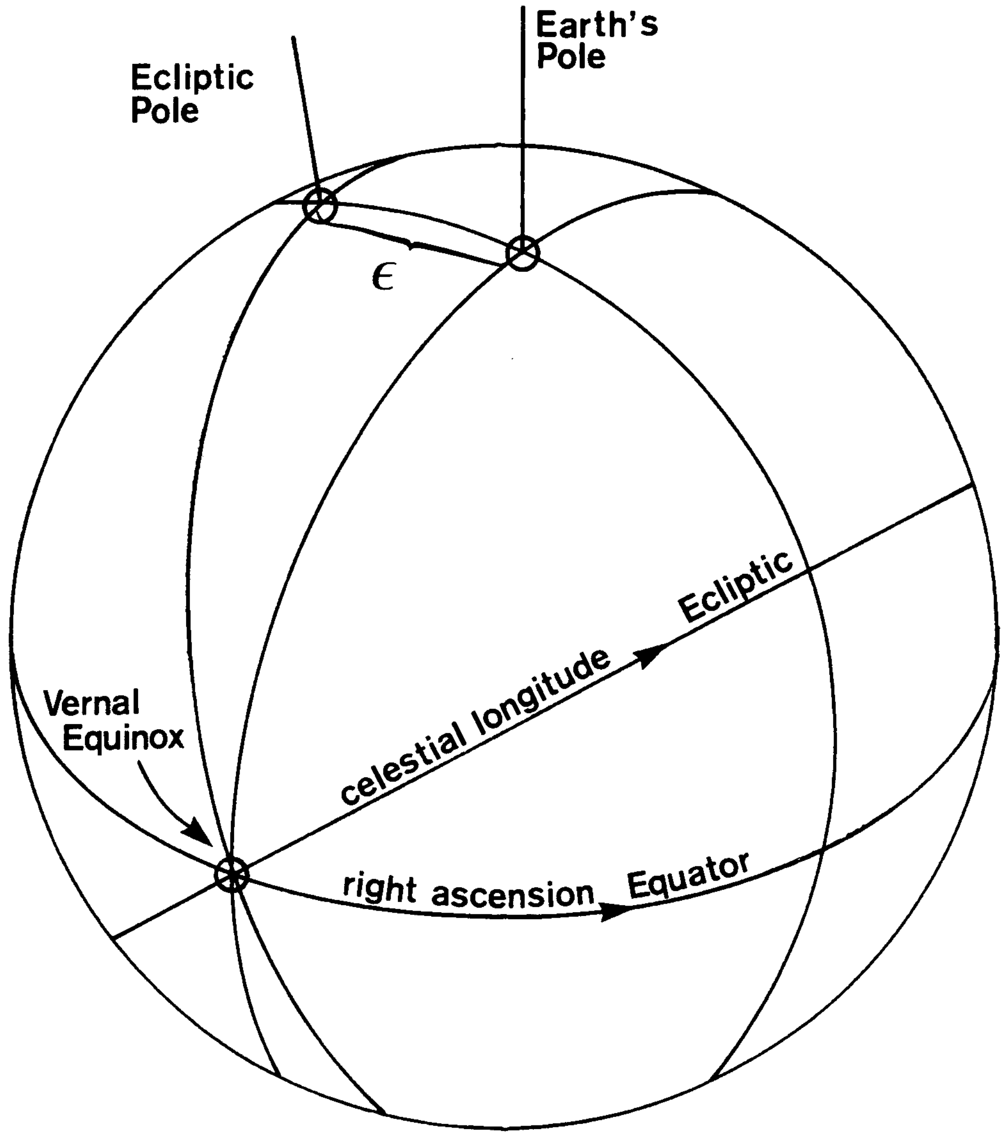

Ecliptic
Pole
Earth's
Pole
ε
Ecliptic
celestial longitude
Vernal
Equinox
right ascension
Equator

FIGURE 5a

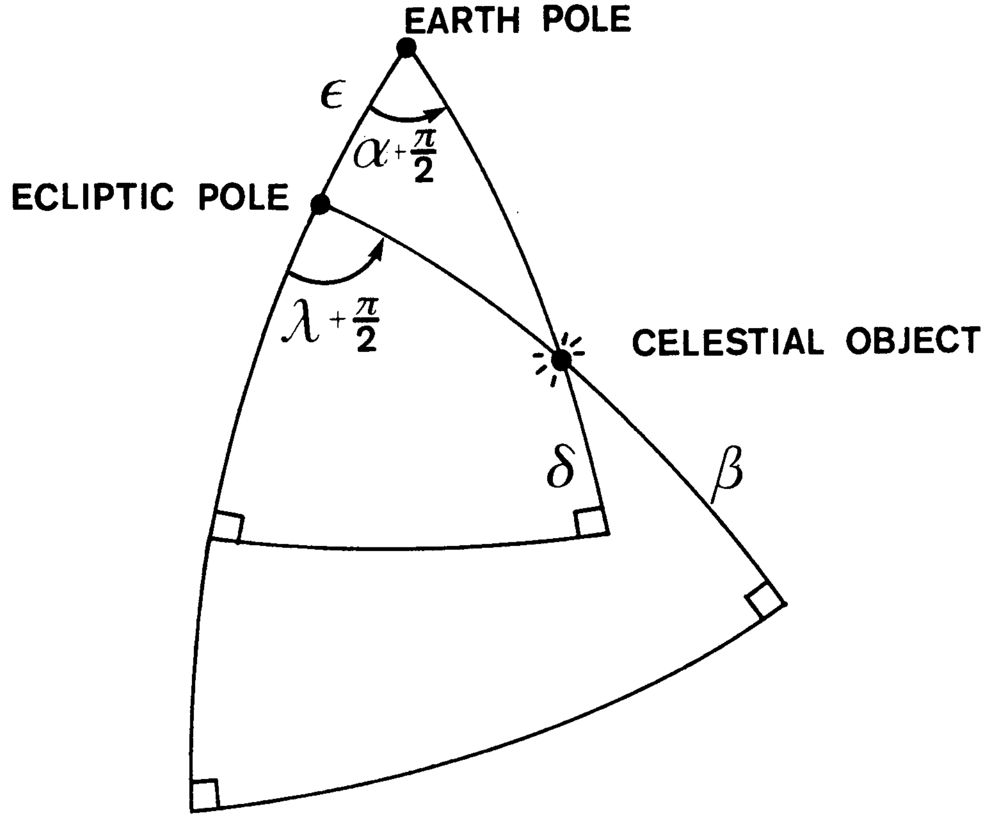
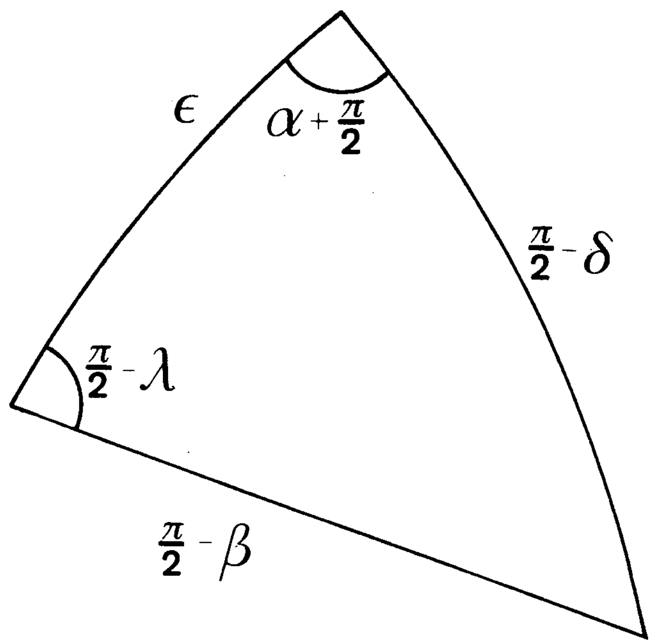

EARTH POLE
ϵ
$\alpha + \frac{\pi}{2}$
ECLIPTIC POLE
$\lambda + \frac{\pi}{2}$
CELESTIAL OBJECT
δ
β

FIGURE 5b



ϵ
$\alpha + \frac{\pi}{2}$
$\frac{\pi}{2} - \delta$
$\frac{\pi}{2} - \lambda$
$\frac{\pi}{2} - \beta$

FIGURE 5c