MASSACHUSETTS INSTITUTE OF TECHNOLOGY

DEPARTMENT OF COMPUTER SCIENCE AND ELECTRICAL ENGINEERING

6.881 COMPUTATIONAL IMAGING

**HWP 3** **Handed out: 2006 Apr 13th** **Due: 2006 Apr 20th**

**Problem 1:** Here we prove a remarkable property of quadratic residue masks. First, the Legendre symbol

$$\left(\frac{k}{n}\right)$$

is defined to be $+1$ when $k$ is a quadratic residue mod $n$ (i.e. if there exists $i$ such that $k = i^2 (\text{mod } n)$), and $-1$ if $k$ is not a quadratic residue. The Legendre symbol has value 0 if $k$ is divisible by $n$, e.g. if $k = 0$. (See MathWorld web site for additional details if needed). Suppose now that the mask size $p$, is a prime.

(a) Show that

$$\left(\frac{n-k}{n}\right) = +\left(\frac{k}{n}\right) \quad \text{for} \quad p = 4n+1$$
$$\left(\frac{n-k}{n}\right) = -\left(\frac{k}{n}\right) \quad \text{for} \quad p = 4n+3$$

That is, the cyclical sequence generated using the Legendre symbol is even when $p$ is a prime of the form $4n+1$, while it is odd when $p$ is a prime of the form $4n+3$. Correspondingly, the Disrete Fourier Transform (DFT) will be real for $p = 4n+1$ and (purely) imaginary for $p = 4n+3$.

(b) Show that if $\{A_l\}$ is the DFT of $\{a_k\}$, then

$$A_l^2 = \begin{cases} +p & \text{when } p = 4n+1 \\ -p & \text{when } p = 4n+3 \end{cases}$$

Hint: if

$$A_l = \sum_{k=0}^{p-1} a_k e^{-j\frac{2\pi kl}{p}}$$

then

$$A_l^2 = \sum_{i=0}^{p-1} a_i e^{-j\frac{2\pi il}{p}} \sum_{j=0}^{p-1} a_j e^{-j\frac{2\pi jl}{p}} = \sum_{i=0}^{p-1}\sum_{j=0}^{p-1} a_i a_j e^{-j\frac{2\pi(i+j)l}{p}}$$

Then separately consider the terms (i) along the diagonal $i = j$, the (ii) cross diagonal $i + j = p$, and (iii) the remainder. For $p = 4n+3$, these add up to $-1$, $-(p-1)$, and 0, respectively, while for $p = 4n+1$, they add up to $-1$, $(p-1)$, and 2.

Note: this doesn't quite prove the amazing property:

$$A_l = \begin{cases} \sqrt{p}a_l & \text{when } p = 4n+1 \\ -j\sqrt{p}a_l & \text{when } p = 4n+3 \end{cases}$$

**Problem 2:** In our discussion of diaphanography (diffuse optical tomography) we have used a resistive grid model for the scattering and absorbing medium. We have also discussed the boundary conditions on the accessible exterior surface of the volume. What are the boundary conditions for the resistive grid model corresponding to:

(a) Ideal totally reflective (matte or specular material applied to the outside of the volume). All photons are returned to the volume.

(b) Ideal totally absorptive (material applied to the outside of the volume, or volume in large dark space). All photons leaving the volume are lost.

(c) Matched termination (material of the same reflectance as a semi-infinite slab of material with same scattering and absorption coefficients). This is equivalent to extending the volume indefinitely.

How is the light level at the photo sensors affected by the choice of boundary condition?

**Problem 3:** Consider a circular disk of radius $R$ in a 2-D version of the diffuse optical tomography (DOT) problem. Suppose we want to estimate the absorption at each point in the disk on a square grid with grid spacing $\epsilon$. Suppose also that the light source positions and the detector positions are evenly spaced around the circle of radius $R$with spacing $\delta$.

(a) From the point of view of collecting at least as many measurements of source-to-detector transmittances as there are unknowns, how large can $\delta$ be relative to $\epsilon$? Keep in mind that reciprocity considerations imply that $N$ source positions and $M$ detector positions yield only $MN/2$ independent measurements. How does the result vary with $R$?

Now consider instead a spherical volume of radius $R$ sampled on a cubic grid with grid spacing $\epsilon$. Source positions are arrayed on the surface in a pattern that yields about $1/\delta^2$ sources per unit area. Detectors are disposed of similarly.

(b) From the point of view of collecting at least as many measurements of source-to-detector transmittances to match the number of unknowns, how large can $\delta$ as a function of $\epsilon$ and $R/\epsilon$? Keep in mind that reciprocity considerations imply that $N$ source positions and $M$ detector positions yield only $MN/2$ independent measurements. How does the largest permitted $\delta$ vary with relative size of the volume (given as $R/\epsilon$)?

**Problem 4:** The diffusion equation approximating radiative transfer in the case where $\mu_s \gg \mu_a$ can be written

$$\Delta\Phi(\mathbf{r}) - k\Phi(\mathbf{r}) + q(\mathbf{r}) = 0$$

where $\Phi(\mathbf{r})$ is the photon density at $\mathbf{r}$, $q(\mathbf{r})$ is a source term proportional to the number of photons injected per unit volume per unit time at $\mathbf{r}$, and

$$k = 3\mu_a\big(\mu_a + (1-f)\mu_s\big)$$

where $\mu_a$ is the linear absorption coefficient, $\mu_s$ is the linear scattering coefficient, and $f$ is the anisotropy factor.

Show that, if the medium is homogeneous, with $k$ constant, and with the source $q(\mathbf{r})$ being zero everywhere except at the origin, a solution of the form

$$\Phi(\mathbf{r}) = \frac{A}{r}e^{-kr}$$

satisfies the equation, with $r = |\mathbf{r}|$ – at least at all points other than the origin.

**Problem 5:** Consider a $2 \times 2$ planar network of resistors connecting nodes labelled 1, 2, 3, 4. We may inject currents at input nodes 1 and 2, and read potentials at output nodes 3 and 4. We are told that the four resistors connecting the nodes have conductances $g_{12}, g_{13}, g_{24}$ and $g_{23}$ — where $g_{ij}$ is the conductance of the resistor connecting node $i$ to node $j$ (Node 1 is not directly connected to node 4, nor is node 2 directly connected to node 3). We are to determine the leakage conductances $g_1, g_2, g_3, g_4$ from each node to ground.

Suppose that we are told that $g_{13} = 1$, $g_{12} = 2$, $g_{24} = 1$ and $g_{34} = 2$.

In a first experiment, we inject current $I_1 = 1$ into node 1 and observe the voltages $V_3 = 23/209$ and $V_4 = 20/209$ at nodes 3 and 4 respectively. In a second experiment, we inject current $I_2 = 1$ into node 2 and observe the voltages $V_3 = 16/209$ and $V_4 = 23/209$ at nodes 3 and 4 respectively. The "inverse task" is to determine $g_1$, $g_2$, $g_3$ and $g_4$ — corresponding to optical absorption at the grid points — from these measurements.

(a) Show that $g_1 = 1$, $g_2 = 2$, $g_3 = 2$ and $g_4 = 1$ will produce the observed measurements.

(b) Would experiments where currents was injected at nodes 3 and 4 and potentials read at nodes 1 and 2 yield any additional information?