University of California at Berkeley

Department of Computer Science and Electrical Engineering

CS-280 Computer Vision

Handed out: 2002 Sep. 18$^{\text{th}}$ **Due on: 2001 Sep. 25$^{\text{th}}$**

Problem 1: Consider the simple image brightness pattern

$$E(x,y) = \frac{1}{2}\left(x^2 + y^2\right)$$

(a) What are the shapes of the isophotes?

(b) Find the brightness gradient, and compute the coefficients of the 2×2 matrix

$$\begin{pmatrix} \iint E_x^2 & \iint E_x E_y \\ \iint E_y E_x & \iint E_y^2 \end{pmatrix}$$

when the rectangular image region under consideration has lower left corner (x_s, y_s) and upper right corner (x_e, y_e).

(c) Remove the common factor $A = (x_e - x_s)(y_e - y_s)$ from the matrix. What is the geometrical significance of A? Show that the matrix simplifies if we place the rectangle symmetrically about the x-axis or y-axis.

(d) What are the eigenvalues and eigenvectors of the 2×2 coefficient matrix?

(e) Explain the significance of the direction of the eigenvectors and the relative magnitudes of the corresponding eigenvalues in terms of the ability to locate the brightness pattern in the rectangular window in the presence of noise.

Problem 2: In the discrete version of the optical flow problem we minimize

$$\sum_i \sum_j \left(u_{i,j}E_x + v_{i,j}E_y + E_t\right)^2$$
$$+\frac{\lambda}{\epsilon^2}((u_{i+1,j} - u_{i,j})^2 + (u_{i,j-1} - u_{i,j})^2(v_{i+1,j} - v_{i,j})^2 + (v_{i,j-1} - v_{i,j})^2)$$

Differentiating w.r.t $u_{k,l}$ and $v_{k,l}$ and setting the results equal to zero yields:

$$(u_{k,l}E_x + v_{k,l}E_y + E_t)E_x = \frac{4\lambda}{\epsilon^2}(\overline{u}_{k,l} - u_{k,l})$$
$$(u_{k,l}E_x + v_{k,l}E_y + E_t)E_y = \frac{4\lambda}{\epsilon^2}(\overline{v}_{k,l} - v_{k,l})$$

where $\overline{u}_{k,l}$ and $\overline{v}_{k,l}$ are local averages of u and v. (Note: the formulation here differs from that in the book in that the regularizer λ appears in a different place).

(a) Rewrite the above as two linear equations in the unknowns $u_{k,l}$ and $v_{k,l}$ by moving everything else to the right hand side of the equations (including the local averages of u and v). Show that the determinant of the 2×2 coefficient matrix equals

$$\frac{4\lambda}{\epsilon^2}\left(E_x^2 + E_y^2 + \frac{4\lambda}{\epsilon^2}\right)$$

When is the determinant zero? How is the stability of the solution of the two equations affected by the local brightness gradient?

(b) Solve the two equations for $u_{k,l}$ and $v_{k,l}$ assuming that the neighboring values of u and v are fixed. Show that

$$u_{k,l} = \overline{u}_{k,l} - \frac{E_x\overline{u}_{k,l} + E_y\overline{v}_{k,l} + E_t}{E_x^2 + E_y^2 + (4\lambda/\epsilon^2)}E_x$$

$$v_{k,l} = \overline{v}_{k,l} - \frac{E_x\overline{u}_{k,l} + E_y\overline{v}_{k,l} + E_t}{E_x^2 + E_y^2 + (4\lambda/\epsilon^2)}E_y$$

(c) How is the local value of optical flow velocity (u, v) determined in areas where brightness is constant?

(d) In which direction are local adjustments in (u, v) driven by non-zero values of the brightness change constraint expression

$$E_x\overline{u}_{k,l} + E_y\overline{v}_{k,l} + E_t$$

(e) Explain why $\lambda = k(\epsilon^2/4)\sigma^2$ might be a reasonable value to try for the regularizing parameter, where σ is the standard deviation of the brightness gradient and k is some constant between zero and one.

Problem 3: Motion in the image plane at a point can be fully specified by a vector with two components. Measurement of brightness and its first spatial and time derivatives at an image point, however, provides only one constraint on these two components, so one is forced to consider an image region, rather than just a single point, in estimation of the optical flow. The larger the region, the less reasonable it is to assume that the optical flow is constant within it. One of the ways of dealing with this is to introduce more parameters to describe the flow within the region.

(a) We may, for example, allow linear variation of flow velocity with position in the patch:

$$u = a + bx + cy \quad \text{and} \quad v = d + ex + fy.$$

There are now six unknown parameters, a, b, c, d, e, and f. Derive six linear equations in these six unknowns based on minimization of

$$\iint (u\,E_x + v\,E_y + E_t)^2 dx\,dy.$$

(b) Determine whether the symmetrical coefficient matrix becomes singular when the brightness gradient has the same direction at all points in the patch, as it does in the simple case when u and v are considered to be constant.

Problem 4: The motion field is particularly simple when a camera is moving without rotation in a fixed environment. All parts of the image then appear to be streaming away from the "focus of expansion"—the projection of the translational motion vector into the image plane.

If the translational motion is $(U, V, W)^T$, then the focus of expansion is at

$$\frac{1}{f}(x_0, y_0) = \frac{1}{W}(U, V).$$

Show that

$$(x - x_0)E_x + (y - y_0)E_y = 0$$

at "critical" points, that is where $E_t = 0$.

Estimate the position of the FOE by minimizing

$$\sum_{i=1}^{n} \left((x_i - x_0)E_{x_i} + (y_i - y_0)E_{y_i}\right)^2,$$

where (x_i, y_i) are the positions of the "critical" points in the image, while (E_{x_i}, E_{y_i}) are the brightness gradients at these points.

Problem 5: The formulation of the optical flow problem as above uses as a measure of "unsmoothness" the sum of squares of first partial derivatives of the components of the optical flow. Consider a sphere of radius R rotating about an axis parallel to the y-axis of the image plane with angular velocity ω. Suppose the sphere is far away from the camera (in relation to its radius) and we approximate perspective projection by parallel projection ("scaled orthography") where

$$x = (f/Z_0)X \quad y = (f/Z_0)Y$$

with $(x, y, f)^T$ being the image coordinates corresponding to the scene point $(X, Y, Z)^T$. Find the motion field components $u(x, y)$ and $v(x, y)$. Where does the measure of "unsmoothness" become large (or even singular)?