Inherent Signal-to-Noise of Random Mask

Well designed coded apertures have an auto-correlation that is an impulse at zero offset plus a perfectly constant background value for other offsets. Correspondingly, their Fourier transform has an impulse at zero frequency plus a perfectly constant background value. However, methods for designing such ideal coded apertures are limited. For example, generally such methods are available only for certain specific sizes of masks and only for specific "fill ratios" (typically approximately 50%). Random masks do not have such constraints.

Note also, that random masks can be incrementally modified in an optimization procedure that flattens out bumps in either the auto-correlation or the power spectrum, as desired. Hence it is of interest to determine what the general properties of random masks are.

Suppose we have n places where a hole may occur with probability ρ. The probability that we have exactly k holes is

$$p_k = \binom{n}{k}\rho^k(1-\rho)^{n-k}$$

As a check, we see that

$$\sum_{k=0}^{n} p_k = \sum_{k=0}^{n}\binom{n}{k}\rho^k(1-\rho)^{n-k} = \big(\rho + (1-\rho)\big)^n = 1$$

and

$$\mu = \sum_{k=0}^{n} k p_k = \rho n \sum_{k=1}^{n}\binom{n-1}{k-1}\rho^{k-1}(1-\rho)^{(n-1)-(k-1)} = \rho n((\rho+(1-\rho))^{n-1} = \rho n$$

The expected value of the number of holes is, not suprisingly, $\mu = \rho n$. Next

$$\sum_{k=0}^{n} k^2 p_k = \rho n(\rho(n-1)+1)$$

and so the variances is

$$\sigma^2 = \sum_{k=0}^{n} k^2 p_k - \mu^2 = \rho(1-\rho)n$$

Note that the variance for hole probability $\rho' = (1-\rho)$ is the same as for hole probability ρ. This makes sense because the variance in the number of holes must equal the variance in the number of places where we did not put holes (since the two add up to the constant n_t). The standard deviation from the mean is $\sigma = \sqrt{\rho(1-\rho)n}$.

Now consider a grid with n_t grid positions, in each of which a hole may occur with probability ρ. Then the expected number of holes is ρn_t.

When we line up a mask with the same pattern, we can, of course, look through a total of ρn_t holes.

When the mask is *not* aligned, we have ρn_t holes which each may line up with a hole with probability ρ. The expected number of places where holes line up is $\rho(\rho n_t) = \rho^2 n_t$. The "signal" is the difference between the number of holes in the pattern and this "pedestal" or "background" value, hence

$$S = \rho n_t - \rho^2 n_t = \rho(1-\rho)n_t$$

The variance in this background value is, from the above,

$$\sigma^2 = \rho(1-\rho)n = \rho(1-\rho)(\rho n_t) = \rho^2(1-\rho)n_t$$

since the number of places where a hole may line up with another is now $n = \rho n_t$ (*not* n_t). As a result the standard deviation in the background is

$$N = \rho\sqrt{(1-\rho)n_t}$$

Hence the signal-to-noise ratio is

$$S/N = \frac{\rho(1-\rho)n_t}{\rho\sqrt{(1-\rho)}\sqrt{n_t}} = \sqrt{1-\rho}\sqrt{n_t}$$

not simply $\sqrt{n_t}$.