## Sample Resistive Grid Inversion — Scattering and Absorption Model

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Here we consider about the simplest possible case of the two-d resistive sheet to get some insight into the more general problem.

(draw your own picture here :)

There are four nodes, arranged in $N = 2$ rows and $M = 2$ columns. On the left are the two nodes used for input ((1) and (2)) and on the right are the two output nodes ((3) and (4)). Four 'horizontal' resistors with conductance $g_{12}$, $g_{13}$, $g_{24}$ and $g_{34}$ connect these four nodes. These resistors represent scattering, and are assumed to be of known value. There is also a 'vertical' leakage path from each of the four nodes to ground — with conductances $g_1$, $g_2$, $g_3$ and $g_4$. These represent absorption, and are the unknowns.

We are to recover the values of the unknown leakage resistors. We perform two experiments: First we inject current at node (1) and measure the potentials on nodes (3) and (4). Call the 'trans-impedance' (ratio of output potential to injected current) observed this way $R_{3,1}$ and $R_{4,1}$. Then we inject instead current at node (2) and again measure the potential on nodes (3) and (4). Call the 'trans-impedance' observed this way $R_{3,2}$ and $R_{4,2}$.

If the grid was $N \times M$ instead of $2 \times 2$ then we would have performed $N$ experiments, each time injecting current on one of the $N$ input nodes and reading out the potential on each of the $N$ output nodes. We then try to recover the $N \times M$ unknown leakage conductances to ground. Clearly there is not enough constraint if $M > N$ since there are then more unknowns than measurements. Conversely if $M < N$, we have redundant information and may want to use a least squares approach to obtain the best possible answer.

Here we deal with the simple case where $M = N = 2$. The node equations in this case are:

$$\begin{aligned}
I_1 &= g_1 V_1 + g_{13}(V_1 - V_3) + g_{12}(V_1 - V_2)\\
I_2 &= g_2 V_2 + g_{12}(V_2 - V_1) + g_{24}(V_2 - V_4)\\
I_3 &= g_3 V_3 + g_{13}(V_3 - V_1) + g_{34}(V_3 - V_4)\\
I_4 &= g_4 V_4 + g_{24}(V_4 - V_2) + g_{34}(V_4 - V_3)
\end{aligned}$$

or

$$\begin{pmatrix}
(g_1 + g_{13} + g_{12}) & -g_{12} & \vdots & -g_{13} & 0\\
-g_{12} & (g_2 + g_{12} + g_{24}) & \vdots & 0 & -g_{24}\\
\cdots & \cdots & \cdot & \cdots & \cdots\\
-g_{13} & 0 & \vdots & (g_3 + g_{13} + g_{34}) & -g_{34}\\
0 & -g_{24} & \vdots & -g_{34} & (g_4 + g_{24} + g_{34})
\end{pmatrix}
\begin{pmatrix} V_1\\ V_2\\ \cdots\\ V_3\\ V_4 \end{pmatrix}
=
\begin{pmatrix} I_1\\ I_2\\ \cdots\\ I_3\\ I_4 \end{pmatrix}$$

Note that all off diagonal elements are negative, and that the unknown leakage conductances all appear on the diagonal. Also, the matrix becomes singular if all leakages conductances are set to zero, since then each row adds up to zero.

Making use of the partitioning indicated above, we can write

$$\begin{pmatrix} G_{11} & \vdots & G_{12}\\ \cdots & \cdot & \cdots\\ G_{21} & \vdots & G_{22} \end{pmatrix}
\begin{pmatrix} V_1\\ V_2\\ \cdots\\ V_3\\ V_4 \end{pmatrix}
=
\begin{pmatrix} I_1\\ I_2\\ \cdots\\ I_3\\ I_4 \end{pmatrix}$$

This partitioning is convenient, since in the experiments we always have $I_3 = 0$ and $I_4 = 0$, and since $V_1$ and $V_2$ are not known.

If we invert this set of equations we get:

$$\begin{pmatrix} C_{11} & \vdots & C_{12}\\ \cdots & \cdot & \cdots\\ C_{21} & \vdots & C_{22} \end{pmatrix}
\begin{pmatrix} I_1\\ I_2\\ \cdots\\ I_3\\ I_4 \end{pmatrix}
=
\begin{pmatrix} V_1\\ V_2\\ \cdots\\ V_3\\ V_4 \end{pmatrix}$$

where

$$
\begin{aligned}
C_{11} &= (G_{11} - G_{12}G_{22}^{-1}G_{21})^{-1} \\
C_{21} &= -G_{22}^{-1}G_{21}C_{11} \\
C_{22} &= (G_{22} - G_{21}G_{11}^{-1}G_{12})^{-1} \\
C_{12} &= -G_{11}^{-1}G_{12}C_{22}
\end{aligned}
$$

Upon multiplying out we obtain

$$C_{11}\begin{pmatrix} I_1 \\ I_2 \end{pmatrix} = \begin{pmatrix} V_1 \\ V_2 \end{pmatrix}$$

where the term involving $C_{12}$ drops out because $I_3 = 0$ and $I_4 = 0$. This equation is of no interest since we don't know $V_1$ and $V_2$. But we also obtain

$$C_{21}\begin{pmatrix} I_1 \\ I_2 \end{pmatrix} = \begin{pmatrix} V_3 \\ V_4 \end{pmatrix}$$

where the term involving $C_{22}$ drops out because $I_3 = 0$ and $I_4 = 0$. The four unkowns $(g_1, g_2, g_3, g_4)$ occur in the matrix $C_{21}$, but we obviously can't solve for them using a single set of measurements. We can however, combine two sets of measurements and obtain:

$$C_{21}\begin{pmatrix} I_{1,1} & I_{1,2} \\ I_{2,1} & I_{2,2} \end{pmatrix} = \begin{pmatrix} V_{3,1} & V_{3,2} \\ V_{4,1} & V_{4,2} \end{pmatrix}$$

where typically we would choose $I_{1,1} = 1$, $I_{2,1} = 0$ for the first experiment and $I_{1,2} = 0$, $I_{2,2} = 1$ for the second. We then obtain

$$C_{21} = \begin{pmatrix} R_{3,1} & R_{3,2} \\ R_{4,1} & R_{4,2} \end{pmatrix}$$

where, provided $I_{2,1} = 0$ and $I_{1,2} = 0$, $R_{3,1} = V_{3,1}/I_{1,1}$, $R_{4,1} = V_{4,1}/I_{1,1}$, and $R_{3,2} = V_{3,2}/I_{1,2}$, $R_{4,2} = V_{4,2}/I_{1,2}$. So from image measurements we can recover the matrix $C_{21}$, and we know that

$$C_{21} = -G_{22}^{-1}G_{21}C_{11}$$

or

$$C_{21}C_{11}^{-1} = -G_{22}^{-1}G_{21}$$

then, since

$$C_{11} = (G_{11} - G_{12}G_{22}^{-1}G_{21})^{-1}$$

we obtain

$$C_{21}(G_{11} - G_{12}G_{22}^{-1}G_{21}) = -G_{22}^{-1}G_{21}$$

We need to manipulate this some more to try and isolate the two matrices $G_{11}$ and $G_{22}$, which contain the unknowns $g_1$, $g_2$, $g_3$, and $g_4$. We see that

$$C_{21}G_{11} = C_{21}G_{12}G_{22}^{-1}G_{21} - G_{22}^{-1}G_{21}$$

or

$$C_{21}G_{11}G_{21}^{-1}G_{22} = C_{21}G_{12} - I$$

or finally

$$G_{11}G_{21}^{-1}G_{22} = G_{12} - C_{21}^{-1}$$

Here $C_{21}^{-1}$ is obtained from experimental measurements, while $G_{11}$ and $G_{22}$ contain the unknown leakage conductances. In our simple example, $G_{12}$ and $G_{21}$ are diagonal.

## Numerical Example

Suppose that the 'horizontal' resistors have conductance as follows: $g_{13} = 1$, $g_{12} = 2$, $g_{24} = 1$, and $g_{34} = 2$. Next assum that the leakage or 'vertical' resistors have conductance $g_1 = 1$, $g_2 = 2$, $g_3 = 2$, and $g_4 = 1$. Then the conductance matrix is

$$\begin{pmatrix} 4 & -2 & \vdots & -1 & 0 \\ -2 & 5 & \vdots & 0 & -1 \\ \cdots & \cdots & \cdot & \cdots & \cdots \\ -1 & 0 & \vdots & 5 & -2 \\ 0 & -1 & \vdots & -2 & 4 \end{pmatrix}$$

and so

$$G_{11}^{-1} = \frac{1}{16}\begin{pmatrix} 5 & 2 \\ 2 & 4 \end{pmatrix} \quad \text{and} \quad G_{22}^{-1} = \frac{1}{16}\begin{pmatrix} 4 & 2 \\ 2 & 5 \end{pmatrix}$$

and so

$$G_{12}G_{22}^{-1}G_{21} = G_{22}^{-1} = \frac{1}{16}\begin{pmatrix} 4 & 2 \\ 2 & 5 \end{pmatrix}$$

while

$$G_{21}G_{11}^{-1}G_{12} = G_{11}^{-1} = \frac{1}{16}\begin{pmatrix} 5 & 2 \\ 2 & 4 \end{pmatrix}$$

so

$$G_{11} - G_{12}G_{22}^{-1}G_{21} = \frac{1}{16}\begin{pmatrix} 60 & -34 \\ -34 & 75 \end{pmatrix}$$

and

$$G_{22} - G_{21}G_{11}^{-1}G_{12} = \frac{1}{16}\begin{pmatrix} 75 & -34 \\ -34 & 60 \end{pmatrix}$$

So in the inverse we have

$$C_{11} = \frac{1}{209}\begin{pmatrix} 75 & 34 \\ 34 & 60 \end{pmatrix} \quad \text{and} \quad C_{22} = \frac{1}{209}\begin{pmatrix} 60 & 34 \\ 34 & 75 \end{pmatrix}$$

and

$$C_{21} = \frac{1}{209}\begin{pmatrix} 23 & 16 \\ 20 & 23 \end{pmatrix} \quad \text{and} \quad C_{12} = \frac{1}{209}\begin{pmatrix} 23 & 20 \\ 16 & 23 \end{pmatrix}$$

Finally

$$G^{-1} = C = \frac{1}{209}\begin{pmatrix} 75 & 34 & \vdots & 23 & 20 \\ 34 & 60 & \vdots & 16 & 23 \\ \cdots & \cdots & \cdot & \cdots & \cdots \\ 23 & 16 & \vdots & 60 & 34 \\ 20 & 23 & \vdots & 34 & 75 \end{pmatrix}$$

In the first experiment we have $I_1 = 1$ and the other node currents are zero so

$$\begin{pmatrix} V_1 \\ V_2 \\ V_3 \\ V_4 \end{pmatrix} = \frac{1}{209}\begin{pmatrix} 75 \\ 34 \\ 23 \\ 20 \end{pmatrix}.$$

In the second experiment we have $I_2 = 1$ and the other node currents are zero so

$$\begin{pmatrix} V_1 \\ V_2 \\ V_3 \\ V_4 \end{pmatrix} = \frac{1}{209}\begin{pmatrix} 34 \\ 60 \\ 16 \\ 23 \end{pmatrix}.$$

Note that we can only measure $V_3$ and $V_4$ in each case. This is the end of the 'forward' problem (finding trans-impedance given leakage conductances).

The 'inverse' task is to recover the unknown leakage conductances. Extracting the relevant parts from the above 'experimental' data we see that

$$C_{21} = \frac{1}{209}\begin{pmatrix} 23 & 16 \\ 20 & 23 \end{pmatrix}.$$

so

$$C_{21}^{-1} = \begin{pmatrix} 23 & -16 \\ -20 & 23 \end{pmatrix}.$$

so

$$G_{12} - C_{21}^{-1} = \begin{pmatrix} -24 & 16 \\ 20 & 24 \end{pmatrix}.$$

and

$$G_{11} = \begin{pmatrix} g_1+3 & -2 \\ -2 & g_2+3 \end{pmatrix} \quad \text{and} \quad G_{22} = \begin{pmatrix} g_3+3 & -2 \\ -2 & g_4+3 \end{pmatrix}.$$

So

$$G_{11}G_{21}^{-1}G_{22} = \begin{pmatrix} g_1+3 & -2 \\ -2 & g_2+3 \end{pmatrix}\begin{pmatrix} g_3+3 & -2 \\ -2 & g_4+3 \end{pmatrix}.$$

So that we get the following equations in the unknown leakage conductances:

$$\begin{aligned}
(g_1+3)(g_3+3)+4 &= 24 \\
2(g_1+3)+2(g_4+3) &= 16 \\
2(g_3+3)+2(g_2+3) &= 20 \\
(g_2+3)(g_4+3)+4 &= 24
\end{aligned}$$

or

$$\begin{aligned}
\bar{g}_1\bar{g}_3 &= 20 \\
\bar{g}_1+\bar{g}_4 &= 8 \\
\bar{g}_2+\bar{g}_3 &= 10 \\
\bar{g}_2\bar{g}_4 &= 20
\end{aligned}$$

where $\bar{g}_1 = g_1+3$, $\bar{g}_2 = g_2+3$, $\bar{g}_3 = g_3+3$, and $\bar{g}_4 = g_4+3$. These equations have only one solution: $\bar{g}_1 = 4$, $\bar{g}_2 = 5$, $\bar{g}_3 = 5$, and $\bar{g}_4 = 4$, that is $g_1 = 1$, $g_2 = 2$, $g_3 = 2$, and $g_4 = 1$.

## Summary

While this shows a solution method for a $2 \times 2$ grid, some of the points noted here also apply to the more general case, although an explicit solution cannot be expected then. In general, the matrix would have to be partitioned into a $3 \times 3$ arrangement corresponding to the fact that in addition to input nodes and output nodes there are then also interior nodes.

## $3 \times 3$ node grid example

Here we have three input nodes (1, 2, 3), three output nodes (7, 8, 9), and three interior nodes (4, 5, 6). In three experiments we apply currents to each of the input nodes in turn, each time reading out all of the output nodes, yielding a total of nine measurements. We try and recover the nine leakage conductances to ground from each of the nine nodes.

It is natural to partition the conductance matrix as follows given that $I_4$, $I_5$, $I_6$, $I_7$, $I_8$, and $I_9$ are always zero, and that we do not measure $V_1$, $V_2$, $V_3$, $V_4$, $V_5$, and $V_6$.

$$
\begin{pmatrix}
G_{11} & \vdots & G_{12} & \vdots & G_{13} \\
\cdots & \cdot & \cdots & \cdot & \cdots \\
G_{21} & \vdots & G_{22} & \vdots & G_{23} \\
\cdots & \cdot & \cdots & \cdot & \cdots \\
G_{31} & \vdots & G_{32} & \vdots & G_{33}
\end{pmatrix}
\begin{pmatrix}
V_1 \\ V_2 \\ V_3 \\ \cdots \\ V_4 \\ V_5 \\ V_6 \\ \cdots \\ V_7 \\ V_8 \\ V_9
\end{pmatrix}
=
\begin{pmatrix}
I_1 \\ I_2 \\ I_3 \\ \cdots \\ I_4 \\ I_5 \\ I_6 \\ \cdots \\ I_7 \\ I_8 \\ I_9
\end{pmatrix}
$$

In detail

$$
\begin{pmatrix}
g'_1 & -g_{12} & -g_{13} & \vdots & -g_{14} & 0 & 0 & \vdots & 0 & 0 & 0 \\
-g_{12} & g'_2 & -g_{23} & \vdots & 0 & -g_{25} & 0 & \vdots & 0 & 0 & 0 \\
-g_{13} & -g_{23} & g'_3 & \vdots & 0 & 0 & g_{36} & \vdots & 0 & 0 & 0 \\
\cdots & \cdots & \cdots & \cdot & \cdots & \cdots & \cdots & \cdot & \cdots & \cdots & \cdots \\
-g_{14} & -0 & 0 & \vdots & g'_4 & -g_{45} & -g_{46} & \vdots & -g_{47} & 0 & 0 \\
0 & -g_{25} & 0 & \vdots & -g_{45} & g'_5 & -g_{56} & \vdots & 0 & -g_{58} & 0 \\
0 & 0 & -g_{36} & \vdots & -g_{46} & -g_{56} & g'_6 & \vdots & 0 & -0 & -g_{69} \\
\cdots & \cdots & \cdots & \cdot & \cdots & \cdots & \cdots & \cdot & \cdots & \cdots & \cdots \\
0 & 0 & 0 & \vdots & -g_{47} & 0 & 0 & \vdots & g'_7 & -g_{78} & -g_{79} \\
0 & 0 & 0 & \vdots & 0 & -g_{58} & 0 & \vdots & -g_{78} & g'_8 & -g_{89} \\
0 & 0 & 0 & \vdots & 0 & 0 & -g_{69} & \vdots & -g_{79} & -g_{89} & g'_9
\end{pmatrix}
\begin{pmatrix}
V_1 \\ V_2 \\ V_3 \\ \cdots \\ V_4 \\ V_5 \\ V_6 \\ \cdots \\ V_7 \\ V_8 \\ V_9
\end{pmatrix}
=
\begin{pmatrix}
I_1 \\ I_2 \\ I_3 \\ \cdots \\ I_4 \\ I_5 \\ I_6 \\ \cdots \\ I_7 \\ I_8 \\ I_9
\end{pmatrix}
$$

where $g'_1 = (g_1 + g_{12} + g_{13} + g_{14})$, $g'_2 = (g_2 + g_{12} + g_{23} + g_{25})$, $g'_3 = (g_3 + g_{13} + g_{23} + g_{36})$, $g'_4 = (g_4 + g_{14} + g_{45} + g_{46} + g_{47})$, $g'_5 = (g_5 + g_{25} + g_{45} + g_{56} + g_{47})$, $g'_6 = (g_6 + g_{36} + g_{46} + g_{56} + g_{69})$, $g'_7 = (g_7 + g_{47} + g_{78} + g_{79})$, $g'_8 = (g_8 + g_{58} + g_{78} + g_{89})$, and $g'_9 = (g_9 + g_{69} + g_{79} + g_{89})$.

We note that $G_{13}$ and $G_{31}$ are all zeros, and $G_{12} = G_{21}$, and $G_{23} = G_{32}$ are diagonal. Also, the sub-matrices appearing on the diagonal are of Toeplitz form. Tpeolitz matrices can be inverted in order $N^2$ (instead of order $N^3$).

We can write the inverse as follows:

$$\begin{pmatrix} C_{11} & \vdots & C_{12} & \vdots & C_{13} \\ \cdots & \cdot & \cdots & \cdot & \cdots \\ C_{21} & \vdots & C_{22} & \vdots & C_{23} \\ \cdots & \cdot & \cdots & \cdot & \cdots \\ C_{31} & \vdots & C_{32} & \vdots & C_{33} \end{pmatrix} \begin{pmatrix} I_1 \\ I_2 \\ I_3 \\ \cdots \\ I_4 \\ I_5 \\ I_6 \\ \cdots \\ I_7 \\ I_8 \\ I_9 \end{pmatrix} = \begin{pmatrix} V_1 \\ V_2 \\ V_3 \\ \cdots \\ V_4 \\ V_5 \\ V_6 \\ \cdots \\ V_7 \\ V_8 \\ V_9 \end{pmatrix}$$

We can use the formula for the inverse of matrix partitioned into four parts twice on this matrix partitioned into nine parts. But it may be a bit much to expect to easily obtain explicit formulae the way we did for the $2 \times 2$ case...

Note that we are only really interested in the bottom left corner ($C_{31}$) of the inverse, given that $I_4$, $I_5$, $I_6$, $I_7$, $I_8$, and $I_9$ are always zero, and that we do not measure $V_1$, $V_2$, $V_3$, $V_4$, $V_5$, and $V_6$. Each experiment yields three measurements and thus three equations of the form

$$C_{31} \begin{pmatrix} I_1 \\ I_2 \\ I_3 \end{pmatrix} = \begin{pmatrix} V_7 \\ V_8 \\ V_9 \end{pmatrix}.$$

By performing three experiments we can find all nine elements of the matrix $C_{31}$. Each of these is a polynomial in the unknown leakage conductances $g_1$, $g_2$, $g_3$, $g_4$, $g_5$, $g_6$, $g_7$, $g_8$, and $g_9$ (or rather, we can cross-multiply to obtain nine such polynomials).

The part of the inverse of this conductance matrix that we need is the lower left corner, $C_{31}$. Using the decomposition rule for partitioned $2 \times 2$ matrices twice, we get

$$C_{31} = G_{33}^{-1} G_{32} (G_{22} - G_{23} G_{33}^{-1} G_{21})^{-1} G_{21} \left(G_{11} - G_{12} (G_{22} - G_{23} G_{33}^{-1} G_{21})^{-1} G_{21}\right)^{-1}$$

Note that the term $(G_{22} - G_{23} G_{33}^{-1} G_{21})^{-1} G_{21}$ appears twice. This can be exploited to save on computation.