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RECOVERY OF A COMPACTLY SUPPORTED FUNCTION STARTING FROM ITS INTEGRALS OVER LINES INTERSECTING A GIVEN SET OF POINTS IN SPACE

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In this note we investigate the following problem of integral geometry. Let the integrals be given of some function $\rho(x) = \rho(x_1, x_2, x_3)$, concentrated in a fixed finite volume V of space, over rays emanating from points of a set M in space (i.e., sources of X-ray radiation are situated at points of the set M).

1) For what kind of mutual arrangement of M and the region V is it possible to recover the function?

2) How can the function $\rho(x_1, x_2, x_3)$ be found effectively in those cases where it is possible in principle?

Here we give a solution for this problem in which the compact support of the function $\rho(x)$ plays an essential role. In fact, it can be shown that if one does not restrict oneself to functions of compact support but considers, for example, all integrable functions, then the problem is solvable only when the sources are arranged along some line. In this case it at once reduces to a plane Radon problem: it suffices to consider planes passing through the line of the sources.

We note that the solution of the analogous problem in the complex space $\mathbf{C}^3$, when integrals of the function are given on complex lines intersecting a complex (algebraic) curve in $\mathbf{C}^3$, is already well known (see [1]). The first problem of this kind, in which a family of complex lines in $\mathbf{C}^3$ intersecting a plane curve of the second order is considered, was solved by Gel'fand and Naĭmark in 1947. This problem lies at the basis of the theory of infinite-dimensional unitary representations of the Lorentz group [3]. Our method is similar to the method of [3].

In the complex case the situation is considerably simplified, namely, the problem always has a solution (compact support of the function ρ is not required), and there exists a simple local inversion formula.

Real problems of integral geometry are the subject of the paper of Gel'fand and Gindikin [4].

In tomography the three-dimensional structure of an object which is described by a material density function $\rho(x_1, x_2, x_3)$ is recovered from a set of "X-ray pictures." More precisely, each source generates a beam of X-rays which passes through the object and is later recorded on a screen. As a result we obtain integrals of the density function over the rays along which the X-radiation propagates.

One can assume that the source generates a beam of rays emanating from some point of space (recently the "size" of this point has successfully been reduced to 10 microns). Until now in tomography it was possible to neglect the fact that the source was located at a finite distance from the object, and assume that we were dealing with parallel pencils of rays. Thus the resulting mathematical problem reduced to the Radon problem.

However, if we wish to investigate the microstructure of the object, it becomes essential that the source be located at a finite distance from it. As a result we arrive at the necessity of investigating the above-described problem of integral geometry.

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I. Relation between the Radon transform and integrals over lines passing through a point. Let

$$\check{\rho}(\omega, p) = \int \rho(x)\delta(\langle \omega, x\rangle - p)\, d^3x,$$

where $\omega = (\omega_1, \omega_2, \omega_3)$ and $|\omega| = 1$, be the Radon transform of the function $\rho(x)$. We assume that

$$(R_1, \rho)(x^0, a) = \int \rho(x^0 + ta)\, dt$$

is the integral of $\rho(x)$ over the line with direction vector $a = (a_1, a_2, a_2)$, $|a| = 1$, passing through the point x^0.

LEMMA 1.

$$(R_1\rho)(x^0, a) = \int \left(\int \frac{\check{\rho}(b, p)}{(p - \langle x^0, b\rangle)^2}\, dp \right) db, \tag{1}$$

where $\langle b, a\rangle = 0$, $|b| = 1$, and db is an average over all unit vectors b orthogonal to the vector a.

PROOF. We reduce (1) to the inversion formula for the Radon transform in the plane. Let $\Pi_{x^0,a}$ be the plane passing through the point x^0 perpendicular to the vector a. We integrate $\rho(x)$ over all lines perpendicular to this plane; as a result we get a function $I_{x^0,a}(\rho)$ defined on $\Pi_{x^0,a}$. Let us denote by $(I_{x^0,a}\rho)(b, p)$, where $\langle b, a\rangle = 0$ and $|b| = 1$, the integral of this function over the line perpendicular to the vector b and located at a distance p from x^0.

The integral is calculated with respect to the standard Euclidean measure on the line. In other words, $(I^\vee_{x^0,a}\rho)(b, p)$ is the Radon transform of $I_{x^0,a}(\rho)$ in the plane $\Pi_{x^0,a}$. By the inversion formula (see [1]) we have

$$(I_{x^0,a}\rho)(x^0) = \int_{\substack{|b|=1 \\ \langle b,a\rangle = 0}} \int \frac{(I^\vee_{x^0,a}\rho)(b, p)}{p^2}\, dp\, db. \tag{2}$$

We note that by definition

$$(I_{x^0,a}\rho)(x^0) = (R_1\rho)(x^0, a), \qquad (I_{x^0,a}\rho)(b, p) = \check{\rho}(b, p + \langle x^0, b\rangle).$$

Substituting into (2), we obtain equality (1). Lemma 1 is proved.

We set

$$(S_{x^0}\rho)(b) = \int \frac{\check{\rho}(b, p)}{(p - \langle x^0, b\rangle)^2}\, dp. \tag{3}$$

LEMMA 2.

$$(S_{x^0}\rho)(b) = \int_{|a|=1} \frac{(R_1\rho)(x^0, a)}{\langle b, a\rangle^2}\, d^2a, \tag{4}$$

where d^2a is an average over all unit vectors.

PROOF. $(S_{x^0}\rho)(b)$ is an even function on the unit sphere consisting of terminal points of the vectors b. Formula (1) signifies that $(R_1\rho)(x^0, b)$ is obtained as a result of averaging this function over the great circle on the sphere perpendicular to the vector a. We must recover an even function on the sphere, knowing its integrals over great circles. This problem reduces to the inversion problem for Radon transforms in the projective plane ([5], Chapter II, §1.7). Formula (4) is a consequence of formula 2.16 in [5].

II. The main result.

1. We have thus shown that integrals of functions over lines passing through x^0 determine the integrals $(S_{x^0}\rho)(b)$ (see (4)) and in turn are determined by these integrals.

Below in subsection 2 we shall indicate conditions on the mutual arrangement of a set M and a region V which allow us to find the Radon transform of a function ρ starting from the integrals $(S_{x^0}\rho)(b)$ (where $x^0 \in M$ and $|b| = 1$). Then, using the Radon transform inversion formula [1], we shall find the unknown function $\rho(x)$:

$$\rho(x) = -\frac{1}{8\pi^2}\int \check{\rho}''_{pp}(b, \langle b, x\rangle)\, d^2b,$$

where d^2b is the average over all unit vectors b.

2. We fix a vector b. We denote by $p_b(V)$ ($p_b(M)$) the projection of the region V (the set M) on the line $x = \lambda b$ (λ is a parameter on this line).

The problem of determining $\check{\rho}(b,p)$ starting from $S_{x^0}\rho$ reduces to solution of the singular integral equation

$$\varphi_b(\lambda) = -\int_{P_b(M)} \frac{\check{\rho}'_p(b,p)}{p-\lambda}\, dp, \tag{5}$$

where $\varphi_b(\lambda) := (S_{x^0}\rho)(b)$ and $\lambda = \langle x^0, b\rangle \in p_b(M)$. In particular, for example, $\check{\rho}(b,p)$ is uniquely determined from $S_{x^0}\rho$ if and only if equation (5) has exactly one solution.

PROPOSITION 1. *If the set $p_b(M)$ contains a line segment (which we denote by $l(b)$) having no points in common with $p_b(V)$, then the function $\check{\rho}(b,p)$ is uniquely determined by the integrals $(R_1\rho)(x^0, a)$.*

PROOF. The function $\varphi_b(\lambda)$ is analytic in λ for $\lambda \in \mathbf{C}\backslash p_b(V)$ since the integrand is different from 0 only for $p \in p_b(V)$.

Using the Sokhotskiĭ-Plemelj formula [7], we obtain from (5)

$$\lim_{\lambda_2\to+0}(\varphi(\lambda_1 + i\lambda_2) - \varphi(\lambda_1 - i\lambda_2)) = -i\pi\check{\rho}'_p(b,\lambda_1),$$

where $\lambda_1 = \operatorname{Re}\lambda$ and $\lambda_2 = \operatorname{Im}\lambda$. Since $\check{\rho}(b,p) = 0$ as $p \to \pm\infty$,

$$\check{\rho}(b,p) = -\int_{-\infty}^{p} \rho'_p(b,\lambda_1)\, d\lambda_1.$$

The proposition is proved.

THEOREM. *If for any vector b the set $p_b(V)$ is a line segment and is contained in (or coincides with) $p_b(M)$, then the function can be found effectively starting from its integrals over lines intersecting M.*

PROOF. Let α (β) be the coordinate of the leftmost (respectively, rightmost) point of $p_b(M)$. We must solve the singular integral equation

$$\varphi_b(\lambda) = -\int_{\alpha}^{\beta} \frac{\check{\rho}'_p(b,p)}{p-\lambda}\, dp, \qquad \lambda \in [\alpha, \beta].$$

We cite the final answer [7]:

$$\check{\rho}'_p(b,p) = \frac{1}{\pi^2}\sqrt{(\rho-\alpha)(p-\beta)}\int_{\alpha}^{\beta} \frac{\varphi_b(\lambda)\, d\lambda}{\sqrt{(\lambda-\alpha)(\lambda-\beta)}(\lambda-p)}.$$

Hence, if one sets $p = \langle b, x \rangle$, then

$$\rho(x) = -\frac{1}{8\pi^4} \int_{|b|=1} \left(\frac{2p - \alpha - \beta}{\sqrt{(p-\alpha)(p-\beta)}} \int_\alpha^\beta \frac{\varphi_b(\lambda)\, d\lambda}{\sqrt{(\lambda-\alpha)(\lambda-\beta)}(\lambda - p)} - \sqrt{(p-\alpha)(p-\beta)} \int_\alpha^\beta \frac{\varphi_b(\lambda)\, d\lambda}{\sqrt{(\lambda-\alpha)(\lambda-\beta)}(\lambda-p)} \right) d^2 b.$$

EXAMPLE. If V is a ball of radius R with center at $(0,0,0)$, then as M one can take two semicircles of radius $R+\varepsilon$, the centers of which lie at $(0,0,0)$: one of them lies in the plane $x_3 = 0$, its endpoints located at $(0, R+\varepsilon, 0)$ and $(0, -(R+\varepsilon), 0)$, while the second is in the plane $x_2 = 0$ with endpoints at $(0, 0, R+\varepsilon)$ and $(0, 0, -(R+\varepsilon))$.

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