# MATHEMATICAL FRAMEWORK OF CONE BEAM 3D RECONSTRUCTION VIA THE FIRST DERIVATIVE OF THE RADON TRANSFORM

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**Abstract**

Either for medical imaging or for non destructive testing, X-ray provides a very accurate mean to investigate internal structures. The object is described by a 3D map f of the local density. The use of a 2D X-ray detector like an image intensifier in front of the ponctual X-ray source defines a cone beam geometry. When the source moves along a curve, the acquisition measurements are modelized by the cone beam X-ray transform of the function f. This same model can be applied to emission tomography when cone beam collimators are used.

The aim of the 3D reconstruction is to recover the original function f. We have established an exact formula between the cone beam X-ray transform and the first derivative of the 3D Radon transform. We propose to use the planes as information vectors to achieve the rebinning from the coordinates system linked to the cone beam geometry, to the spherical coordinates system of the Radon domain. Then the reconstruction diagram is to compute and to invert the first derivative of the 3D Radon transform.

In this publication, we describe the mathematical framework of this reconstruction diagram. We emphasize the special case of the circular acquisition trajectory.

Key-words : Cone beam. Three-dimensional reconstruction. Radon transform. X-ray. Transmission tomography. Gamma-ray. Emission tomography.



## 1. INTRODUCTION

In 3D image reconstruction, the object to investigate is characterized by the 3D map f of a physical parameter, the local density in transmission tomography or the local activity in emission tomography. For a given curve $\Gamma$, the cone beam X-ray transform Xf does associate to each half line starting of one point on the curve, the integral of the function f over this half line. The cone beam X-ray transform does modelize the acquisition system for tomographic imaging devices where a focal point S moves along the acquisition trajectory $\Gamma$ and a 2D detector on the opposite side provides a parametrization of the half-lines starting from S, crossing the object, and intersecting the detector. For X-ray transmission acquisitions, the X-ray source defines the focal point S and an image intensifier can be used as 2D detector. For gamma-ray emission acquisitions, the point S is the focal point of a cone beam collimator and the 2D detector is a gamma camera. The main technological interests are to speed the acquisition by the use of a 2D detector and to improve the spatial resolution by the magnification effect induced by the cone beam geometry.

The cone beam X-ray transform describes a new generation of 3D tomographic imaging devices using 2D detectors. Some examples in X-ray medical imaging are the Dynamic Spatial Reconstructor [ROBB (1985)] to investigate the beating heart and lungs, or the TRIDIMOS project developped at the LETI for the CNES to measure the bone mineral content of lumbar vertebrae [GRANGEAT (1989)], or the MORPHOMETRE project of GE-CGR, with the LETI as partner, to image vessel trees and bone structures [SAINT-FELIX et al. (1990), GRANGEAT (1990c)] or a 3D X-ray microtomograph to investigate small objects like biopsies [MORTON et al. (1990)]. In nuclear medicine, cone beam collimators are used with large gamma cameras to focalize the acquisition on small region of interest like the brain [JASZCZAK et al. (1988)] or the heart [GULLBERG et al. (1989)]. Moreover, it enables to put a X-ray source at the focal point in order to reconstruct the attenuation map from cone beam transmission measurements [MANGLOS et al. (1990)], to correct the auto-attenuation effect of the gamma-rays. Finally, in X-ray Non Destructive Testing, cone beam 3D tomographs are developped to control small pieces like ceramic materials [FELDKAMP et al. (1984), VICKERS et al. (1989)]. The LETI is one partner of the Evaluation of Voludensitometric Analysis (EVA) project of the EEC [RIZO and GRANGEAT (1989), SIRE et al. (1990)].

The reconstruction algorithm has to recover the original function f from its cone beam X-ray transform Xf. We will only review the transform methods. This inverse problem has been studied in the following publications [KIRILLOV (1961), HAMAKER et al. (1980), FINCH and SOLMON (1980, 1983), TUY (1983), FINCH (1985)]. TUY has proposed a first inversion formula using the homogeneous extension of functions. All these studies have given mathematical results but could not lead directly to a numerical implementation.

So the first approach has been to generalize to the 3D cone beam geometry the 2D fan beam reconstruction algorithm. This direct processing is based on a convolution cone beam backprojection algorithm. This has first been suggested by FELDKAMP [FELDKAMP et al. (1984)]. This idea has induced large developments [WEBB et al. (1987), JACQUET (1988)]. BRUCE SMITH has expressed the mathematical steps to reach this convolution cone beam backprojection formula [SMITH B. (1985)]. He gives a survey on cone beam tomography in [SMITH B. (1990)]. But the hypothesis the trajectory has to fullfill are too restrictive. The only general framework which can deal with the largest class of trajectories is to use the Radon transform. The first attempts have been formulated in [MINERBO (1979), GRANGEAT (1985, 1986a, 1986b)]. But an approximation was used. To get an exact relation, the solution is to work either with the first derivative R'f of the Radon transform [GRANGEAT (1987b)] or with the Hilbert transform HDRf of the first derivative of the Radon transform. This second solution can be derived from [SMITH B. (1985, 1987, 1990), KUDO and SAITO (1990a, 1990b)]. The main advantage of our first solution is that it involves only partial differentiation which means local filtering, whereas the second needs a global filtering to achieve the Hilbert transform.

In our oral communication at Oberwolfach [GRANGEAT et al. (1990b)], we have presented and compared reconstructions with both FELDKAMP's algorithm and ours. In this written publication, we describe the mathematical framework of our reconstruction algorithm via the first derivative R'f of the Radon transform.

After some preliminary definitions in chapter 2, we describe how to compute the first derivative R'f of the 3D Radon transform. We first introduce in chapter 3 the fundamental relation between the cone beam X-ray transform Xf and the first derivative R'f of the Radon transform. Then we give in chapter 4 the expression of this fundamental relation in the coordinates system of the detection plane, in order to prepare the numerical implementation. In the chapter 5, we give the necessary and sufficient condition on the trajectory to describe all the Radon domain of the object support and we introduce a geometrical description of set of planes. Finally, we emphasize the special case of the circular trajectory in the chapter 6. We depict the shadow zone in the Radon domain and its processing by interpolation. We introduce the rebinning equation.

Then we describe in the chapter 7 the inversion of the first derivative R'f of the Radon transform using two steps, a first convolution backprojection step to compute the filtered rebinned X-ray transform and a second backprojection step to recover the original object function f.



## 2. PRELIMINARY DEFINITIONS

### 2.1. The cone beam X-ray transform Xf

Let us take a curve $\Gamma$ in $\mathbb{R}^3$. For a given point S on $\Gamma$ and for a unit vector $\vec{u}$ in $S^2$, we use $(S,\vec{u})$ to characterize the half-line of direction $\vec{u}$, with S as origin point. For a given function f on $\mathbb{R}^3$, the cone beam X-ray transform Xf associates to each half-line $(S,\vec{u})$ with S moving along $\Gamma$, the integral of f over this half line :

$$Xf(S,\vec{u}) = \int_{a=0}^{+\infty} f(S + a.\vec{u})\, da \qquad (2.1)$$

for $S \in \Gamma$ and $\vec{u} \in S^2$.

It is supposed throughout that the support $\Omega$ of f is a compact convex subset of $\mathbb{R}^3$ and that the function f is an element of $C^2(\Omega)$ the space of two times differentiable functions, with continuous second differential functions, vanishing outside $\Omega$. The discussion over more general functions spaces is not in the scope of this paper.

For the curve $\Gamma$, we don't need strong regularity conditions. For mechanical feasibility, we will assume that $\Gamma$ is a piecewise $C^1$ curve, which means that $\Gamma$ is the image in $\mathbb{R}^3$ of the union of a finite number of intervals in $\mathbb{R}$, under a map $C^1$ on each interval. We suppose the curve $\Gamma$ lies outside the support $\Omega$.

On a physical device, we will have the X-ray source S on one side of the object and the two-dimensional X-ray detector on the opposite side, as it is represented on the figure 1. This acquisition geometry will move around the object. We choose an origin point 0 for the object. If the motion of the acquisition geometry as a fixed point, for instance a center of rotation, this will be selected as origin point 0. We will assume that the front side of the detector is a plane $P_dX$. Each point $A_d$ of this plane $P_dX$ defines an unique half-line issued from S. So these points can be used to parametrize the cone beam X-ray transform for each unit vector directed toward this plane.

To simplify, we assume that this plane is perpendicular to the line (S,O). To avoid the use of scaling factor, we will consider the detection plane PX which we define as the plane crossing the origin O and perpendicular to the line (S,O). Thus this detection plane PX is parallel to the detector front side $P_dX$.

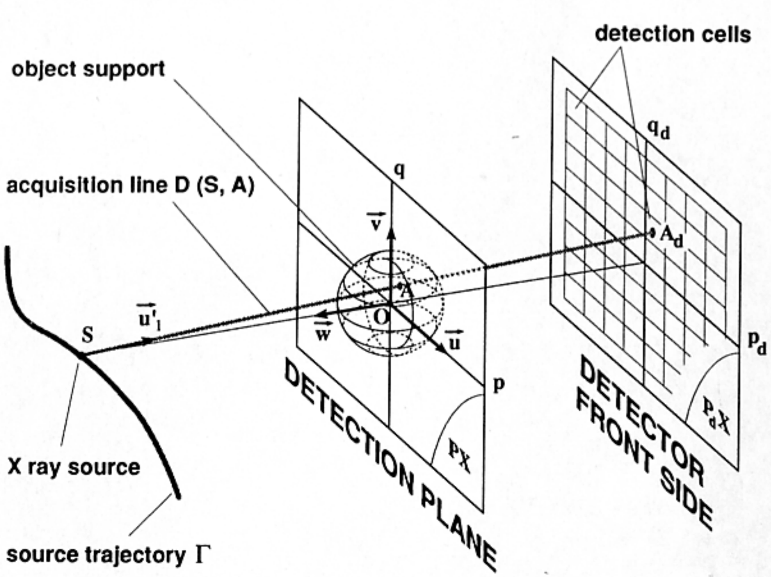


**Figure 1 : The cone beam X-ray transform**

To describe more precisely the acquisition procedure, we give now a new definition of the cone beam X-ray transform Xf :

$$Xf(S,A) = \int_{a=0}^{+\infty} f\left(S + a.\vec{u'_1}\right) da \qquad (2.2)$$

for $S \in \Gamma$ , $A \in PX$ and with $\vec{u'_1} = \dfrac{\vec{SA}}{\|\vec{SA}\|}$ .

In this definition, we assume that the support Ω of f belongs to the half-space described by the half-lines (S,A) when A moves within the detection plane PX, for each position of the source S on the curve Γ.



## 2.2. The first derivative R'f of the Radon transform

Let us set $\left(O,\vec{i},\vec{j},\vec{k}\right)$ the orthonormal object reference system in $\mathbb{R}^3$ and (x,y,z) the cartesian coordinates system to parametrize the object points M.

Each plane P is characterized by one of its orthogonal unit vector $\vec{n}$ and by its algebraic distance to the origin O (cf figure 2) :

$$M \in P\left(\rho,\vec{n}\right) \iff \vec{OM}.\vec{n} = \rho \qquad \left(\rho,\vec{n}\right) \in \mathbb{R} \times S^2 \qquad (2.3)$$

We use the spherical coordinates system (θ,φ) to parametrize the unit vector $\vec{n}$. The colatitude angle θ is the angle between the axis $(O,\vec{z})$ and the vector $\vec{n}$. The longitude angle φ is the angle between the axis (O,x) and the orthogonal projection of $\vec{n}$ on the $\left(\vec{i},\vec{j}\right)$ vectorial plane.

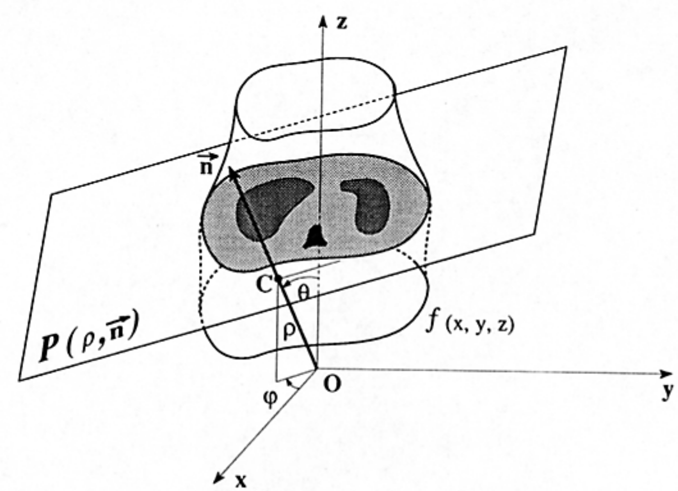


**Figure 2 : The 3D Radon transform parameters**

For a given object function f on $\mathbb{R}^3$, its 3D Radon transform Rf is defined as the set of its integrals over the planes of $\mathbb{R}^3$ [NATTERER (1986)] :

$$Rf\left(\rho,\vec{n}\right) = \iint_{M \in P(\rho,\vec{n})} f(M)\ dM \qquad (2.4)$$

We define the first derivative R'f of the Radon transform as the partial derivative of Rf with respect to the algebraic distance $\rho$ :

$$R'f\left(\rho,\vec{n}\right) = \frac{\partial Rf}{\partial \rho}\left(\rho,\vec{n}\right) \qquad (2.5)$$

We shall remark that the definition set of the planes can be restricted to the planes which do intersect the function support $\Omega$, because Rf and R'f are null on the others.

Let us now write the inversion formula of the 3D Radon transform [NATTERER (1986)] :

$$f(M) = -\frac{1}{8\,\pi^2} \cdot \int_{S^2} \frac{\partial^2 Rf}{\partial \rho^2}\left(\vec{OM}.\vec{n},\vec{n}\right) d\vec{n} \qquad (2.6)$$

From this, we deduce the inversion formula of the first derivative of the 3D Radon transform :

$$f(M) = -\frac{1}{8\,\pi^2} \cdot \int_{S^2} \frac{\partial R'f}{\partial \rho}\left(\vec{OM}.\vec{n},\vec{n}\right) d\vec{n} \qquad (2.7)$$

## 3. THE FUNDAMENTAL RELATION

Let us take a source position S on the curve $\Gamma$ and a given unit vector $\vec{n}$ in $S^2$.
Let us define the weighted cone beam X-ray transform Yf :

$$Yf(S,A) = \frac{\|\vec{SO}\|}{\|\vec{SA}\|} .\ Xf(S,A) \quad A \in PX \qquad (3.1)$$

We define the integration plane $P\left(\vec{OS}.\vec{n},\vec{n}\right)$ as the uniq· · plane perpendicular to the vector $\vec{n}$ and crossing S. The point C orthogonal projection of the origin O on this integration plane is called its characteristic point.



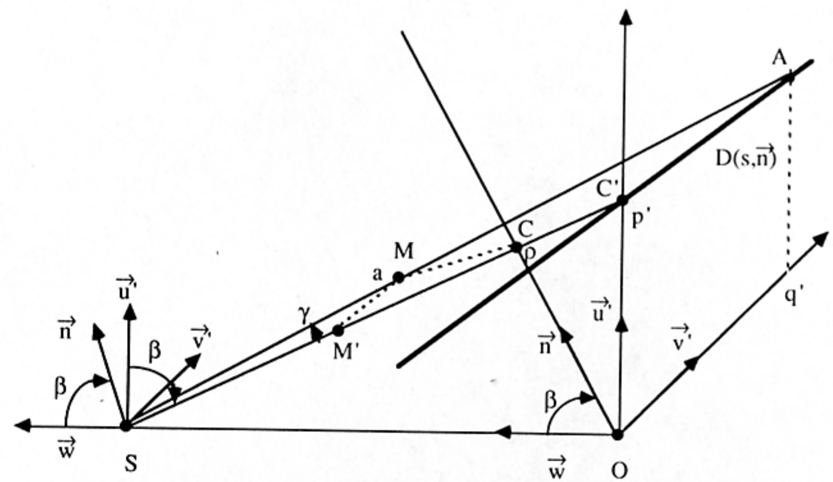


**Figure 3 : Parameters of the fundamental relation**

We note $\vec{w}$ the unit vector which gives the direction of the source point S from the origin O :

$$\vec{w} = \frac{\vec{OS}}{\|\vec{OS}\|}$$

If the vectors $\vec{n}$ and $\vec{w}$ are not parallel, the integration plane $P\left(\vec{OS}.\vec{n},\vec{n}\right)$ does intersect the detection plane PX along a straight line $D\left(S,\vec{n}\right)$ that we call the integration line.

We define the following $(\vec{u}', \vec{v}')$ orthonormal vectors as vector basis of the detection plane :

$$\begin{cases} \vec{v}' = \dfrac{\vec{w} \wedge \vec{n}}{\| \vec{w} \wedge \vec{n} \|} \\ \\ \vec{u}' = \vec{v}' \wedge \vec{w} \end{cases} \tag{3.2}$$

where $\wedge$ represents the direct vector product.

We remark that $(\vec{u}', \vec{v}', \vec{w})$ is an orthonormal vector basis of $\mathbb{R}^3$. Moreover $\vec{v}'$ is perpendicular to $\vec{n}$ and to $\vec{w}$ and so is parallel to the integration line $D(S, \vec{n})$. We use $\vec{v}'$ as unit vector to define its orientation.

Let us note C' the orthogonal projection of the origin O on this integration line. We define $SYf(S, \vec{n})$ as the integration of the weighted cone beam X-ray transform Yf over the integration line $D(S, \vec{n})$ :

$$SYf(S, \vec{n}) = \int_{A \in D(S, \vec{n})} Yf(S, A) \; dA \tag{3.3}$$

$$= \int_{q_1 = -\infty}^{+\infty} Yf\left(S, C' + q'.\vec{v}'\right) dq'$$

Let us write (p', q') the coordinates system in the $(O, \vec{u}', \vec{v}')$ reference system of the detection plane. If we consider all the integration lines parallel to $\vec{v}'$, we can use p' as a parameter of $SYf(S, \vec{n})$.

Then we can introduce **the fundamental relation between the cone beam X-ray transform Xf and the first derivative R'f of the Radon transform** :

$$\frac{\| \vec{OS} \|^2}{\| \vec{OS} \wedge \vec{n} \|^2} \cdot \frac{\partial SYf}{\partial p'} (S, \vec{n}) = R'f\left(\vec{OS}.\vec{n}, \vec{n}\right) \tag{3.4}$$

This formula and its proof has been published in 1987 in our thesis [GRANGEAT (1987)] which we consider as the reference publication. This formula has been introduced first in our talk [GRANGEAT (1986)] in the last meeting at Oberwolfach :



Theory and Application of Radon Transforms. In 1984, we have proved a first relation equivalent to this, using the homogeneous extension of Xf, as it is defined in Tuy's paper [TUY (1983)]. It was written in a private communication with D. FINCH and D. SOLMON [GRANGEAT (1984)] which is reproduced in our thesis [GRANGEAT (1987)].

We will give here the proof of this Grangeat's formula. The reader should refer to the figure 3 to follow the presentation.

Let us define two coordinates systems to parametrize the points M in $\mathbb{R}^3$.

The first is the acquisition cartesian coordinates system (x',y',z') associated to the $(S, \vec{u}', \vec{v}', \vec{w})$ reference system with the origin at the source point S.

The second is a spherical coordinates system to parametrize the cone beam geometry. The origin is at S and $\vec{v}'$ is the polar axis. Let us note M' the orthogonal projection of M on the $(S, \vec{w}, \vec{u}')$ plane.

We define the longitude parameter $\beta$ as the angle between $\vec{SM'}$ and $\vec{u}'$ and the latitude parameter $\gamma$ as the angle between $\vec{SM}$ and $\vec{SM'}$. The parameter a describes the distance between the point M and the source point S.

We remind that we have assumed the points M are restricted to the subspace $\mathbb{R}^2 \times \mathbb{R}^-$ in the acquisition cartesian coordinates system, which corresponds to :

$$(a, \beta, \gamma) \in \mathbb{R}^+ \times [0, \pi] \times \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$$

The cone beam coordinate system is related to the acquisition cartesian coordinate system by the following equations :

$$\begin{cases} x' = a.\cos\gamma.\cos\beta \\ y' = a.\sin\gamma \\ z' = -\,a.\cos\gamma.\sin\beta \end{cases} \tag{3.5}$$

The angular parameter $\beta$ can be used to describe the integration lines instead of p'. We have :

$$p' = \| \vec{OS} \| . \operatorname{cotg} \beta \tag{3.6}$$

and so : $\dfrac{d\beta}{dp'} = -\dfrac{\sin^2\beta}{\| \vec{OS} \|}$

But on the integration plane, we have :

$$\sin^2\beta = \frac{\|\overrightarrow{OS}\wedge\vec{n}\|^2}{\|\overrightarrow{OS}\|^2}$$

So we get :

$$\frac{\|\overrightarrow{OS}\|^2}{\|\overrightarrow{OS}\wedge\vec{n}\|^2} \cdot \frac{\partial SYf(S,\vec{n})}{\partial p'} = -\frac{1}{\|\overrightarrow{OS}\|} \cdot \frac{\partial SYf}{\partial\beta}(S,\vec{n}) \qquad (3.7)$$

Let us express the function $SYf(S,\vec{n})$ in the cone beam coordinates system.

For a given half-line starting from S and crossing the detection plane at the point A of coordinates (p',q'), we have :

$$q' = \|\overrightarrow{SC'}\| \,.\, \mathrm{tg}\,\gamma \qquad (3.8)$$

If we use the angular parameters $(\beta,\gamma)$ to parametrize the half-lines issued from S, we get :

$$SYf(S,\vec{n}) = \int_{\gamma=-\frac{\pi}{2}}^{+\frac{\pi}{2}} Yf(S,\beta,\gamma) \,.\, \frac{\|\overrightarrow{SC'}\|}{\cos^2\gamma} \,.\, d\gamma \qquad (3.9)$$

But :
$$\frac{\|\overrightarrow{SO}\|}{\|\overrightarrow{SA}\|} = \frac{\|\overrightarrow{SO}\|}{\|\overrightarrow{SC'}\|} \cdot \frac{\|\overrightarrow{SC'}\|}{\|\overrightarrow{SA}\|}$$

$$= \frac{\|\overrightarrow{SO}\|}{\|\overrightarrow{SC'}\|} \,.\, \cos\gamma$$

So :
$$Yf \,.\, \frac{\|\overrightarrow{SC'}\|}{\cos^2\gamma} = Xf \,.\, \frac{\|\overrightarrow{SO}\|}{\cos\gamma}$$

As :
$$Xf\,(S,\beta,\gamma) = \int_{a=0}^{+\infty} f(a,\beta,\gamma)\, da$$



We get the following relation :

$$SYf(S,\vec{n}) = \int_{\gamma=-\frac{\pi}{2}}^{+\frac{\pi}{2}} \int_{a=0}^{+\infty} f(a,\beta,\gamma) \,.\, \frac{\|\overrightarrow{SO}\|}{\cos\gamma} \; da\, d\gamma \qquad (3.10)$$

Now we introduce the basic relation, induced by the moment effect if we derive f with respect to the angular variable $\beta$. For all the points M of the integration plane, we have :

$$\frac{\partial f(M)}{\partial\beta} = -\frac{\partial f(M)}{\partial\rho} \,.\, a \,.\, \cos\gamma \qquad M \in P(\overrightarrow{OS}.\vec{n},\vec{n}) \qquad (3.11)$$

It can be stated with reference to the cartesian coordinates system :

$$\frac{\partial f}{\partial\beta} = -\, a.\cos\gamma.\sin\beta.\frac{\partial f}{\partial x'} - a.\cos\gamma.\cos\beta.\frac{\partial f}{\partial z'}$$

If $\beta$ is the longitude of the integration plane, the coordinates of its orthogonal unit vector $\vec{n}$ in the $(\vec{u}',\vec{v}',\vec{w})$ basis are :

$$\vec{n} \begin{vmatrix} \sin\beta \\ 0 \\ \cos\beta \end{vmatrix}$$

And so the partial derivative in the direction of $\vec{n}$ is given by :

$$\frac{\partial f}{\partial\rho} = \sin\beta.\frac{\partial f}{\partial x} + \cos\beta.\frac{\partial f}{\partial z}$$

So we get the moment formula (3.11). We can now conclude the demonstration. We differentiate the relation (3.10) and we apply the differentiation operation before the integration. We get :

$$\frac{1}{\|\overrightarrow{SO}\|} \cdot \frac{\partial SYf}{\partial\beta}(S,\vec{n}) = -\int_{\gamma=-\frac{\pi}{2}}^{+\frac{\pi}{2}} \int_{a=0}^{+\infty} \frac{\partial f}{\partial\beta}(a,\beta,\gamma) \,.\, \frac{1}{\cos\gamma} \; da\; d\gamma$$

We apply the moment formula (3.11) :

$$\frac{1}{\|\vec{SO}\|} \cdot \frac{\partial SYf}{\partial \beta}\left(S,\vec{n}\right) = - \int_{\gamma=-\frac{\pi}{2}}^{+\frac{\pi}{2}} \int_{a=0}^{+\infty} \frac{\partial f}{\partial \rho}(a,\beta,\gamma) \,.\, a \quad da \ d\gamma$$

The second member of this equation is the integration in polar coordinates of the partial derivative of f in the direction of the unit vector $\vec{n}$. If we apply the differentiation operator after the integration, we get the first derivative of the Radon transform :

$$\frac{\partial Rf}{\partial \rho}\left(\vec{OS}.\vec{n},\vec{n}\right) = \frac{\partial}{\partial \rho}\left[\iint_{M \in P(\vec{OS}.\vec{n},\vec{n})} f(M)\ dM\right]$$

$$= \int_{\gamma=-\frac{\pi}{2}}^{+\frac{\pi}{2}} \int_{a=0}^{+\infty} \frac{\partial f}{\partial \rho}(a,\beta,\gamma) \,.\, a \quad da \ d\gamma \qquad (3.12)$$

And so :

$$\frac{1}{\|\vec{SO}\|} \cdot \frac{\partial SYf}{\partial \beta}\left(S,\vec{n}\right) = - \frac{\partial Rf}{\partial \rho}\left(\vec{OS}.\vec{n},\vec{n}\right)$$

If we use now the relation (3.7), we get the fundamental relation (3.4).

## 4. EXPRESSIONS OF THE FUNDAMENTAL RELATION IN THE COORDINATES SYSTEM OF THE DETECTION PLANE

We want now to prepare the numerical computation of the fundamental relation.

We refer to the schematic representations of the figures 1 and 4. We consider that the acquisition system provides a sampling of the function Xf on a regular rectangular grid. Then we define the $(O,\vec{u},\vec{v})$ orthonormal reference system such that the vector $\vec{u}$ is parallel to the rows of the grid, the vector $\vec{v}$ to the columns, and such that $\left(\vec{u},\vec{v},\vec{w}\right)$ represents a direct orthonormal basis, with :

$$\vec{w} = \frac{\vec{OS}}{\|\vec{OS}\|}$$



We note (p,q) the coordinates of the points A on the detection plane.

For a given unit vector $\vec{n}$, we have defined in the last paragraph the $(O,\vec{u}',\vec{v}')$ orthonormal reference system of the detection plane, such that the integration line $D(S,\vec{n})$ is parallel to the axis $(O,\vec{v}')$.

We call $\alpha$ the rotation angle around $\vec{w}$ to go from the $\left(\vec{u},\vec{v}\right)$ basis to the $\left(\vec{u}',\vec{v}'\right)$ basis.

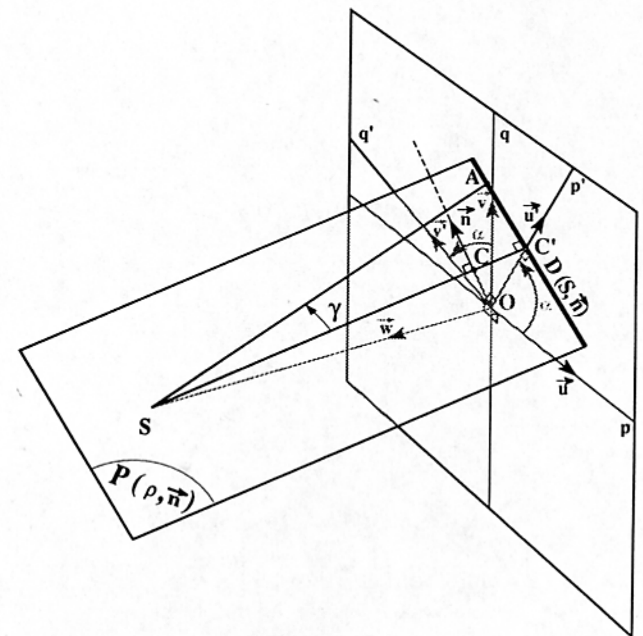


**Figure 4 : Coordinates systems on the detection plane**

In the (p',q') coordinates system associated to $(O,\vec{u}',\vec{v}')$, we can express the fundamental relation (3.4) in the following way, writting the differentiation operator inside the integration operator :

$$R'f\left(\vec{OS}.\vec{n},\vec{n}\right) = \frac{\|\vec{OS}\|^2}{\|\vec{OS}\wedge\vec{n}\|^2} \int_{q'=-\infty}^{+\infty} \frac{\partial Yf}{\partial p'}(S,A(q'))\ dq' \qquad (4.1)$$

The computation of the integration operator is equivalent to the computation of the X-ray transform Xf of a function f in two-dimensions from its representation on a rectangular sampling grid. This operation is called the reprojection. In our case we call it the integration processing. To get an efficient implementation, we use the algorithm described by JOSEPH [JOSEPH (1982)]. The principle is to sample the integration line either using the intersection points A(p) with the columns of the projection grid or using the intersection points A(q) with the rows of the projection grid. Let us describe as pep, respectively peq, the sampling distances between the columns, respectively the rows, of the grid. In order to get the sampling points as near as possible, we will choose either the first solution, if the following condition is fullfilled, or the second :

$$|\cos\gamma| \,.\, pep \leq |\sin\alpha| \,.\, peq \qquad (4.2)$$

The first case corresponds to the implementation of the first following equation (4.3), and the second case to the relation (4.4) :

case 1 :

$$R'f\left(\vec{OS}.\vec{n},\vec{n}\right) = \frac{\|\vec{OS}\|^2}{\|\vec{OS}\wedge\vec{n}\|^2}\cdot\frac{1}{|\sin\gamma|}\cdot\int_{p=-\infty}^{+\infty}\frac{\partial Yf}{\partial p'}(S,A(p))\,dp \qquad (4.3)$$

case 2 :

$$R'f\left(\vec{OS}.\vec{n},\vec{n}\right) = \frac{\|\vec{OS}\|^2}{\|\vec{OS}\wedge\vec{n}\|^2}\cdot\frac{1}{|\cos\gamma|}\cdot\int_{q=-\infty}^{+\infty}\frac{\partial Yf}{\partial p'}(S,A(q))\,dq \qquad (4.4)$$

In order to achieve only one time the differentiation on the weighted cone beam X-ray transform, for all the integration planes $P\left(\vec{OS}.\vec{n},\vec{n}\right)$, we split the differentiation operator :

$$\frac{\partial Yf}{\partial p'}(S,A) = \cos\alpha \,.\, D_pYf(S,A) + \sin\alpha \,.\, D_qYf(S,A)$$

where $D_p$ and $D_q$ are the differentiation operators along the rows and the columns of the grid :

$$D_pYf = \frac{\partial Yf}{\partial p}$$

$$D_qYf = \frac{\partial Yf}{\partial q}$$



Finally, we get the expressions of the fundamental relation which are used to achieve the numerical implementation :

case 1 :

$$R'f\left(\vec{OS}.\vec{n},\vec{n}\right) = C_1.\left[\cos\alpha \,.\int_{p=-\infty}^{+\infty} D_pYf(S,A(p))\,dp + \sin\alpha \,.\int_{p=-\infty}^{+\infty} D_qYf(S,A(p))\,dp\right]$$

$$\text{with}\quad C_1 = \frac{\|\vec{OS}\|^2}{\|\vec{OS}\wedge\vec{n}\|^2}\cdot\frac{1}{|\sin\alpha|} \qquad (4.5)$$

case 2 :

$$R'f\left(\vec{OS}.\vec{n},\vec{n}\right) = C_2.\left[\cos\alpha \,.\int_{q=-\infty}^{+\infty} D_pYf(S,A(q))\,dq + \sin\alpha \,.\int_{q=-\infty}^{+\infty} D_qYf(S,A(q))\,dq\right]$$

$$\text{with}\quad C_2 = \frac{\|\vec{OS}\|^2}{\|\vec{OS}\wedge\vec{n}\|^2}\cdot\frac{1}{|\cos\alpha|} \qquad (4.6)$$

## 5. FROM THE CONE BEAM X-RAY TRANSFORM Xf TO THE FIRST DERIVATIVE R'f OF THE RADON TRANSFORM.

For a given source position S and a given unit vector $\vec{n}$, the fundamental relation (3.4) allows to compute the first derivative R'f of the Radon transform on the integration plane $P\left(\vec{OS}.\vec{n},\vec{n}\right)$. This can be applied to every unit vector $\vec{n}$ in $S^2$. It means that we can assign its value R'f for every plane crossing S. The case of the unit vector $\vec{n}$ parallel to the source vector $\vec{SO}$ is a singularity only because we use the detection plane to parametrize the cone beam X-ray transform.

The basic idea to design the reconstruction algorithm is to achieve a rebinning operation. But in the three-dimensional space $\mathbb{R}^3$, the set of straight lines intersecting the object support is a four-dimensional space and the set of straight lines intersecting a curve is a three-dimensional space. So there is no change of parameters possible. Whereas the set of planes intersecting the object support and the set of planes intersecting the curve are both three-dimensional spaces. So it becomes possible to achieve a rebinning operation, if we use the planes as information vectors.

We assign to each plane its integral value R'f. Thanks to the fundamental relation (3.4), this value R'f can be computed from the cone beam X-ray transform Xf for the planes which cross the trajectory $\Gamma$ at least at one source position S. The condition on the trajectory to apply the rebinning is to describe all the first derivative R'f of the Radon transform. As the function f is null out of its support $\Omega$, the definition set of R'f can be restricted to the planes which cross the object support $\Omega$, because R'f is null for the others.

Now we can express **the necessary and sufficient condition on the curve $\Gamma$** : the curve $\Gamma$, which defines the cone beam X-ray transform Xf, allows to compute the first derivative R'f of the Radon transform if and only if every plane which does intersect the object support $\Omega$ does intersect the curve $\Gamma$ at least at one point.

This condition is quiet equivalent to the KIRILLOV-TUY condition [KIRILLOV (1961), TUY (1983)], but is more general because we don't need that the planes should not be tangent to the curve at their intersection points. Such a condition have been proposed in [SMITH B. (1985)] using the reconstruction diagram via the Hilbert transform of R'f.

We have defined the cone beam X-ray transform for a source point moving on a curve $\Gamma$. But it can be generalized to a source S moving on a surface $\Sigma$. Then the same necessary and sufficient condition on the surface $\Sigma$ can be stated : the surface $\Sigma$ allows to compute the R'f transform if and only if every plane which does intersect the object support $\Omega$ does intersect the curve $\Gamma$ at least at one point.

Let us introduce now a geometrical description of the planes. The reader should refer to the figures 2 and 5.

We define as the characteristic point C of a plane P, the orthogonal projection of the origin O on P. This description is one-one for the planes which don't cross the origin O. The parameters $(\rho,\theta,\varphi)$ of the Radon transform correspond to the spherical coordinates of the characteristic points C. We have the following relations :

$$\begin{cases} \overrightarrow{OC} = \rho.\vec{n} \\ M \in P(\rho,\vec{n}) \Leftrightarrow \overrightarrow{OM}.\vec{n} = \rho \end{cases} \qquad (5.1)$$

Let us take a point M different from the origin O. We note $\mathcal{A}(M)$ the set of the characteristic points C of the planes crossing the point M. This set $\mathcal{A}(M)$ is defined by the following equation :

$$C \in \mathcal{A}(M) \quad \Leftrightarrow \quad \overrightarrow{CM}.\overrightarrow{CO} = 0 \qquad (5.2)$$



It is equivalent to define $\mathcal{A}(M)$ as the set of the points C such that MCO is a right angle. So $\mathcal{A}(M)$ is the spherical surface of diameter (O,M).

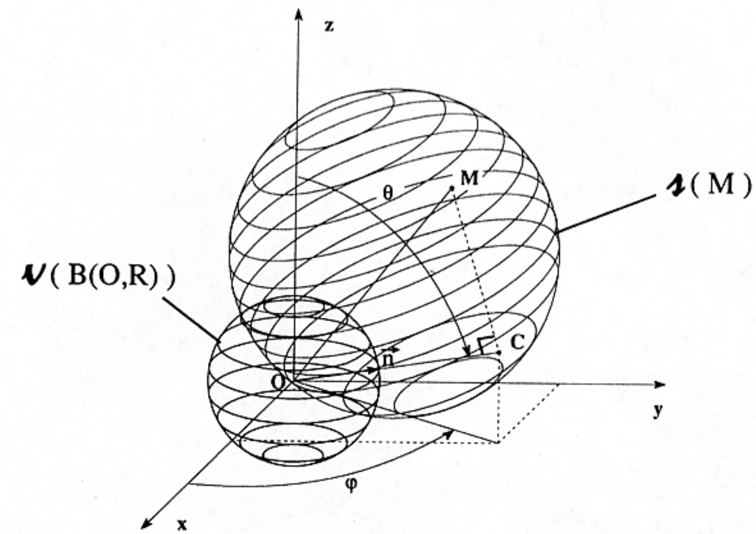


**Figure 5 : Characteristic sets $\mathcal{A}$(M) for the point M and $\mathcal{V}$(B(O,R)) for the ball B(O,R).**

For the curve $\Gamma$, and respectively for the support $\Omega$, we define the characteristic set $\mathcal{V}(\Gamma)$, and respectively $\mathcal{V}(\Omega)$, as the set of the characteristic points of the planes which intersect the curve $\Gamma$, and respectively the support $\Omega$. This sets are defined by the following relations :

$$\mathcal{V}(\Gamma) = \bigcup_{S \in \Gamma} \mathcal{A}(S) \qquad (5.3)$$

$$\mathcal{V}(\Omega) = \bigcup_{M \in \Omega} \mathcal{A}(M) \qquad (5.4)$$

Then we give a geometrical description of the necessary and sufficient condition on the curve $\Gamma$. Every plane which does intersect the object support $\Omega$ does intersect the curve $\Gamma$ at least at one point if and only if :

$$\mathcal{V}(\Omega) \subset \mathcal{V}(\Gamma)$$

For instance, let us take as support $\Omega$ the ball B(O,R) centred on the origin O and with radius R. Then we have :

$$\mathcal{V}(B(O,R)) = B(O,R) \qquad (5.5)$$

The ball B(O,R) is its self characteristic set.

We define the following oscillating curve in the object reference system [GRANGEAT (1987a)] :

$$\begin{cases} x = \left[RSOU^2 - [A \cdot \cos(2\psi)]^2\right]^{\frac{1}{2}} \cdot \sin\psi \\ y = -\left[RSOU^2 - [A \cdot \cos(2\psi)]^2\right]^{\frac{1}{2}} \cdot \cos\psi \\ z = A \cdot \cos(2\psi) \end{cases} \qquad (5.6)$$

If we have the following inequalities, then the necessary and sufficient condition is fullfilled [GRANGEAT (1987b)] :

$$R \leq A \quad \text{and} \quad \sqrt{3} \cdot A \leq RSOU$$

## 6. THE SPECIAL CASE OF THE CIRCULAR TRAJECTORY

The circular trajectory is the most convenient to design. Moreover it has a revolution symetry which induces simplifications and efficiency in the reconstruction processing. But it is a special case because it doesn't fullfill the necessary and sufficient condition on the acquisition curves.

Indeed, if $\Gamma$ is a circular trajectory, the characteristic set $\mathcal{V}(\Gamma)$ is a torus (cf. figure 6). It is the hull of the set of spheres $\mathcal{S}(S)$ of diameter (O,S), for S moving along the circular trajectory $\Gamma$. If the support $\Omega$ is the ball B(O,R), we see on the figure 6 that the necessary and sufficient condition is not fullfilled : $\mathcal{V}(\Omega)$ is not included within $\mathcal{V}(\Gamma)$. There exists a shadow zone in the characteristic set $\mathcal{V}(\Omega)$, which corresponds to the



planes almost parallel to the trajectory plane and which don't cross the circular trajectory $\Gamma$.

Then, from the cone beam X-ray transform Xf we get an incomplete description of the first derivative R'f of the Radon transform. As in 2D for a limited angular acquisition, it does induce artefacts on the reconstructed function. To lower these artefacts, it is necessary to fill R'f by interpolation on this shadow zone. This ability to process the shadow zone is the main difference between the FELDKAMP'S algorithm [FELDKAMP et al. (1984)] and ours [GRANGEAT (1987b)].

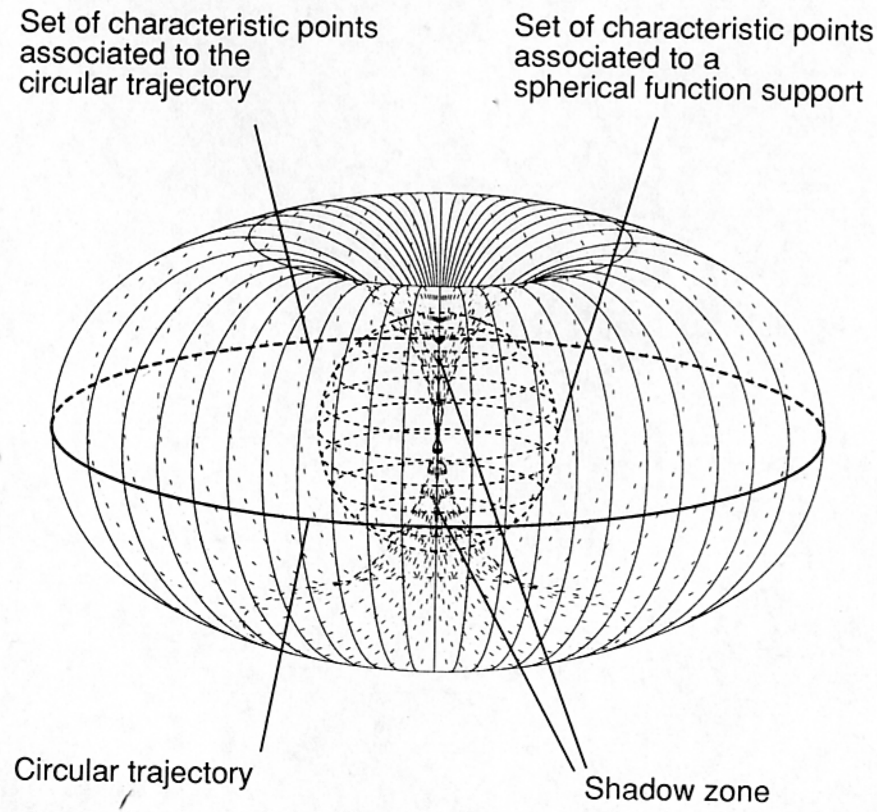


**Figure 6 : Shadow zone in the Radon domain associated to a circular trajectory**

The easiest way to achieve the interpolation on the shadow zone is to consider on each meridian plane, specified by a longitude angle φ, and on each circle, specified by a given distance ρ to the origin, the two upper and respectively the two lower limit points of the shadow zone. Then, on these circles we proceed either to a nearest neighbour interpolation or to a more sophisticated one like a linear interpolation or a B-spline interpolation.

In the inversion formula (7.1) of R'f, the values on this shadow zone are weighted by sin θ and the colatitude angle θ is small. So a high precision on the interpolated data is not necessary.

Let us now describe the rebinning operation (cf. figure 7). We consider as definition set for the angular parameters of the unit vectors $\vec{n}$ :

$$\theta \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$$

$$\varphi \in [0,2\pi] \qquad (6.1)$$

This definition set describes a half of the unit sphere $S^2$.

Let us note $(O,\vec{i},\vec{j},\vec{k})$ the object reference system. The cartesian coordinates of the unit vector $\vec{n}$ are :

$$\vec{n} \begin{vmatrix} \sin\theta \,.\, \cos\varphi \\ \sin\theta \,.\, \sin\varphi \\ \cos\theta \end{vmatrix} \qquad (6.2)$$

The source point S is moving along a circular trajectory, centred on the origin O, of radius RSOU and with $(O,\vec{k})$ as rotation axis.

We define the acquisition coordinates system $(O,\vec{u},\vec{v},\vec{w})$ as :

$$\begin{cases} \vec{w} = \dfrac{\vec{OS}}{\|\vec{OS}\|} \\ \vec{u} = \vec{k} \wedge \vec{w} \\ \vec{v} = \vec{k} \end{cases} \qquad (6.3)$$

$(O,\vec{u},\vec{v})$ is an orthonormal reference system for the detection plane.



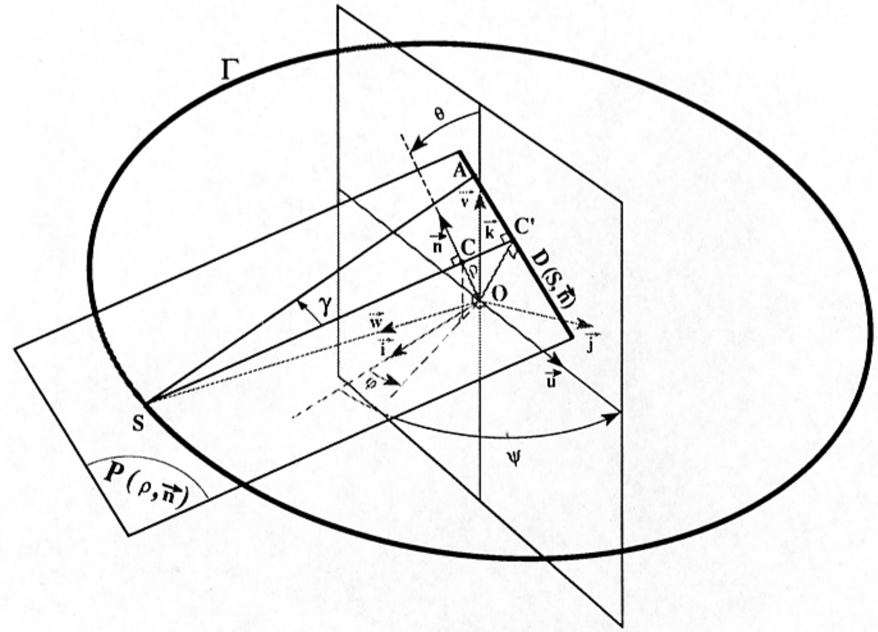


**Figure 7 : Reference systems for the circular trajectory**

The vector $\vec{u}$ belongs to the vectorial plane associated to the $(\vec{i},\vec{j})$ vectorial basis. We use as angular parameter ψ to describe the source position S, the angle between the vectors $\vec{u}$ and $\vec{i}$. The cartesian coordinates of the source position S within the object reference system are :

$$S \begin{vmatrix} RSOU \,.\, \sin\psi \\ -RSOU \,.\, \cos\psi \\ 0 \end{vmatrix} \qquad (6.4)$$

The principle of our algorithm is to fill the Radon domain. For each plane $P(\rho,\vec{n})$,we must find one source position S which belongs to that plane, such that we can apply the fundamental relation (3.4). This source position S is given by the following equation :

$$\vec{OS}.\vec{n} = \rho \iff RSOU \,.\, \sin\theta \,.\, \sin(\psi-\varphi) = \rho \qquad (6.5)$$

We check that this equation has a solution if and only if :

$$|\rho| \leq RSOU \, . \, |\sin\theta| \tag{6.6}$$

This relation defines the torus $\mathcal{V}(\Gamma)$ (cf. figure 6) which is the characteristic set of the circular trajectory $\Gamma$. Each plane which doesn't fullfill this relation belongs to the shadow zone.

Each plane which does intersect the circular trajectory has two intersection points. The first is given by the rebinning equation :

$$\psi = \varphi + \arcsin\left[\frac{\rho}{RSOU \, . \, \sin\theta}\right] \tag{6.7}$$

The second is given by the equation :

$$\psi = \varphi + \pi - \arcsin\left[\frac{\rho}{RSOU \, . \, \sin\theta}\right] \tag{6.8}$$

We have the following redundancy in the definition set of the unit vectors $\vec{n}$ :

$$\vec{n}(\theta,\varphi) = \vec{n}(-\theta,\varphi+\pi) \qquad \theta \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right], \ \varphi \in [0,\pi]$$

So we don't loose any solution if we consider only the first equation (6.7) that we have called the rebinning equation.

For a given $(\rho,\theta)$ couple of parameters, the rebinning from the source coordinate system to the Radon coordinate system is a rotation with the constant angle :

$$-\arcsin\left[\frac{\rho}{RSOU \, . \, \sin\theta}\right]$$

So the rebinning algorithm is very efficient for a circular trajectory [GRANGEAT (1986b), GRANGEAT (1987b)].

Finally, we remark that if the distance RSOU between the source S and the origin O increases to infinity, the cone beam X-ray transform becomes the parallel beam X-ray transform. We can check that there is no more shadow zone. Moreover, no rebinning is necessary, as it is described by the limit of the rebinning equation (6.7) :

$$\psi = \varphi$$



## 7. FROM THE FIRST DERIVATIVE R'f OF THE RADON TRANSFORM TO THE OBJECT FUNCTION f.

The reconstruction of the object function f from the first derivative R'f of its Radon transform is given by the inversion formula (2.7). If we use the definition set $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right] \times [0,2\pi]$ for the angular parameters $(\theta,\varphi)$ of the unit vector $\vec{n}$, we get :

$$f(M) = -\frac{1}{8\pi^2} \cdot \int_{\theta=-\frac{\pi}{2}}^{\frac{\pi}{2}} \int_{\varphi=0}^{2\pi} \frac{\partial R'f}{\partial\rho}\left(\vec{OM}.\vec{n},\vec{n}\right) . \, |\sin\theta| \ d\theta \, d\varphi \tag{7.1}$$

To achieve an efficient implementation of this inversion formula [MARR and al. (1980)], we must split the computation of this double integral in two steps, first an integration over $\theta$ for all the meridian planes and second an integration over $\varphi$ for all the axial planes. For the numerical aspects, we refer to [LOUIS (1983), GRANGEAT (1987b)].

Let us introduce the rebinned X-ray transform $\tilde{X}f$ as the acquisition in parallel geometry along straight lines perpendicular to the axis $(O,\vec{k})$. We note $D(\varphi,B)$ the straight line perpendicular to the meridian plane of longitude $\varphi$, with B as intersection point (cf. figure 8). Then the rebinned X-ray transform $\tilde{X}f$ is defined by :

$$\tilde{X}f(\varphi,B) = \int_{M \in D(\varphi,B)} f(M) \ dM \tag{7.2}$$

Let us define the filtered rebinned X-ray transform $HD\tilde{X}f$ as the convolution of the rebinned X-ray transform $\tilde{X}f$ with the classical ramp filter HD along all the lines perpendicular to the axis $(O,\vec{k})$. The two-dimensional Fourier transform $F_2$ [HD] of this filter HD is given by :

$$F_2 \, [HD] \ (\nu_r,\nu_z) = |\nu_r| \tag{7.3}$$

if $(\nu_r,\nu_z)$ are the frequential coordinates in the directions respectively perpendicular and parallel to the axis $(O,\vec{k})$.

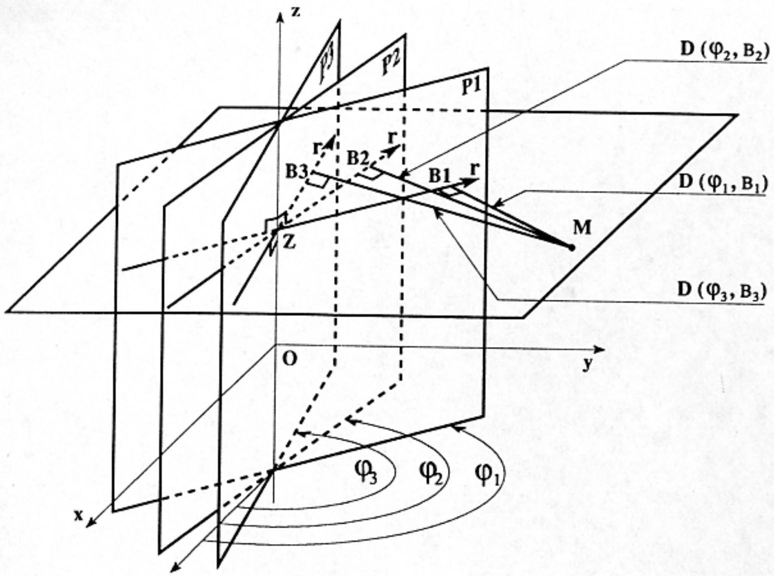


**Figure 8 : The rebinned X-ray transform $\tilde{X}f$**

The same relation can be defined if this frequential plane is parametrized with the polar coordinates (ν,θ) :

$$F_2\,[HD]\,(\nu,\theta) = |\,\nu\,|\,.\,|\,\sin\theta\,| \qquad (7.4)$$

On each meridian plane of longitude φ, we get the following relation for the first convolution backprojection step :

$$HD\tilde{X}f\,(\varphi,B) = -\frac{1}{4\pi^2}\,.\int_{\theta=-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{\partial R'f}{\partial\rho}\left(\overrightarrow{OB}.\vec{n},\vec{n}\right).\,|\,\sin\theta\,|\quad d\theta \qquad (7.5)$$



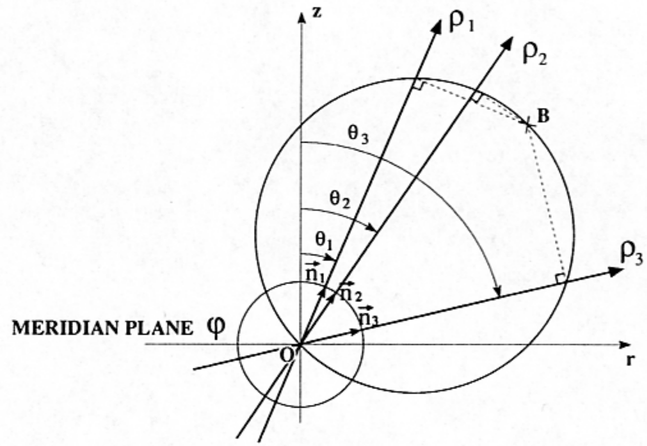


**Figure 9 : The first convolution backprojection step on each meridian plane**

The relation (7.5) can be proved from the central silice theorems. If $\hat{f}$ is the three-dimensional Fourier transform of the function f, we have the folloving relations :

$$Rf\left(\rho,\vec{n}\right) = \int_{\nu=-\infty}^{+\infty} \hat{f}\left(\nu,\vec{n}\right).\,e^{2i\pi\nu\rho}\,d\nu \qquad (7.6)$$

$$\tilde{X}f\,(\varphi,B) = \int_{\theta=-\frac{\pi}{2}}^{\frac{\pi}{2}}\int_{\nu=-\infty}^{+\infty} \hat{f}\left(\nu,\vec{n}\right).\,e^{2i\pi\nu\overrightarrow{OB}.\vec{n}}\,.\,|\,\nu|\quad d\theta\;d\nu \qquad (7.7)$$

After the filtering, we get :

$$\frac{\partial R'f}{\partial\rho}\left(\rho,\vec{n}\right) = -\,4\pi^2\,.\int_{\nu=-\infty}^{+\infty} \hat{f}\left(\nu\,,\vec{n}\right).\,e^{2i\pi\nu\rho}\,.\,\nu^2\,d\nu \qquad (7.8)$$

$$HD\tilde{X}f\,(\varphi,B) = \int_{\theta=-\frac{\pi}{2}}^{\frac{\pi}{2}}\int_{\nu=-\infty}^{+\infty} \hat{f}\left(\nu,\vec{n}\right).\,e^{2i\pi\nu\overrightarrow{OB}.\vec{n}}\,.\,\nu^2\cdot|\,\sin\theta\,|\quad d\theta\;d\nu \qquad (7.9)$$

If we replace the integration over v in the second relation (7.9) by the $\frac{\partial R'f}{\partial \rho}$ function defined by the first relation (7.8), we get the first convolution backprojection formula (7.5).

The second backprojection step is the backprojection of the filtered rebinned X-ray transform $HD\tilde{X}f$ on each axial plane :

$$f(M) = \frac{1}{2} \cdot \int_{\varphi = 0}^{2\pi} HD\tilde{X}f\,(\varphi, B(\varphi, M))\; d\varphi \qquad (7.10)$$

where $B(\varphi,M)$ is the orthogonal projection of the point M on the meridian plane of longitude $\varphi$.

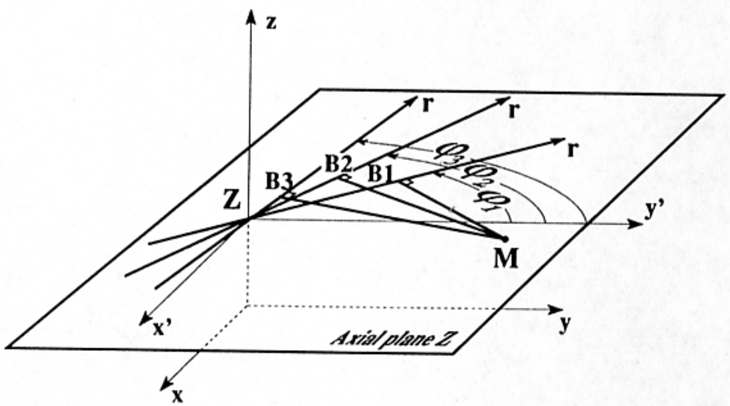


**Figure 10 : The second backprojection step on each axial plane**

This relation is the classical inversion formula of the X-ray transform in parallel geometry.

For the implementation of these formulas, we consider a given number NPHI of meridian angles $\varphi$, which is equal to the number NPSI of source positions for a circular trajectory. If this number is even, we have a redundancy in the description of the Radon domain because :

$$\vec{n}\,(\theta,\varphi) = \vec{n}\,(-\theta,\varphi + \pi) \qquad (\theta,\varphi) \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right] \times [0,\pi] \qquad (7.11)$$



To reduce by a factor 2 the number of backprojections, we first compute the average value $AR'f\,(\theta,\varphi)$ for opposite meridians :

$$A'Rf(\theta,\varphi) = \frac{1}{2}\,[\,R'f(\theta,\varphi) + R'f\,(-\theta,\varphi + \pi)] \qquad (7.12)$$

$$\text{For } (\theta,\varphi) \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right] \times [0,\pi]$$

Then the first convolution backprojection step can be restricted to the longitude angles $\varphi$ between 0 and $\pi$, and the second backprojection step becomes :

$$f(M) = \int_{\varphi = 0}^{\pi} HD\tilde{X}f\,(\varphi, B(\varphi, M))\; d\varphi \qquad (7.13)$$

## 8. CONCLUSION

In this publication, we have described the mathematical framework of our cone beam 3D reconstruction algorithm via the first derivative of the Radon transform. It is based first on the fundamental relation (3.4) between the cone beam X-ray transform Xf and the first derivative R'f of the Radon transform, second on the rebinning operation on planes from the acquisition coordinates system to the Radon spherical coordinates system, third on the inversion of the first derivative R'f of the Radon transform to recover the original function f. This inversion diagram gives a direct reconstruction algorithm for a large class of acquisition trajectories. The induced large family of cone beam 3D tomographic devices associated to this reconstruction process have been patented [GRANGEAT (1987a)]. The description of the mathematical theory and of the numerical processing is presented in our thesis [GRANGEAT (1987b)].

The numerical implementation of the algorithm does proceed in three steps : first the differentiation integration operations on each projection and the rebinning,to compute the first derivative R'f of the Radon transform from the cone beam X-ray transform Xf, second the parallel backprojection step on each meridian to get the filtered rebinned X-ray transform $HD\tilde{X}f$, from R'f, third the parallel backprojection step on each axial plane to compute the function f from $HD\tilde{X}f$. Each step is a sequential processing of elementary 2D files. So this algorithm is well suited for parallelization and vectorization. It uses only the four fondamental operations of reconstruction algorithms that are convolution, rebinning, backprojection, reprojection. So it can be implemented on reconstruction processors [CAQUINEAU and AMANS (1990a), CAQUINEAU (1990b)]. Today, at the LETI, we have developped a software for the cone beam 3D reconstruction via the first derivative of the Radon transform, that we call RADON [GRANGEAT et al. (1990a), SIRE et al. (1990)]. For the moment, it does process the cone beam acquisitions only for a circular trajectory. We have take care to the vectorization of the algorithm. The experimental comparison between the FELDKAMP'S reconstruction software and this RADON software is under study [RIZO and ELLINGSON (1990a), RIZO et al. (1990b)]. The next extension of the RADON software will be to process the acquisitions with a double circular trajectory [RIZO et al. (1990c)]. So we conclude that all this mathematical framework has lead at the LETI to large algorithmical and technological developments.

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