ON A PROBLEM OF I. M. GEL’FAND

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Let there be given a curve $K$ in $n$-dimensional (real or complex) space. The set of all straight lines which intersect $K$ is an $n$-dimensional (real or complex manifold $M$. To every rapidly decreasing and indefinitely differentiable function $f(x)$ on the original space we can associate a function $\hat{f}(m)$ or the manifold $M$: $\hat{f}(m)$ is equal to the integral of $f(x)$ along the line $m \in M$. We show that the function $f(x)$ can be recovered from $\hat{f}(m)$ if and only if the curve $K$ intersects almost all hyperplanes $(\xi, \xi') = 0$. In the case where the order of the intersection is the same for almost all hyperplanes it turns out to be possible to give an explicit formula expressing $f(x)$ from $\hat{f}(m)$. This result answers a question posed by I. M. Gel’fand in [1].

In the following we restrict ourselves to the case of the complex space $C^n$ (for real space the same formulas hold with obvious changes). Let the curve $K$ be given in parametric form by $x = \phi(\lambda)$, where $\lambda$ is a complex parameter. We denote by $g(\alpha, \lambda)$ the integral of $f(x)$ on the line passing through the point $\phi(\lambda)$ in the direction of the vector $\alpha$:

$$g(\alpha, \lambda) = \int f(\phi(\lambda) + t\alpha) \, dt \, d\bar{t}.$$ (1)

Our problem is to recover the function $f(x)$, knowing $g(\alpha, \lambda)$. We denote by $G(\beta, \lambda)$ the Fourier transform of the function $g(\alpha, \lambda)$ with respect to the variable $\alpha$.* Then

$$G(\beta, \lambda) = \int g(\alpha, \lambda) e^{iR(\alpha, \beta)} \, d\alpha \, d\bar{\alpha} = \int f(\phi(\lambda) + t\alpha) e^{iR(\alpha, \beta)} \, dt \, d\alpha \, d\bar{\alpha} =$$

$$= \int f(\phi(\lambda)) e^{iR(\phi(\lambda), \beta)} \, d\tau \, d\tau' \, d\xi \, d\bar{\xi} =$$

$$= \int f(\phi(\lambda)) e^{iR(\phi(\lambda), \beta)} \, d\tau \, d\tau' \, d\xi \, d\bar{\xi} = \int \hat{f}(\beta) \, d\tau \, d\tau' \, d\xi \, d\bar{\xi},$$

where $\hat{f}(\beta)$ is the Fourier transform of the function $f(x)$. We introduce the functions $\Phi(\beta, \gamma) = \hat{f}(\beta) | \tau |^{2n-1}$ and $F(\beta, \omega) = \int \Phi(\beta, \gamma) e^{-iR(\omega, \gamma)} \, d\tau \, d\bar{\tau}$.

The above computations show that

$$G(\beta, \lambda) = F(\beta, (\phi(\lambda), \beta)).$$ (2)

It is clear that the functions $f$ and $F$, just like the functions $g$ and $G$ are connected by an invertible transform. So it suffices to study the connection between $F$ and $G$. The relation (2) shows that if we know the function $G$, we can determine the value $F(\beta, \omega)$ only in case we have $(\phi(\lambda), \beta) = \alpha$ for some $\lambda$. This condition means that the curve $K$ intersects the hyperplane $(\alpha, \beta) = \omega$.

Thus, in order to be able to get the function $F$ back from $G$ (and thereby get $f$ back from $g$) it is necessary and sufficient that $K$ should intersect almost every hyperplane in the case of a plane curve in real three-dimensional space this result was obtained earlier by I. Ya. Vahutinskii.

Assume now that the curve $K$ intersects almost all hyperplanes in exactly $l$ points. This condition is satisfied, for example, if $K$ is an algebraic curve. For almost every $\beta$ we can divide the domain of the parameter $\lambda$ in $l$ parts $\Lambda_1, \ldots, \Lambda_l$ in such a way that for $\lambda \in \Lambda_i$, the range of $\omega = (\phi(\lambda), \beta)$ is almost the whole complex plane. Now we use the expression of $G$ through $F$:

*The function $g(\alpha, \lambda)$ is homogeneous: $g(t\alpha, \lambda) = |t|^{-2}g(\alpha, \lambda)$. So its Fourier transform is a homogeneous generalized function (see e.g. [2], Chapter 3).

**It is easy to see that for $\beta \neq 0$ this integral converges since $f$ is a rapidly decreasing function.
\[ f(x) = (2\pi)^{-2n-2} \sum_{\lambda \in \Lambda} G(\beta, \lambda) e^{i \Re \theta (\beta - x)} \frac{D(\omega, \omega)}{D(\lambda, \lambda)} d\beta d\bar{\beta} d\omega d\bar{\omega} \]

and substitute into it the expression (2) of \( F \). We obtain
\[ f(x) = (2\pi)^{-2n-2} \sum_{\lambda \in \Lambda} g(\alpha, \lambda) e^{i \Re \theta (\beta - x)} \left( \left| \frac{\partial \omega^2}{\partial x} \right| - \left| \frac{\partial \omega^2}{\partial \lambda} \right| \right) d\alpha d\bar{\alpha} d\beta d\bar{\beta} d\lambda d\bar{\lambda} = \]
\[ = \frac{1}{(2\pi)^{-2n-2}} \sum_{\lambda \in \Lambda} Dg(x - \varphi(\lambda), \lambda) d\lambda d\bar{\lambda}, \]

where
\[ Dg(x, \lambda) = \sum \left( \frac{\partial \phi_i}{\partial x} \frac{\partial \phi_j}{\partial \lambda} - \frac{\partial \phi_i}{\partial \lambda} \frac{\partial \phi_j}{\partial x} \right) \frac{\partial^2 g(\alpha, \lambda; x)}{\partial x_i \partial x_j}. \]

In the case of an algebraic curve \( \varphi \) depends analytically on \( \lambda \). Therefore \( \partial \phi_i / \partial \lambda = 0 \), and the operator \( D \) has the simpler expression
\[ D = \sum \frac{\partial \phi_i}{\partial x} \frac{\partial \phi_j}{\partial \lambda} - \frac{\partial \phi_i}{\partial \lambda} \frac{\partial \phi_j}{\partial x}. \]

Our formula
\[ f(x) = \frac{1}{(2\pi)^{-2n-2}} \int Dg(x - \varphi(\lambda), \lambda) d\lambda d\bar{\lambda}, \quad \text{(3)} \]

admits a simpler geometric interpretation: In order to find the value \( f(x) \) we have to know the integral of the function on the lines passing through the point \( x \) (and intersecting \( K \)), and on the lines close to these.

Formula (3) can be rewritten without using the parametric representation of the curve. We note that on the curve \( x = \varphi(\lambda) \) the differential form \( \frac{\partial \phi_i}{\partial \lambda} d\lambda d\bar{\lambda} \) coincides with \( dx_i d\bar{x}_j \). Thus, denoting by \( h(x, \alpha) \) the integral of the function \( f \) along the line passing through the point \( x \) in the direction \( \alpha \), we get
\[ f(x) = \frac{1}{(2\pi)^{-2n-2}} \sum R h(\varphi - x, \lambda) dx_id\bar{x}_j. \quad \text{(4)} \]

In the case where \( K \) is a "hyperbola" in \( \mathbb{C}^3 \) given by equations \( z_1, z_2 = 1, z_3 = 0 \), formula (4) was found by I. M. Gel’fand \[1\].

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BIBLIOGRAPHY


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