ON A PROBLEM OF I. M. GEL'FAND

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Let there be given a curve K in n-dimensional (real or complex) space. The set of all straight lines which intersect K is an n-dimensional (real or complex manifold M. To every rapidly decreasing and indefinitely differentiable function $f(x)$ on the original space we can associate a function $\hat{f}(m)$ on the manifold M: $\hat{f}(m)$ is equal to the integral of $f(x)$ along the line $m \in M$. We show that the function $f(x)$ can be recovered from $\hat{f}(m)$ if and only if the curve K intersects almost all hyperplanes $(x, \xi) = c$. In the case where the order of the intersection is the same for almost all hyperplanes it turns out to be possible to give an explicit formula expressing $f(x)$ from $\hat{f}(m)$. This result answers a question posed by I. M. Gel'fand in [1].

In the following we restrict ourselves to the case of the complex space C^n (for real space the same formulas hold with obvious changes). Let the curve K be given in parametric form by $x = \phi(\lambda)$, where λ is a complex parameter. We denote by $g(\alpha, \lambda)$ the integral of $f(x)$ on the line passing through the point $\phi(\lambda)$ in the direction of the vector α:

$$g(\alpha, \lambda) = \int f(\phi(\lambda) + t\alpha)\, dt\, d\bar{t}. \tag{1}$$

Our problem is to recover the function $f(x)$, knowing $g(\alpha, \lambda)$. We denote by $G(\beta, \lambda)$ the Fourier transform of the function $g(\alpha, \lambda)$ with respect to the variable α.* Then

$$G(\beta, \lambda) = \int g(\alpha, \lambda)\, e^{i\,\mathrm{Re}(\alpha, \beta)}\, d\alpha\, d\bar{\alpha} = \iint f(\varphi(\lambda) + t\alpha)\, e^{i\,\mathrm{Re}(\alpha, \beta)}\, dt\, d\bar{t}\, d\alpha\, d\bar{\alpha} =$$

$$= \iint f(\gamma)\, e^{i\,\mathrm{Re}\left(\frac{\gamma - \varphi(\lambda)}{t}, \beta\right)} |t|^{-2n}\, d\gamma\, d\bar{\gamma}\, dt\, d\bar{t} =$$

$$= \iint f(\gamma)\, e^{\mathrm{R}i\mathrm{e}(\gamma - \varphi(\lambda), \tau\beta)} |\tau|^{2n-4}\, d\gamma\, d\bar{\gamma}\, d\tau\, d\bar{\tau} = \int \tilde{f}(\tau\beta) |\tau|^{2n-4} e^{-i\,\mathrm{Re}(\varphi(\lambda), \beta)}\, d\tau\, d\bar{\tau}.$$

where $\tilde{f}(\beta)$ is the Fourier transform of the function $f(x)$. We introduce the functions $\Phi(\beta, \tau) = \tilde{f}(\tau\beta) |\tau|^{2n-4}$ and $F(\beta, \omega) = \int \Phi(\beta, \tau)\, e^{-i\,\mathrm{Re}\,\omega\tau} d\tau\, d\bar{\tau}$.** The above computations show that

$$G(\beta, \lambda) = F(\beta, (\phi(\lambda), \beta)). \tag{2}$$

It is clear that the functions f and F, just like the functions g and G are connected by an invertible transform. So it suffices to study the connection between F and G. The relation (2) shows that if we know the function G, we can determine the value $F(\beta, \omega)$ only in case we have $(\phi(\lambda), \beta) = \omega$ for some λ. This condition means that the curve K intersects the hyperplane $(x, \beta) = \omega$.

Thus, in order to be able to get the function F back from G (and thereby get f back from g) it is necessary and sufficient that K should intersect almost every hyperplane. In the case of a plane curve in real three-dimensional space this result was obtained earlier by I. Ya. Vahutinskiĭ.

Assume now that the curve K intersects almost all hyperplanes in exactly l points. This condition is satisfied, for example, if K is an algebraic curve. For almost every β we can divide the domain of the parameter λ in l parts $\Lambda_1, \cdots, \Lambda_l$ in such a way that for $\lambda \in \Lambda_i$, the range of $\omega = (\phi(\lambda), \beta)$ is almost the whole complex plane. Now we use the expression of f through F:

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*The function $g(\alpha, \lambda)$ is homogeneous: $g(t\alpha, \lambda) = |t|^{-2} g(\alpha, \lambda)$. So its Fourier transform is a homogeneous generalized function (see e.g. [2], Chapter 3).

**It is easy to see that for $\beta \neq 0$ this integral converges since f is a rapidly decreasing function.

$$f(x) = (2\pi)^{-2n-2} \iint F(\beta, \omega)\, e^{i \operatorname{Re}[\omega - (\beta, x)]}\, d\beta\, d\bar{\beta}\, d\omega\, d\bar{\omega}$$

and substitute into it the expression (2) of F. We obtain

$$f(x) = (2\pi)^{-2n-2} \iint_{\lambda \in \Lambda_i} G(\beta, \lambda)\, e^{i \operatorname{Re}(\varphi(\lambda) - x, \beta)} \frac{D(\omega, \bar{\omega})}{D(\lambda, \bar{\lambda})}\, d\beta\, d\bar{\beta}\, d\lambda\, d\bar{\lambda} =$$

$$= \frac{1}{l} (2\pi)^{-2n-2} \iiint g(\alpha, \lambda)\, e^{i \operatorname{Re}(\varphi(\lambda) - x + \alpha, \beta)} \left(\left| \frac{\partial \omega}{\partial \lambda} \right|^2 - \left| \frac{\partial \omega}{\partial \bar{\lambda}} \right|^2 \right) d\alpha\, d\bar{\alpha}\, d\beta\, d\bar{\beta}\, d\lambda\, d\bar{\lambda} =$$

$$= \frac{-1}{4l\pi^2} \int Dg(x - \varphi(\lambda), \lambda)\, d\lambda\, d\bar{\lambda},$$

where $Dg(\alpha, \lambda) = \sum \left(\frac{\partial \varphi_i}{\partial \lambda} \frac{\partial \bar{\varphi}_j}{\partial \bar{\lambda}} - \frac{\partial \varphi_i}{\partial \bar{\lambda}} \frac{\partial \bar{\varphi}_j}{\partial \lambda} \right) \frac{\partial^2 g(\alpha, \lambda)}{\partial \alpha_i \partial \bar{\alpha}_j}$.

In the case of an algebraic curve ϕ depends analytically on λ. Therefore $\partial \phi_i / \partial \bar{\lambda} = 0$, and the operator D has the simpler expression

$$D = \sum \frac{\partial \varphi_i}{\partial \lambda} \frac{\partial \bar{\varphi}_j}{\partial \bar{\lambda}} \frac{\partial^2}{\partial \alpha_i \partial \bar{\alpha}_j}.$$

Our formula

$$f(x) = \frac{-1}{4l\pi^2} \int Dg(x - \varphi(\lambda), \lambda)\, d\lambda\, d\bar{\lambda} \tag{3}$$

admits a simpler geometric interpretation: In order to find the value $f(x)$ we have to know the integral of the function on the lines passing through the point x (and intersecting K), and on the lines close to these.

Formula (3) can be rewritten without using the parametric representation of the curve. We note that on the curve $x = \phi(\lambda)$ the differential form $\frac{\partial \phi_i}{\partial \lambda} \frac{\partial \bar{\phi}_j}{\partial \bar{\lambda}} d\lambda\, d\bar{\lambda}$ coincides with $dx_i d\bar{x}_j$. Thus, denoting by $h(\alpha, x)$ the integral of the function f along the line passing through the point x in the direction α, we get

$$f(x_0) = \frac{-1}{4l\pi^2} \int_K \sum \frac{\partial^2 h(x_0 - x, x)}{\partial \alpha_i \partial \bar{\alpha}_j} dx_i\, d\bar{x}_j. \tag{4}$$

In the case where K is a "hyperbola" in C^3 given by equations $z_1, z_2 = 1, z_3 = 0$, formula (4) was found by I. M. Gel'fand [1].

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