Theory of three dimensional reconstruction.
1. Conditions for a complete set of projections

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Conditions are given under which the set of projections of the three-dimensional function contains complete information about it.

1. INTRODUCTION

The optics of the electron microscope are such that the microphotograph of a biomacromolecule is the projection of its density distribution, as discussed in ref. 1. The problem of recovering the density of the object from the given projections is the basic problem of electron microscopy of biomacromolecules.

In ref. 2 the authors considered the case of coaxial projection when the two projected directions are perpendicular to the fixed axis. The problem of recovering the structure was accomplished, in this case, by using Radon's equation from ref. 3.

The aim of this article is to study the general case, when the projection directions occupy a certain domain on the sphere of directions.

2. POISING THE PROBLEM

Mathematically the problem is formulated as follows. Let \( \rho(x) \) be a finite function describing the structure of some object. The projection of this object along the line passing through \( x \) and having the direction \( \tau \) is given by the integral

\[
\rho(x, \tau) = \int_{-\infty}^{\infty} \rho(x + \tau t) \, dt.
\]  

(1)

The function \( \rho(x, \tau) \) is not a function of the three-dimensional vector \( x \), but of the two-dimensional vector \( x - \tau(x\tau) \), perpendicular to the vector \( \tau \). It does not depend on the sign of \( \tau \), and it describes the structure of the two-dimensional projection of the given object. For brevity, we denote it by \( \rho(x, \tau) \).

Suppose we know the projections \( \{\tau\} \) along a continuous set of directions \( \{\tau\} \). The problem of decoding the structure of the object under consideration is reduced to the solution of the integral equation (1) defined for all \( x \) and all \( \tau \in \{\tau\} \), i.e., to the finding of a recovery operator \( R\{\tau\} \).

\[
\rho(x) = R\{\tau\} \rho(x, \tau).
\]  

(2)

3. CONDITIONS FOR THE EXISTENCE OF A SOLUTION

We shall first show that for sets \( \{\tau\} \) the problem has a solution. In what follows these sets will be called complete sets. Take the unit sphere of directions \( S \). Each vector \( \tau \in \{\tau\} \) is represented on it by a point \( P_\tau \), and the set \( \{\tau\} \) is a certain closed domain \( G_\tau \).

We now show that in order to recover \( \rho(x) \) it is necessary and sufficient that \( G_\tau \) have points in common with any arc of a great circle. To prove this we note that if some projection \( \rho(x, \tau) \) is known, then the Fourier transform of the function \( \rho(x) \) can be found for all directions lying in the plane orthogonal to the vector \( \tau \). Whence it follows, for example, that if a certain arc of a great circle has at least one point in common with \( G_\tau \), then the Fourier transform of the function \( \rho(x) \) can be found in the direction perpendicular to the plane of this circle. If any arc of a great circle has points in common with \( G_\tau \), then the Fourier transform of the function \( \rho(x) \) can be found for any direction, and therefore in this case it will be possible to recover \( \rho(x) \).

Now let the domain \( G_\tau \) be such that there is an arc \( L \) of the great circle which does not have points in common with \( G_\tau \). Since \( G_\tau \) is closed, each point of this arc \( L \) has its own \( \tau \)-neighborhood in which there are no points of \( G_\tau \). This means that the Fourier transform of the function \( \rho(x) \) cannot be found for the directions lying in some neighborhood of the normal to the plane of \( L \), and therefore \( \rho(x) \) cannot be recovered.

The set \( G_\tau \) corresponding to the complete set \( \{\tau\} \) is called complete. This follows from the assertion proved above.

If the set \( G_\tau \) contains diametrically opposed points, then it is complete. Such a domain can be any curve joining diametrically opposed points. In the simplest case it will be a great semicircle.

If a two-dimensional domain \( G_\tau \) is complete, but does not contain diametrically opposed points, then the boundary \( \partial G_\tau \) of \( G_\tau \) is complete.

4. FORM OF THE VARIOUS PROJECTIONS \( \{\tau\} \)

We now show that if the projections of \( \rho(x) \) are known along some curve \( \Gamma \), joining the arbitrary points \( P_1 \) and \( P_2 \) on the sphere of directions, then it is possible to determine the projection of \( \rho(x) \) along the shortest curve \( \Gamma_0 \) of the arc of the great circle joining these points.

Let \( \{\tau_0\} \) be an arbitrary projection lying on \( \Gamma_0 \), and let \( \{\tau\} \) be the set of directions corresponding to \( \Gamma \). Project each two-dimensional function \( \rho(x, \tau) \) from the set \( \{\tau\} \) along the direction \( \tau = \tau_0 \). This projection coincides with the projection of the function \( \rho(x, \tau_0) \) along the direction \( \tau = \tau_0 \). This coincidence with the projection of the function \( \rho(x, \tau_0) \) along the direction \( \tau = \tau_0 \), and both coincide with the convolution \( \rho(x) \delta(x, n) \), where \( n = \|\tau_0\| \|\tau_0\|^{-1}, \delta(0x) = \delta(0) \) being the one-dimensional \( \delta \)-function. As \( \tau \) moves along \( \Gamma \) the vector \( n \) (up to sign) runs over all possible directions in the plane perpendicular to the vector \( \tau_0 \). This means that
the projections of the two-dimensional function \( \rho(x, \tau) \) are known, and therefore \( \rho(x, \tau_0) \) can be recovered by Radon's equation.

Let us obtain the appropriate equation. Let \( \rho(x, \tau) = \psi(x, [\tau, n]) \).

\[
\psi(x, [\tau, n]) = \int \psi(x + [\tau, n] \eta) \, d\eta = \int_{\mathbb{R}^2} \rho(x + [\tau, n] \eta, \tau) \, d\eta.
\]  

Radon's equation yields

\[
\psi(x) = -\frac{\Delta}{2\pi^2} \int d\eta_0 \int d\tau \psi(\eta_0, [\tau, n]) \ln |t - nx|,
\]  

where \( \Delta \) is the Laplace operator. We substitute (3) in

\[
\rho(x, \tau) = -\frac{\Delta}{2\pi^2} \int d\eta_0 \int d\tau \left( \rho(x', \tau) \ln |nx' - nx| \right),
\]  

where \( \Sigma_\tau \) is the plane which passes through the coordinate origin and perpendicular to the vector \( \tau \). Now we express the integration with respect to the azimuth of the vector \( n \) in the form of an integral along \( \Gamma \):

\[
d\eta_0 = \frac{\tau_0 \tau}{|\tau_0 \times \tau|^2} \, dl,
\]  

where \( dl \) is the element of length along \( \Gamma \), \( \tau = d\tau/dl \), and \( \tau_0 \) is the cross product of the vectors \( \tau_0, \tau \). Finally,

\[
\rho(x, \tau_0) = \frac{1}{2\pi^2} \int_{\mathbb{R}^2} \frac{\rho(x') \tau \rho(x', \tau)}{|\tau_0 \times \tau|^2} \, dl_0.
\]  

Now let some closed simple (nonself-intersecting) curve \( \Gamma \) be given on \( S \). Consider that domain \( G_0 \) of the sphere which is bounded by \( \Gamma \), and has solid angle \( \Omega \) with respect to the center of the sphere.

Any projection \( \{\tau_0\} \), lying inside \( G \), can be found from known projection \( \{\tau\} \) along the boundary of the domain. To prove this, it suffices to pass an arc of a great circle through the point \( P_0 \), where the arc meets the boundary \( S \) at the points \( P_1 \) and \( P_2 \), and then to use the results of Sec. 4.

**Symmetry**

If the object has point-symmetry group \( g \), then the function \( \rho(x) \) is invariant with respect to the transformations of the group \( F_k \) of this group:

\[
\rho(F_k x) = \rho(x) \quad (k = 1, 2, \ldots, p),
\]  

where \( p \) is the order of the group \( g \). From this relation, from the definition of the projection, it follows that the projections in directions forming the star of this group are related by the equation

\[
\rho(x, F_k \tau) = \rho(F_k^{-1} x, \tau).
\]  

is assumed that the projections of the object have radial symmetry.
3) m3, p = 24.
One vertex of the triangle is at the point where the 3 axis leaves S, and the other two are at the points where the 2 axes leave S.

4) m3m, p = 48.
The triangle is defined by the points where the 4, 3, and 2 axes leave S.

5) 53m, p = 120.
The triangle is defined by the points where the 5, 3, and 2 axes leave S.

It is easy to prove that for all the groups listed above, except for n and nm, any side of the spherical triangles so determined is complete. Also complete is any line containing some vertex of the triangle. For the groups m3m, m3m, and n/mnm, moreover, any line joining any vertex with the opposite side is complete. For the groups m3 and n/m the line joining the point where the 3 or n axis leaves the sphere with the opposite side is complete.

For the group nm those lines are complete which join the vertices of the triangle with the 2 axis, and for the group n the arc $\Gamma_0 = 2\pi/n$ of the great circle perpendicular to the n axis is complete. Any two-dimensional domain containing the complete curves $\Gamma$ listed above is complete.

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1) If the sets $\{\tau\}$ and $\{-\tau\}$ pass continuously from one to the other, then we will take $\{\tau\}$ to mean the set $\{\tau\} + \{-\tau\}$. In the opposite case, the set $\{\tau\}$ does not contain vectors which differ only by sign.
2) A domain is any continuous connected two-dimensional or one-dimensional set of points on the sphere. A closed domain is, as usual, the union of the set and its frontier.
3) When $G_\tau$ is multiply connected, its boundary is complete which is closest to the domain $G - \tau$, which is the image of the set $\{-\tau\}$.

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