INSTITUTE OF PHYSICS PUBLISHING INVERSE PROBLEMS

Inverse Problems **20** (2004) 239–242 PII: S0266-5611(04)66672-X

Reconstruction from ray integrals with sources on a curve

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Received 28 July 2003, in final form 23 October 2003
Published 5 December 2003
Online at stacks.iop.org/IP/20/239 (DOI: 10.1088/0266-5611/20/1/014)

Abstract
A new formula is given for the reconstruction of a function in 3D space from a family of its ray integrals. The cost of the reconstruction is two-fold integration.

1. The formula

Let $\mathbf{E}$ be a three-dimensional Euclidean space and f be a bounded function with compact support in $\mathbf{E}$. Take a point y, a unit vector v in $\mathbf{E}$ and consider the ray $R(y, v)$ with the source y and the direction v. The ray integral of f is

$$g(y, v) = \int_0^\infty f(y + tv)\,\mathrm{d}t.$$

The problem of reconstruction of the function f from data of ray integrals with sources on a curve $\Gamma \subset \mathbf{E}$ is of practical importance; see, for example, [7]. Several methods are known [1–3, 6, 8]. Grangeat's method [3] gives the first derivative of the 3D Radon transform of f by one-fold integration. Combining with the Radon inversion formula yields a reconstruction by means of three-fold integration. Katsevich [5] has adapted this method for numerical implementation by means of segmentation of Γ and local reduction to two-fold integration. He introduced special weights to cope with multiple intersection of a hyperplane with Γ.

We state here a global two-fold integral reconstruction formula from data of ray integrals $g(y, v)$, $y \in \Gamma$. The function f can be evaluated on any chord of Γ. No special weight is necessary.

Theorem 1. *Let* $\Gamma = \{y = y(s), 0 \leqslant s \leqslant 1\}$ *be a* C^1*-curve in* $\mathbf{E}$*. An arbitrary function* $f \in C^2(\mathbf{E})$ *such that* $\operatorname{supp} f \Subset \mathbf{E}\backslash\Gamma$ *can be recovered in any point* $x \in\ ]y(0), y(1)[$ *from its ray integrals* g *by*

$$2\pi^2 f(x) = -\int_0^1 \frac{\mathrm{d}\gamma}{r(s)} \int \frac{\partial}{\partial\phi} g(y(s), v) \frac{\mathrm{d}\phi}{\sin\phi} + \int_0^1 \frac{r'(s)\,\mathrm{d}s}{r^2(s)} \int g(y(s), v) \frac{\mathrm{d}\phi}{\sin\phi} + \frac{1}{r(s)} \int g(y(s), v) \frac{\mathrm{d}\phi}{\sin\phi} \Big|_{s=0}^{s=1}. \qquad (1.1)$$

Here $r(s) = |y(s) - x|$, $\mathrm{d}\gamma$ *is the angle a piece of* Γ *is seen from* x, $v = v(s, \phi)$ *is a unit vector in the plane* $T_{y(s)}$ *spanned by* $y(s) - x$ *and* $y'(s)$ *with the polar angle* ϕ *which is defined by the conditions* $\phi(y(s) - x) = 0$, $\sin\phi(y'(s)) \geqslant 0$.

The integrals with the density $\mathrm{d}\phi/\sin\phi$ are taken as principal value integrals over the interval $[0, 2\pi]$. The set of points s, where the vectors $y(s) - x$ and $y'(s)$ are collinear, is neglected in the s-integral (it has zero measure). The first inner integral does not depend on the orientation of Γ and so the density $\mathrm{d}\gamma$ does. The same is true for the second and the third terms, since changing the orientation of Γ multiplies the inner integral by -1.

2. Proof

For an arbitrary point $y \neq x$ in $\mathbf{E}$ and a plane T that contains x and y we consider the integral

$$I(y) \doteq \int_0^{2\pi} g(y, v(\phi)) \frac{\mathrm{d}\phi}{\sin\phi}$$

taken over the unit circle in T.

Lemma 2. *We have* $\langle y - x, \nabla_y \rangle I(y) = 0$.

Choose an orientation in T and set $v = v(\phi)$, $u = v(\phi + \pi/2)$. We have

$$\langle y - x, \nabla_y \rangle g(y, v) = -|y - x| \sin\phi \left.\frac{\partial g(y + pu, v)}{\partial p}\right|_{p=0}$$

and

$$\begin{aligned}
\langle y - x, \nabla_y \rangle \int_0^{2\pi} g(y, v) \frac{\mathrm{d}\phi}{\sin\phi} &= \int \langle y - x, \nabla_y \rangle g(y, v) \frac{\mathrm{d}\phi}{\sin\phi} \\
&= -|y - x| \int \left.\frac{\partial g(y + pu(\phi), v)}{\partial p}\right|_{p=0} \mathrm{d}\phi \\
&= |y - x| \int \int_0^{\infty} \left.\frac{f(y + pu + \tau v)}{\partial p}\right|_{p=0} \mathrm{d}\tau \, \mathrm{d}\phi \\
&= |y - x| \int \int \frac{\partial f(y + \tau v)}{\partial\phi} \frac{\mathrm{d}\tau}{\tau} \mathrm{d}\phi \\
&= |y - x| \int_0^{\infty} \frac{\mathrm{d}\tau}{\tau} \int_0^{2\pi} \frac{\partial}{\partial\phi} f(y + \tau v(\phi)) \, \mathrm{d}\phi = 0.
\end{aligned}$$

The exterior integral converges at $\tau = 0$ since f vanishes in the neighbourhood of y.

Now we apply a homotopy of Γ to a curve $\Gamma(\infty)$ in the infinite sphere. Set $y(s, t) \doteq x + t(y(s) - x)$, $t > 0$ and consider the derivative

$$I(s, t) \doteq \langle y'(s), \nabla_y \rangle \int_0^{2\pi} g(y(s, t), v) \frac{\mathrm{d}\phi}{\sin\phi}.$$

Taking the t-derivative

$$\frac{\mathrm{d}}{\mathrm{d}t} I(s, t) = \langle y(s) - x, \nabla_y \rangle \langle y'(s), \nabla_y \rangle \int_0^{2\pi} g(y(s, t), v) \frac{\mathrm{d}\phi}{\sin\phi},$$

we can change order of the operators $\langle y'(s), \nabla_y \rangle$ and $\langle y(s) - x, \nabla_y \rangle$. The result vanishes because of lemma 2. It follows that $I(s, 1) = I(s, t)$ for any s, $0 \leqslant s \leqslant S$, $t > 0$. Substituting

$$\langle y'(s), \nabla \rangle g(y(s), v(s, \phi)) = \frac{\partial}{\partial s} g(y(s), v(s, \phi)) - \frac{\mathrm{d}\gamma}{\mathrm{d}s} \frac{\partial}{\partial\phi} g(y(s), v(s, \phi)),$$

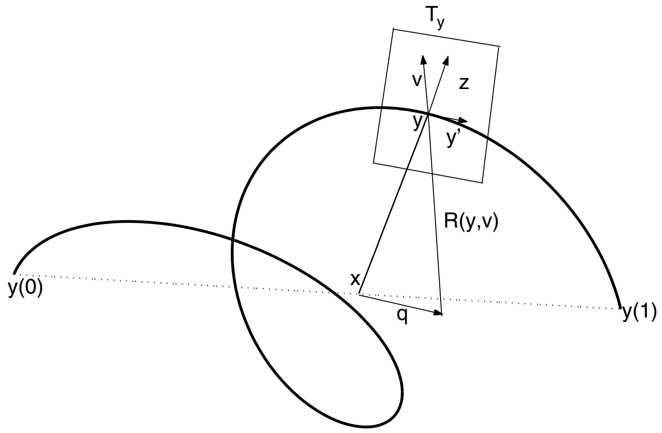

Figure 1. The curve Γ and the tangent plane T through x and y; $z = y - x$, $q = R\sin\phi$.

yields

$$\int_0^S I(s,1)\,\mathrm{d}s = -\int_\Gamma \frac{\mathrm{d}\gamma}{r(s)} \int_0^{2\pi} \frac{\partial}{\partial\phi} g(y(s),v)\frac{\mathrm{d}\phi}{\sin\phi} + \int_\Gamma \frac{1}{r(s)}\frac{\partial}{\partial s}\int_0^{2\pi} g(y(s),v)\frac{\mathrm{d}\phi}{\sin\phi}.$$

Integrating by parts in the last term, we get the right-hand side of (1.1). The integral $\int I(s,t)\,\mathrm{d}s$ is equal to the right-hand side of (1.1) for the curve $\Gamma(t)$ given by the equation $y = y(s,t)$. The second and third terms tend to zero as $t \to \infty$ since $I(y) \to 0$ as $|y| \to \infty$. The density $\mathrm{d}\gamma$ is the curve element of projection $\Gamma(\infty)$ of the curve $\Gamma(t)$ to the infinite 'sky' sphere $\mathbb{S}^2$. For large t the support of the function $g(y(s,t), v(s,\phi))$ is contained in the interval $|\phi| \leqslant C/R$, where $R = t(y(s) - x)$ and C does not depend on s. Setting $q = R\sin\phi$ in the first term yields (see figure 1)

$$-\int_0^1 \mathrm{d}\gamma \int \frac{\partial}{\partial q} g\left(x - q\frac{z'(s)}{|z'(s)|}, z(s)\right)\frac{\mathrm{d}q}{q} + \mathrm{O}(R^{-1}). \tag{2.1}$$

Here $z = z(s)$ is a parameterization of the curve $\Gamma(\infty) \subset \mathbb{S}^2$ and $z(1) = -z(0)$. Therefore the image of $\Gamma(\infty)$ under the natural projection $\mathbb{S}^2 \to \mathbb{P}(\mathbf{E})$ is a closed curve Θ that is not homotopic to a point. By (3.1) the main term of (2.1) is equal to $2\pi^2 f(x)$, which completes the proof of theorem 1.

Remark. It is easy to check that the integral $I(y)$ is equal to the Hilbert transform in the direction $(y - x)^\perp$ of the Radon transform of f in the plane T_y, see [4]. This gives another proof of lemma 2.

3. Sources at infinity

Take a curve Θ in the projective plane $\mathbb{P}(\mathbf{E})$ and consider data of line integrals $g(x,\theta) = \int_{\mathbb{R}} f(x + t\theta)\,\mathrm{d}s$, $\theta \in \Theta$ for lines that meet Θ at infinity. The function f can be reconstructed as follows.

Theorem 3. *Let $\Theta \subset \mathbb{P}(\mathbf{E})$ be a closed C^1-curve that is not homotopic to a point. Let $\theta = \theta(s)$, $0 \leqslant s \leqslant 1$ be the equation of this curve in the unit sphere. The integral*

$$f(x) = -\frac{1}{2\pi^2}\int_0^1 \mathrm{d}s \int_{-\infty}^{\infty} \frac{\partial}{\partial q} g\left(x - q\frac{\theta'(s)}{|\theta'(s)|}, \theta(s)\right)\frac{\mathrm{d}q}{q} \tag{3.1}$$

gives a reconstruction of the function $f \in C^2(\mathbf{E})$ with compact support.

This fact was proved in [6], remark 3 under the assumption that $|\theta'| = 1$. The general case follows by the coordinate change $\tilde{\theta}(s) = (\int_0^s \theta'(\sigma)\,\mathrm{d}\sigma)^{-1}\theta(s)$.

If Θ is a projective line, then (3.1) coincides with the inversion formula for the Radon transform in each plane H such that $\mathbb{P}(H) = \Theta$.

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