Appendix A

Translation of Radon's 1917 Paper*

ON THE DETERMINATION OF FUNCTIONS FROM THEIR INTEGRALS ALONG CERTAIN MANIFOLDS

If one integrates a function of two variables x, y—a *point-function* $f(P)$ in the plane—that satisfies suitable regularity conditions, along an arbitrary straight line g, then the values $F(g)$ of these integrals define a *line-function*. The problem that is solved in part A of this paper is the inversion of this functional transformation. That is, answers to the following questions are given: Is every line-function that satisfies suitable regularity conditions obtainable by this process? If this is the case, is the point-function f then uniquely determined by F and how can it be found?

The problem of finding a line-function $F(g)$ from the mean values over its points $f(P)$, which is in a sense the dual problem, is solved in part B.

Finally, in part C, certain generalizations that arise particularly from considering non-Euclidian manifolds as well as higher-dimensional spaces are briefly discussed.

Interesting in themselves, the treatment of these problems is gaining even more interest because of the fact that there are numerous relations between this subject and the theory of the logarithmic and the Newtonian potential. These will be pointed out in the appropriate places.

A. DETERMINATION OF A POINT-FUNCTION IN THE PLANE FROM ITS INTEGRALS ALONG STRAIGHT LINES

1. Let $f(x, y)$ be a real function defined for all real points $p = [x, y]$ that satisfies the following regularity conditions:

(a_1) $f(x, y)$ is continuous.

(b_1) The following double integral, which is to be taken over the whole plane, is convergent:

$$\iint \frac{|f(x, y)|}{\sqrt{x^2 + y^2}}\, dx\, dy.$$

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(c_1) For an arbitrary point $P = [x, y]$ and any $r \geqslant 0$, let

$$\bar{f}_P(r) = \frac{1}{2\pi}\int_0^{2\pi} f(x + r\cos\phi,\ y + r\sin\phi)\, d\phi.$$

Then for every point P,

$$\lim_{r\to\infty} \bar{f}_P(r) = 0.$$

Thus the following theorems hold.

Theorem I. The integral of f along the straight line g with the equation $x\cos\phi + y\sin\phi = p$, given by

(I)

$$F(p, \phi) = F(-p, \phi + \pi) = \int_{-\infty}^{+\infty} f(p\cos\phi - s\sin\phi,\ p\sin\phi + s\cos\phi)\, ds$$

is "in general" well-defined. This means that on any circle those points that have tangent lines for which F does not exist form a set of linear measure zero.

Theorem II. If the mean value of $F(p, \phi)$ is formed for the tangent lines of the circle with center $P = [x, y]$ and radius q:

$$\text{(II)}\qquad \bar{F}_P(q) = \frac{1}{2\pi}\int_0^{2\pi} F(x\cos\phi + y\sin\phi + q, \phi)\, d\phi,$$

then this integral is absolutely convergent for all P, q.

Theorem III. The value of f is completely determined by F and can be computed as follows:

$$\text{(III)}\qquad f(P) = -\frac{1}{\pi}\int_0^{\infty} \frac{d\bar{F}_P(q)}{q}.$$

Here the integral is to be understood in the Stieltjes sense and it can also be defined by the formula:

$$\text{(III}'\text{)}\qquad f(P) = \frac{1}{\pi}\lim_{\varepsilon\to 0}\left(\frac{\bar{F}_P(\varepsilon)}{\varepsilon} - \int_{\varepsilon}^{\infty} \frac{\bar{F}_P(q)}{q^2}\, dq\right).$$

Before starting with the proof of these theorems, we note that conditions a_1–c_1 are invariant under rigid motions of the plane. Thus we can always consider the point [0, 0] to represent an arbitrary point of the plane.

Now the double integral

(1) $$\iint_{x^2+y^2>q^2} \frac{f(x,y)}{\sqrt{x^2+y^2-q^2}}\,dx\,dy$$

is seen to converge absolutely. Using the transformation

$$x = q\cos\phi - s\sin\phi, \qquad y = q\sin\phi + s\cos\phi,$$

it becomes

$$\int_0^{2\pi} d\phi \int_0^{\infty} f(q\cos\phi - s\sin\phi, q\sin\phi + s\cos\phi)\,ds$$
$$= \int_0^{2\pi} d\phi \int_{-\infty}^{0} f(q\cos\phi - s\sin\phi, q\sin\phi + s\cos\phi)\,ds,$$

so that its value can also be expressed as

$$\frac{1}{2}\int_0^{2\pi} d\phi \int_{-\infty}^{+\infty} f(q\cos\phi - s\sin\phi, q\sin\phi + s\cos\phi)\,ds = \frac{1}{2}\int_0^{2\pi} F(q,\phi)\,d\phi.$$

From well-known properties of absolute convergent double integrals, theorems I and II follow.

In order to derive formula (III), one can choose the following path: Introducing polar coordinates in (1) yields

$$\int_q^{\infty} dr \int_0^{2\pi} \frac{f(r\cos\phi, r\sin\phi)}{\sqrt{r^2-q^2}}\,d\phi$$

or, using the mean value notation from c_1:

$$2\pi\int_q^{\infty} \frac{\bar{f}_0(r)\,dr}{\sqrt{r^2-q^2}}.$$

Comparing this with the value of (1) obtained before,

(2) $$\bar{F}_0(q) = 2\int_q^{\infty} \frac{\bar{f}_0(r)\,dr}{\sqrt{r^2-q^2}}.$$

Introducing the variables $r^2 = v$, $q^2 = u$, this integral equation of the first kind can easily be solved by the well-known method of Abel, which yields formula (III) for

$$\bar{f}_0(0) = f(0,0).$$

However, it seems to be hard to derive this without placing further restrictions on f; therefore, we prefer a direct verification.

To prove the equality of the expressions (III) and (III′), it first must be shown that

$$\lim_{q\to\infty} \frac{\bar{F}_0(q)}{q} = 0.$$

Because of (2),

$$\left|\frac{\bar{F}_0(q)}{q}\right| \leqslant \frac{2}{q}\left|\int_q^{2q} \frac{\bar{f}_0(r)\,r\,dr}{\sqrt{r^2-q^2}}\right| + \frac{2}{q}\left|\int_{2q}^{\infty} \frac{\bar{f}_0(r)\,dr}{\sqrt{r^2-q^2}}\right|$$
$$\leqslant 2\sqrt{3}\,|\bar{f}_0(t)| + \frac{4}{\sqrt{3}}\int_{2q}^{\infty} |\bar{f}_0(r)|\,dr \qquad (q < t < 2q)$$

and this converges to zero as $q \to \infty$ because of b_1 and c_1.

Introducing (2), the right-hand side of (III′) is transformed into

$$\frac{2}{\pi}\lim_{\varepsilon\to 0}\left(\frac{1}{\varepsilon}\int_{\varepsilon}^{\infty} \frac{r\bar{f}_0(r)}{\sqrt{r^2-\varepsilon^2}}\,dr - \int_{\varepsilon}^{\infty}\frac{dq}{q^2}\int_q^{\infty}\frac{r\bar{f}_0(r)}{\sqrt{r^2-q^2}}\,dr\right).$$

If the order of integration is interchanged in the second integral, one can integrate with respect to q and see that this integral is an absolute convergent double integral that justifies the interchange. Moreover, one finds the value

$$\frac{2}{\pi}\lim_{\varepsilon\to 0}\int_{\varepsilon}^{\infty} \frac{\bar{f}_0(r)}{r\sqrt{r^2-\varepsilon^2}}\,dr$$

for the preceding expression which yields, in fact, $\bar{f}_0(0) = f(0,0)$, as can be shown without difficulty.

2. Let $F(p,\phi) = F(-p, \phi+\pi)$ be a line-function satisfying the following regularity conditions:

(a_2) F and its derivatives F_p, F_{pp}, F_{ppp}, F_ϕ, $F_{p\phi}$, $F_{pp\phi}$ are continuous for all $[p,\phi]$.

(b_2) F, F_ϕ, pF_p, $pF_{p\phi}$ and pF_{pp} converge to zero uniformly in ϕ as $p \to \infty$.

(c_2) The integrals

$$\int_0^{\infty} F_{pp}\ln p\,dp,\ \int_0^{\infty} F_{ppp}\,p\ln p\,dp,\ \int_0^{\infty} F_{pp\phi}\,p\ln p\,dp$$

converge absolutely and uniformly in ϕ.

Then we can prove the following theorem.

Theorem IV. If $f(P)$ is formed according to (III) or (III′), then it satisfies conditions a_1, b_1, c_1 and its integrals along straight lines yield the given $F(p, \phi)$. Due to theorem III, it is the only such function.

Introducing polar coordinates, we get

$$f(\rho\cos\psi, \rho\sin\psi) = -\frac{1}{2\pi^2}\int_0^\infty \frac{dp}{p}\int_0^{2\pi} F_p(\rho\cos\omega + p, \omega + \psi)\,d\omega$$
$$= \frac{1}{2\pi^2}\int_0^\infty \ln p\,dp\int_0^{2\pi} F_{pp}(p + \rho\cos\omega, \omega + \psi)\,d\omega$$

since

$$\int_0^{2\pi} F_p(\rho\cos\omega + p, \omega + \psi)\,d\omega = \int_0^{2\pi} F_p(\rho\cos\omega, \omega + \psi)\,d\omega$$
$$+ \int_0^{2\pi} d\omega \int_0^p F_{pp}(\rho\cos\omega + t, \omega + \psi)\,dt$$

and the first term is equal to zero because of $F(p, \phi) = F(-p, \phi + \pi)$. Thus the product of the integral with $\ln p$ converges to zero as $p \to 0$. From the same property of F, it also follows that

$$(3)\quad f(\rho\cos\psi, \rho\sin\psi) = \frac{1}{2\pi^2}\int_0^\pi d\omega\int_{-\infty}^{+\infty} F_{pp}(p, \omega + \psi)\ln|p - \rho\cos\omega|\,dp.$$

Now it suffices to show

$$(4)\qquad \int_{-\infty}^{+\infty} f(\rho, 0)\,d\rho = F\left(0, \frac{\pi}{2}\right),$$

since the conditions a_2–c_2 are invariant under rigid motions. We let

$$F(p, \phi) = F\left(p, \frac{\pi}{2}\right) + \cos\phi \cdot G(p, \phi).$$

G satisfies regularity conditions that can be easily specified. According to this decomposition, $f(\rho, 0)$ is split into two parts $f_1(\rho)$ and $f_2(\rho)$ that have to be investigated separately. Because of

$$\int_0^\pi \ln|p - \rho\cos\omega|\,d\omega = \begin{cases} \pi\ln\dfrac{|p| + \sqrt{p^2 - \rho^2}}{2}, & |p| > |\rho| \\ \pi\ln\dfrac{|\rho|}{2}, & |p| \leqslant |\rho| \end{cases}$$

it follows that

$$f_1(\rho) = \frac{1}{2\pi^2}\int_0^\pi d\omega\int_{-\infty}^{+\infty} F_{pp}\left(p, \frac{\pi}{2}\right)\ln|p - \rho\cos\omega|\,dp$$
$$= \frac{1}{2\pi}\int_{|\rho|}^\infty F_{pp}\left(p, \frac{\pi}{2}\right)\ln\frac{|p| + \sqrt{p^2 - \rho^2}}{|\rho|}\,dp$$
$$+ \frac{1}{2\pi}\int_{-\infty}^{-|\rho|} F_{pp}\left(p, \frac{\pi}{2}\right)\ln\frac{|p| + \sqrt{p^2 - \rho^2}}{|\rho|}\,dp.$$

Now, this is absolutely integrable from $-\infty$ to $+\infty$ with respect to ρ, which can be seen from interchanging the order of integration. The value of the integral is

$$\int_{-\infty}^{+\infty} f_1(\rho)\,d\rho = \frac{1}{2\pi}\int_{-\infty}^{+\infty} F_{pp}\left(p, \frac{\pi}{2}\right)\int_{-|p|}^{+|p|}\ln\frac{|p| + \sqrt{p^2 - \rho^2}}{|\rho|}\,d\rho\,dp$$
$$= \frac{1}{2}\int_{-\infty}^{+\infty} F_{pp}\left(p, \frac{\pi}{2}\right)|p|\,dp = F\left(0, \frac{\pi}{2}\right).$$

As to $f_2(\rho)$, we will show that it is also absolutely integrable and yields zero when integrated from $-\infty$ to $+\infty$.

We can write $f_2(\rho)$ as follows:

$$f_2(\rho) = \frac{1}{2\pi^2}\int_0^\pi d\omega\int_{-\infty}^{+\infty} G_{pp}(p, \omega)\ln|p - \rho\cos\omega| \cdot \cos\omega\,d\omega$$
$$= \frac{1}{2\pi^2}\int_0^\pi d\omega\int_{-\infty}^{+\infty} G_{pp}(p, \omega)\left[\ln\left|\frac{p - \rho\cos\omega}{\rho\cos\omega}\right|\cos\omega + \frac{\rho p\cos^2\omega}{1 + \rho^2\cos^2\omega}\right]dp$$

since the integral of the additional terms is zero and in this form integration with respect to ρ leads to an *absolutely* convergent threefold integral. This is so because of

$$\int_{-\infty}^{+\infty}\left|\cos\omega\ln\left|\frac{p - \rho\cos\omega}{\rho\cos\omega}\right| + \frac{\rho p\cos^2\omega}{1 + \rho^2\cos^2\omega}\right|d\rho$$
$$= |p|\int_{-\infty}^{+\infty}\left|\ln\left|1 - \frac{1}{\tau}\right| + \frac{p^2\tau}{1 + p^2\tau^2}\right|d\tau = \lambda(p)$$

with

$$\lim_{|p| \to \infty}\frac{\lambda(p)}{|p|\ln|p|} = 2.$$

The integration with respect to ρ yields the value

$$\int_{-\infty}^{+\infty} f_2(\rho)\, d\rho = 0,$$

which completes the proof of (4).

Now it remains to show that f satisfies the conditions a_1–c_1.

The continuity follows from the representation (3) because of the assumptions a_2–c_2. Condition b_1 is also satisfied since

$$\int_{-\infty}^{+\infty} |f(\rho\cos\psi, \rho\sin\psi)|\, d\rho$$

is integrable with respect to ψ, as is easily seen. To show that c_1 holds, we form

$$\begin{aligned}
\bar{f}_0(\rho) &= \frac{1}{2\pi}\int_0^{2\pi} f(\rho\cos\psi, \rho\sin\psi)\, d\psi \\
&= \frac{1}{4\pi^3}\int_0^{\pi} d\omega \int_0^{2\pi} d\psi \int_{-\infty}^{+\infty} F_{pp}(p, \psi)\ln|p - \rho\cos\omega|\, dp \\
&= \frac{1}{4\pi^2}\int_0^{2\pi} d\psi \Bigg[\int_{-\infty}^{-|\rho|} F_{pp}(p, \psi)\ln\frac{|p| + \sqrt{p^2 - \rho^2}}{2}\, dp \\
&\quad + \int_{|\rho|}^{+\infty} F_{pp}(p, \psi)\ln\frac{|p| + \sqrt{p^2 - \rho^2}}{2}\, dp \\
&\quad + F_p(\rho, \psi)\ln\frac{\rho}{2} - F_p(\rho, \psi)\ln\frac{\rho}{2}\Bigg],
\end{aligned}$$

from which the validity of c_1 can be seen. This completes the proof of theorem IV.

B. DETERMINATION OF A LINE-FUNCTION FROM ITS POINT MEAN VALUES

3. Let $F(p, \phi) = F(-p, \phi + \pi)$ be a line-function satisfying the following regularity conditions:

(a_3) F, F_ϕ, F_p are continuous, $|F_\phi| < M$ for all p, ϕ.

(b_3) $F_p \ln|p|$ is convergent to zero uniformly in ϕ as $p \to \infty$.

(c_3) $\int_{-\infty}^{+\infty} |F_p| \ln|p|\, dp$ is uniformly convergent in ϕ.

Again these conditions are invariant under rigid motions. We form the point mean value of $F(p, \phi)$ for $P = [x, y]$:

(5) $$f(x, y) = \frac{1}{\pi}\int_{-\pi/2}^{+\pi/2} F(x\cos\phi + y\sin\phi, \phi)\, d\phi.$$

Then the following theorem holds.

Theorem V. F is uniquely determined by specifying f; that is

(V) $$F\left(0, \frac{\pi}{2}\right) = -\frac{1}{2\pi}\int_{-\infty}^{+\infty}\frac{dx}{x}\int_{-\infty}^{+\infty} f_x(x, y)\, dy,$$

where the Cauchy principal value is to be taken for the integral with respect to x. The value of F for any other straight line can be determined from this formula by means of a suitable rigid motion.

To prove this, we first deduce from (5) that

(6) $$\int_{-A}^{B} f_x(x, y)\, dy = \frac{1}{\pi}\int_{-\pi/2}^{+\pi/2} d\phi \int_{-A}^{B} F_p(x\cos\phi + y\sin\phi, \phi)\cos\phi\, dy,$$

where A, B are two positive constants.

Now, as we have done already earlier, we let

$$F(p, \phi) = F(p, 0) + \sin\phi\, G(p, \phi),$$

where $G(p, \phi)$ is bounded in the domain of integration and has the limit zero as $p \to \infty$. From

$$\begin{aligned}
&\int_{-A}^{B} G_p(x\cos\phi + y\sin\phi, \phi)\cos\phi\sin\phi\, dy \\
&\quad = [G(x\cos\phi + B\sin\phi, \phi) - G(x\cos\phi - A\sin\phi, \phi)]\cos\phi
\end{aligned}$$

it follows that the second term of (6) tends to zero as $A \to \infty$, $B \to \infty$ thus leaving only the first one to be investigated. Performing the analogous integration, one sees that in this first term the integral with respect to ϕ also tends to zero as $A \to \infty$, $B \to \infty$ if the integration is carried out over an interval which does *not* contain $\phi = 0$. Therefore, it remains to consider

$$\lim_{\substack{A\to\infty \\ B\to\infty}} \frac{1}{\pi}\int_{-\varepsilon}^{+\varepsilon} d\phi \int_{-A}^{B} F_p(x\cos\phi + y\sin\phi, 0)\cos\phi\, dy, \qquad 0 < \varepsilon < \frac{\pi}{2}.$$

This integral can be written as

$$\frac{1}{\pi}\int_{-\varepsilon}^{+\varepsilon} d\phi \int_{x\cos\phi - A\sin\phi}^{x\cos\phi + B\sin\phi} F_p(p, 0)\cot\phi\, dp.$$

Then, assuming A and B sufficiently large and interchanging the order of integration, after some computations one obtains the value

$$\frac{1}{\pi}\int_{x\cos\varepsilon-B\sin\varepsilon}^{x\cos\varepsilon+B\sin\varepsilon}\ln\frac{(B^2+x^2)\sin\varepsilon}{\left|Bp-x\sqrt{B^2+x^2-p^2}\right|}F_p(p,0)\,dp$$

$$+\frac{1}{\pi}\int_{x\cos\varepsilon-A\sin\varepsilon}^{x\cos\varepsilon+A\sin\varepsilon}\ln\frac{(A^2+x^2)\sin\varepsilon}{\left|Ap-x\sqrt{A^2+x^2-p^2}\right|}F_p(p,0)\,dp.$$

It is sufficient to determine the limit of the second integral as $A\to\infty$. We write it as follows:

$$\frac{1}{\pi}\ln(A\sin\varepsilon)[F(x\cos\varepsilon+A\sin\varepsilon,0)-F(x\cos\varepsilon-A\sin\varepsilon,0)]$$

$$+\frac{1}{\pi}\int_{x\cos\varepsilon-A\sin\varepsilon}^{x\cos\varepsilon+A\sin\varepsilon}\ln\frac{1}{|p-x|}F_p(p,0)\,dp$$

$$+\frac{1}{\pi}\int_{x\cos\varepsilon-A\sin\varepsilon}^{x\cos\varepsilon+A\sin\varepsilon}\ln\frac{\left|Ap+x\sqrt{A^2+x^2-p^2}\right|}{A|p+x|}F_p(p,0)\,dp.$$

Since in the last integral the logarithm tends to zero *uniformly* as $A\to\infty$, the limit follows:

$$-\frac{1}{\pi}\int_{-\infty}^{+\infty}F_p(p,0)\ln|p-x|\,dp$$

which leads to the limit of (6):

$$\int_{-\infty}^{+\infty}f_x(x,y)\,dy=-\frac{2}{\pi}\int_{-\infty}^{+\infty}F_p(p,0)\ln|p-x|\,dp.$$

It should be noted here that the latter expression represents the boundary values of the imaginary part of a regular analytic function in the upper half plane whose real part has the boundary values $2F(x,0)$.

If we now form

$$-\int_{-\infty}^{+\infty}\frac{dx}{x}\int_{-\infty}^{+\infty}f_x(x,y)\,dy=\frac{2}{\pi}\int_0^{\infty}\frac{dx}{x}\int_{-\infty}^{+\infty}F_p(p,0)\ln\left|\frac{p-x}{p+x}\right|dx$$

in the spirit of formula (V), then this double integral is absolutely convergent and leads directly to formula (V) since

$$\int_0^{\infty}\ln\left|\frac{p-x}{p+x}\right|\frac{dx}{x}=-\frac{\pi^2}{2}\operatorname{sgn}p.$$

4. Now let f be a point-function with the following regularity properties:

(a_4) f and its derivatives up to the the second order are continuous.

(b_4) The expressions $f(x,y)$, $\sqrt{x^2+y^2}\ln(x^2+y^2)f_x(x,y)$, $\sqrt{x^2+y^2}\ln(x^2+y^2)f_y(x,y)$ approach zero as $x^2+y^2\to\infty$.

(c_4) The integrals

$$\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}D_1 f\frac{\ln(x^2+y^2)}{\sqrt{x^2+y^2}}\,dx\,dy$$

and

$$\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}D_2 f\ln(x^2+y^2)\,dx\,dy,$$

where $D_1 f$ means any first and $D_2 f$ any second derivative, are absolutely convergent.

Again these conditions are invariant under rigid motions. Then the following theorem holds.

Theorem VI. The line-function formed from f according to (V) has the point mean values $f(x,y)$.

It is sufficient to show the proof for the origin. For an arbitrary straight line through the origin, (V) yields after an integration by parts:

$$F(0,\phi)=\frac{1}{2\pi}\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}\left[f_{xx}\cos^2\phi+2f_{xy}\sin\phi\cos\phi+f_{yy}\sin^2\phi\right]$$

$$\cdot\ln|x\cos\phi+y\sin\phi|\,dx\,dy$$

or, after introducing polar coordinates ρ,ψ:

$$F(0,\phi)=\frac{1}{2\pi}\int_0^{\infty}\rho\,d\rho\int_0^{2\pi}\left[\frac{\partial^2 f}{\partial\rho^2}\cos^2(\phi-\psi)\right.$$

$$+2\frac{\partial^2 f}{\partial\rho\partial\psi}\frac{\sin(\phi-\psi)\cos(\phi-\psi)}{\rho}+\frac{\partial^2 f}{\partial\psi^2}\frac{\sin^2(\phi-\psi)}{\rho^2}$$

$$+\frac{\partial f}{\partial\rho}\frac{\sin^2(\phi-\psi)}{\rho}$$

$$\left.-2\frac{\partial f}{\partial\psi}\frac{\sin(\phi-\psi)\cos(\phi-\psi)}{\rho^2}\right]\ln|\rho\cos(\phi-\psi)|\,d\psi.$$

In order to form the point mean value for $[0, 0]$, the integration with respect to ϕ from 0 to 2π can be carried out under the double integral, and then one has to divide by 2π. The term containing $\partial^2 f/\partial\psi^2$ that appears during this computation cancels when integrating with respect to ψ and there remains

$$\frac{1}{2\pi}\int_0^{2\pi} d\psi \int_0^{\infty}\left[\frac{1}{4}\left(\rho\frac{\partial^2 f}{\partial\rho^2}-\frac{\partial f}{\partial\rho}\right)+\tfrac{1}{2}\ln\frac{\rho}{2}\,\frac{\partial}{\partial\rho}\left(\rho\frac{\partial f}{\partial\rho}\right)\right]d\rho,$$

which indeed reduces to $f(0,0)$.

In order to show the uniqueness of F, it remains to show that the conditions a_3–c_3 are satisfied, which makes it obviously necessary to place further restrictions on f.

5. Here the following remark, for which I am indebted to Mr. W. Blaschke, who also posed the problem, should be made: Both problems treated here are closely related to the theory of the Newtonian potential. That is, if we consider the transition from a point-function f to its mean values F along straight lines as a linear functional transformation

$$F = Rf$$

and similarly the transition from a line-function F to its point mean values v

$$v = BF,$$

then it is natural to consider the composed transformation $H = BR$ defined by

$$v = Hf = B[Rf] = BRf.$$

It can now be readily seen that Hf is nothing but the Newtonian potential in the points of the plane that is covered with a mass of density $(1/\pi)f$. According to a remark made by G. Herglotz, this can be used to construct the inverse of the transformation H; this leads to

$$f(P) = H^{-1}v = -\frac{1}{2}\int_0^{\infty}\frac{d\bar{v}_P(r)}{r} = -\frac{1}{4\pi}\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}\frac{\Delta v(x',y')}{r_{PP'}}dx'\,dy',$$

where $\bar{v}_P$ is a notation for a mean value analogous to the previously introduced notations and Δ is the Laplacian operator.

Now we could think of performing the inversion of R and H, which was done directly in 1–3 by means of

$$R^{-1} = H^{-1}B \quad \text{and} \quad B^{-1} = RH^{-1}.$$

In fact, I first found the inversion formula (IV) in this way. However, carrying out this thought in a strict manner seems to be more difficult than the direct verification, and it even fails in the non-Euclidian cases, which will be soon discussed.

Finally, we remark that the regularity conditions assumed in parts A and B are of course by no means the most general ones. This can be shown with simple examples.

C. GENERALIZATIONS

6. A far-reaching generalization of the problem treated in part A could be formulated as follows: Let S be a surface on which a line-element ds is defined by any means, and a twice infinite family of curves C is given on S. Then, a point-function on the surface is to be determined from the integrals $\int f\,ds$ along the curves C.

The nearest specialization is obtained by taking a non-Euclidian plane for S, the corresponding line-element for ds, and the corresponding straight lines for the curves C. In the elliptic case, the problem can be carried over to the geometry on a sphere. Interpreting in a well-known fashion a diametrical pair of points on the sphere as a point in the elliptic plane, there results the problem of the determination of an even function on the sphere (i.e., a function with the same value in diametrical points) from its integrals along the great circles. Minkowski was the first to deal with this problem in principle*) and he solved it by expansions in terms of spherical functions. Later P. Funk computed Minkowski's solution and he has shown how to obtain this solution from the Abel integral equation.† This is the method to which I owe the solution of problem A. Funk's solution is analogous to (III) with the exception that the sinus of the spherical radius appears in the denominator and to the integral there is added the value of F at the pole of the corresponding great circle divided by π. In the hyperbolic plane, the solution of the problem is analogous to (III) too:

$$f(P) = -\frac{1}{\pi}\int_0^{\infty}\frac{d\bar{F}_p(q)}{\sinh q}$$

(here the measure of curvature is assumed to be $= -1$). This can be shown to be in total agreement with the derivation of (III) indicated in 1.

In both cases, the question analogous to B can be posed also. In the elliptic geometry, nothing new results because of the absolute polarity, and in the hyperbolic case a solution analogous to (V) does not seem to exist.

*Gesammelte Abhandlungen **II**, pp. 277ff.
†Math. Ann., **74**, pp. 283–288.

A second specialization results if (in the Euclidian or in the non-Euclidian geometry) the circles with constant radius are taken for the curves C. Here Minkowski's method using spherical functions can be applied on the sphere and so the problem can be solved to a certain degree. However, it is interesting that in this case the uniqueness of the solution can be lost. The reason for this is that for certain radii ρ defined by the zeros of the Legendre polynomials of even order there exist even functions on the sphere that do not vanish identically, but whose integrals along any circle with spherical radius ρ are zero. In the Euclidian case, the spherical functions are replaced by the integral theorem of the Bessel functions. Here there are always functions which do not vanish identically but whose integrals along any circle with fixed radius yield zero. If this radius is one then these functions are (in polar coordinates)

$$J_n(x_\nu \rho)\cos n\phi, \qquad J_n(x_\nu \rho)\sin n\phi,$$

and linear combinations, where x_ν is a zero of J_0. In the hyperbolic case, the Bessel functions are replaced by so-called conical functions for which the corresponding integral theorem has been proven by Weyl.* The results are analogous to the Euclidian case.

7. The results in parts A and B can be generalized in another direction by passing on to higher-dimensional spaces. In a Euclidian space $\mathbf{R}^n$, one can try to determine a point-function $f(p) = f(x_1, x_2, \ldots, x_n)$ from its integrals $F(\alpha_1, \ldots, \alpha_n, p)$ over all hyperplanes $\alpha_1 x_1 + \cdots + \alpha_n x_n = p$, $(\alpha_1^2 + \cdots + \alpha_n^2 = 1)$. Following a procedure analogous to that applied in 1, we form the mean value $\bar{F}_0(q)$ of F over the tangent-planes of the sphere with center $[0, 0, \ldots, 0]$ and radius q. It is given by the $(n-1)$-fold integral:

$$\bar{F}_0(q) = \frac{1}{\Omega_n}\int F(\alpha, q)\, d\omega,$$

where $d\omega$ is the surface element and $\Omega_n = (2\pi^{n/2})/(\Gamma(n/2))$ is the surface area of the n dimensional sphere $\alpha_1^2 + \cdots + \alpha_n^2 = 1$.

$\bar{F}_0$ can be represented as an n-fold integral over f:

$$\bar{F}_0(q) = \frac{\Omega_{n-1}}{\Omega_n} \iint_{x_1^2 + \cdots x_n^2 > q^2} f(x_1, x_2, \ldots, x_n) \times \frac{(x_1^2 + \cdots + x_n^2 - q^2)^{(n-3)/2}}{(x_1^2 + \cdots + x_n^2)^{(n-2)/2}}\, dx_1 \cdots dx_n \tag{7}$$

*Gött. Nachr., 1910, p. 454.

or, using an already often used mean value notation:

$$\bar{F}_0(q) = \Omega_{n-1}\int_q^\infty \bar{f}_0(q)(r^2 - q^2)^{(n-3)/2}\, r\, dr.$$

This formula is analogous to (1) and has corresponding consequences. The substitution $r^2 = v$, $q^2 = u$ leads to the integral equation

$$\Phi(u) = \frac{\Omega_{n-1}}{2}\int_u^\infty \phi(v)(v - u)^{(n-3)/2}\, dv.$$

If n is *even*, we get the same equation as (2) by differentiating $((n/2) - 1)$ times, and from this,

$$\phi(0) = f(0, 0, \ldots, 0)$$

can be found. Thus, for a given F, the formation of F differentiations and one integration is necessary. If n is *odd*, then this integration is omitted, since we now get from differentiating $((n-1)/2)$ times:

$$\phi(0) = \frac{2(-1)^{(n-1)/2}}{\Omega_{n-1}\left(\frac{n-3}{2}\right)!}\Phi^{((n-3)/2)}(0).$$

The three-dimensional case is particularly simple, but this case can also be treated using a method analogous to 5 that yields very elegant results. From (7), the point mean value of F for $q = 0$ follows:

$$\bar{F}_0 = \frac{1}{2}\iiint \frac{f(x, y, z)}{\sqrt{x^2 + y^2 + z^2}}\, dx\, dy\, dz.$$

This equation can be considered the Newtonian potential of the space covered with a mass of density $\frac{1}{2}f$. Therefore, it follows that

$$f(x, y, z) = -\frac{1}{2\pi}\Delta\bar{F},$$

where $\bar{F}$ stands for the point mean value of F.

Here also the problem analogous to B can be solved. Using the method indicated in 5, one finds for a plane-function F with known point mean values f that

$$F(E) = -\frac{1}{2\pi}\iint \Delta f\, d\sigma,$$

where $d\sigma$ is the surface area element of the plane E. Δ is the Laplacian operator for the three-dimensional space, and the integration is to be taken over the whole plane E.