Image Reconstruction from Cone-Beam Projections: Necessary and Sufficient Conditions and Reconstruction Methods

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Abstract—Previously unknown sufficient conditions, a necessary condition, and reconstruction methods for image reconstruction from cone-beam projections are developed. A sufficient condition developed is contained in the following statement.

Statement 5: If one every plane that intersects the object, there exists at least one cone-beam source point, then the object can be reconstructed.

Reconstruction methods for an arbitrary configuration of source points that satisfy Statement 5 are derived. By requiring additional conditions on the configuration of source points, a more efficient reconstruction method is developed. It is shown that when the configuration of source points is a circle, Statement 5 is not satisfied. In spite of this, several suggestions are made for reconstruction from a circle of source points.

Introduction

IMAGE reconstruction from cone-beam projection data has been considered in recent years. For instance, an X-ray CT scanner has been designed and built for temporial three-dimensional studies of the heart which use cone-beam data obtained at a number of source points on a circle [13]. In positron emission tomography, fully three-dimensional data collection was used to lower the radiation dosage to the patient and reduce imaging time [18]-[20]. This three-dimensional data collection scheme could be cased as knowing the cone-beam data at each source point on a cylinder (a cylinder without its' two parallel end sections). Hence, there are several applications for reconstruction methods that use cone-beam data.

Perhaps the first formula given in the literature which enables one to reconstruct from cone-beams was given in [15]. Since the formula given in [15] required parallel projections, the cone-beam data would first have to be rebinned into parallel projections. Methods that do not require rebinning were given in [3], [4], [16]. All these methods required the knowledge of the cone-beam data at each source point on a sphere. However, obtaining and processing the cone-beam data at each source point on a sphere would be a difficult and lengthy task. Moreover, there are situations in which a sphere of source points are not known [18]-[21]. Hence, it is desirable to reconstruct from something less than a "sphere" of source points."

Hence, we are lead to consider the following. What configuration of cone-beam source points can we reconstruct an object from? Moreover, we want a reconstruction method which can be employed when the data from such a configuration of source points is known. The purpose of this paper is to develop (previously unknown) necessary and sufficient conditions that will specify which configuration of source points we can reconstruct from and reconstruction methods that can be employed when these conditions are satisfied.

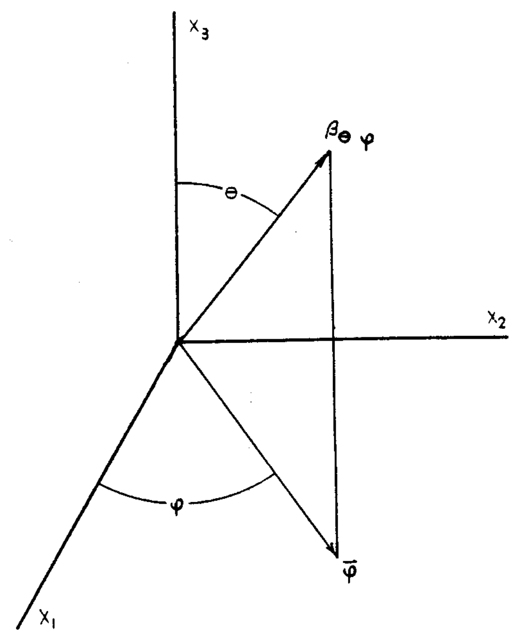

Fig. 1. The unit vector $\beta_{\theta,\varphi}$ defined by θ and φ.

We start in Part I by developing a sufficient condition which is similar to the one stated for the reconstruction formula given in [1]. Two better (less restrictive) sufficient conditions are developed along with reconstruction methods that can be employed when the sufficient conditions are satisfied. We then verify that one of the sufficient conditions is necessary too. In Part II, the reconstruction method only stated in [1] is derived and substantial improvements in it are made. Finally, in Part III, the conditions and reconstruction methods developed in the paper are applied to the particular case of a circle of source points. Throughout this paper we avoid giving cumbersome mathematical details and try to motivate the results intuitively.

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I. Notation and Preliminary Definitions

Let $x \cdot y$ denote the inner product of the vectors x and y. The object to be reconstructed is denoted by $f(x)$ where $x = (x_1, x_2, x_3)^T$. The support of the object is assumed to be a ball of radius R. We define $\beta_{\theta,\varphi}$ to be a unit vector in $\mathbb{R}^3$ that is represented parametrically by θ and φ. Explicitly we define

$$\beta_{\theta,\varphi} = (\cos\varphi \sin\theta, \sin\varphi \sin\theta, \cos\theta)^T.$$

The variables θ and φ are shown in Fig. 1. The greek letter β without the subscript θ, φ will denote a vector with arbitrary length. The following equations define the n-dimensional Fourier transform and its inverse respectively.

$$\tilde{f}(X) = {}_X\mathcal{F}_x\{f(x)\} = \int_{\mathbb{R}^n} f(x)\, e^{-jx\cdot X}\, dx \tag{1.1}$$

$$f(x) = {}_x\mathcal{F}_X^{-1}\{\tilde{f}(X)\} = \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} \tilde{f}(X)\, e^{jx\cdot X}\, dX. \tag{1.2}$$

The following equations define the two-dimensional Radon transform and its inverse, respectively.

For $x = (x_1, x_2)^T$, $\vec{\theta} = (\cos\theta, \sin\theta)^T$ and $\vec{\theta}_\perp = (\sin\theta, -\cos\theta)^T$ we have

$$P(l, \theta) = \int_{-\infty}^{\infty} f(l\vec{\theta} + s\vec{\theta}_\perp)\, ds \tag{1.3}$$

$$f(x) = \frac{1}{2\pi^2} \int_0^\pi \int_{-R}^{R} \frac{1}{x\cdot\vec{\theta} - l} \frac{\partial P}{\partial l}(l, \theta)\, dl\, d\theta. \tag{1.4}$$

In (1.4) the Cauchy principal value of the inner integral is taken. By performing an integration by parts (see (4)–(8) in [17] for details) it is seen that (1.4) is equivalent to the following. Let

$$F_\epsilon(t) = \begin{cases} \dfrac{1}{\epsilon^2} & \text{for } |t| < \epsilon \\[2ex] \dfrac{-1}{t^2} & \text{for } |t| \geq \epsilon \end{cases}$$

then

$$f(x) = \frac{1}{2\pi^2} \int_0^\pi \lim_{\epsilon\to 0} \int_{-R}^{R} F_\epsilon(X\cdot\theta - l)\, P(l, \theta)\, dl\, d\theta. \tag{1.5}$$

Equation (1.5) is essentially (35) and (36) in [17]. For a fuller theoretical understanding of (1.5) see [17], [2], [5] and [6]. For numerical analysis considerations of (1.5) see [17]. (Also related are [24] and [25].) Now, for a given function h, the reader familiar with the tomographic literature will recall that

$${}_l\mathcal{F}_\omega^{-1}\{|\omega|\, \tilde{h}(\omega)\} = \frac{1}{\pi}\frac{1}{l} * \frac{d}{dl} h(l). \tag{1.6}$$

See [26] or [5], [6] for details. Using once again the same reasoning that showed that (1.4) and (1.5) were equivalent, we see that

$${}_l\mathcal{F}_\omega^{-1}\{|\omega|\, \tilde{h}(\omega)\} = \lim_{\epsilon\to 0} \int_{-\infty}^{\infty} F_\epsilon(l - \tau)\, h(\tau)\, d\tau. \tag{1.7}$$

To simplify notation in what follows we will write the right-hand side of (1.7) as

$$\int_{-\infty}^{\infty} \frac{1}{(l-\tau)^2} h(\tau)\, d\tau$$

or more simply as

$$\frac{1}{l^2} * h(l).$$

Recall that the three-dimensional Radon transform involves the integrals of f over planes. If the vectors $\beta_{\theta,\varphi}, \beta_{\perp_1}\beta_{\perp_2}$ form a orthonormal set, then the three-dimensional Radon transform and its inverse can be written as

$$\check{f}(\beta_{\theta,\varphi}, l) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(l\beta_{\theta,\varphi} + s\beta_{\perp_1} + t\beta_{\perp_2})\, ds\, dt \tag{1.8}$$

$$f(x) = \frac{-1}{(2\pi)^2} \int_0^\pi \int_0^\pi \frac{\partial^2}{\partial l^2} \cdot \check{f}(\beta_{\theta,\varphi}, l)\Big|_{l = x\cdot\beta_{\theta,\varphi}} \sin\theta\, d\theta\, d\varphi. \tag{1.9}$$

In the two-dimensional case the "projection theorem" or the "central slice theorem" is well known. In what follows we will need the following extension of the projection theorem.

$$\tilde{f}(\omega\beta_{\theta,\varphi}) = {}_\omega\mathcal{F}_l\{\check{f}(\beta_{\theta,\varphi}, l)\}. \tag{1.10}$$

Part I—Sufficient Conditions, Necessary Condition, and Reconstruction from an Arbitrary Configuration of Cone-Beam Source Points

II. A New Function

The following definition is basic in the sequel.

$$F(\beta_{\theta,\varphi}, l) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \tilde{f}(\omega\beta_{\theta,\varphi})\, |\omega|\, e^{j\omega l}\, d\omega. \tag{2.1}$$

The domain of F is $[0, \pi) \times [0, \pi) \times [-\infty, \infty]$. The following propositions and proofs will motivate a further meaning of F.

Proposition:

$$F(\beta_{\theta,\varphi}, l) = \frac{-1}{\pi} \int_{-R}^{R} \frac{1}{(l-t)^2} \check{f}(\beta_{\theta,\varphi}, t)\, dt. \tag{2.2}$$

(To avoid a possible misinterpretation, recall that the right-hand side of (2.2) involves integrals of f over planes, not lines.)

Proof: Using the "extension" of the projection theorem in (2.1) we obtain

$$F(\beta_{\theta,\varphi}, l) = \frac{1}{2\pi} \int_{-\infty}^{\infty} {}_\omega\mathcal{F}_l\{\check{f}(\beta_{\theta,\varphi}, l)\}\, |\omega|\, e^{j\omega l}\, d\omega. \tag{2.3}$$

Using (1.7) we obtain

$$F(\beta_{\theta,\varphi}, l) = \frac{-1}{\pi} \frac{1}{l^2} * \check{f}(\beta_{\theta,\varphi}, l). \tag{2.4}$$

We recognize (2.4) to be (2.2).

Proposition:

$$f(x) = \frac{-1}{4\pi^3} \int_0^\pi \int_0^\pi \int_{-\infty}^{\infty} \frac{1}{(l-t)^2} F(\beta_{\theta,\varphi}, t) \bigg|_{l = x\cdot\beta_{\theta,\varphi}} dt \sin\theta \, d\theta \, d\varphi \tag{2.5}$$

Proof: From (2.3) we obtain

$$_{\omega}\mathcal{F}_l\{F(\beta_{\theta,\varphi}, l)\} = {}_{\omega}\mathcal{F}_l\{\check{f}(\beta_{\theta,\varphi}, l)\} \; |\omega|. \tag{2.6}$$

Multiplying both sides of (2.6) by $|\omega|$ and applying $_l\mathcal{F}_\omega^{-1}$ to both sides we obtain

$$\frac{1}{\pi}\frac{1}{l^2} * F(\beta_{\theta,\varphi}, l) = \frac{\partial^2}{\partial l^2} \check{f}(\beta_{\theta,\varphi}, l). \tag{2.7}$$

Thus, we can write

$$\frac{-1}{4\pi^3} \int_0^\pi \int_0^\pi \frac{1}{l^2} * F(\beta_{\theta,\varphi}, l) \bigg|_{l = x\cdot\beta_{\theta,\varphi}} \sin\theta \, d\theta \, d\varphi$$

$$= \frac{-1}{(2\pi)^2} \int_0^\pi \int_0^\pi \frac{\partial^2}{\partial l^2} \check{f}(\beta_{\theta,\varphi}, l) \bigg|_{l = x\cdot\beta_{\theta,\varphi}} \sin\theta \, d\theta \, d\varphi. \tag{2.8}$$

The proof is completed by observing the right-hand side of (2.8) is the left-hand side of (1.9).

In what follows, we will consider F on both its whole domain and on the subset $[0, \pi) \times [0, \pi) \times [-R, R]$. To distinguish which one we are talking about, we define $F_R(\beta_{\theta,\varphi}, l)$ to be the restriction of $F(\beta_{\theta,\varphi}, l)$ to the domain $[0, \pi) \times [0, \pi) \times [-R, R]$. That is, we define the domain of $F_R(\beta_{\theta,\varphi}, l)$ to be $[0, \pi) \times [0, \pi) \times [-R, R]$ and define $F_R(\beta_{\theta,\varphi}, l)$ equal to $F(\beta_{\theta,\varphi}, l)$ on that domain.

III. Development of a Sufficient Condition

In this section, we develop a sufficient condition similar to the one stated for the reconstruction formula in [1]. To do this, we need a function that describes the cone-beam projection data at an arbitrary source point. By letting the position of the source point be denoted by the vector Φ, and the direction of the line along which $f(x)$ is integrated be denoted by the vector a for $||\alpha|| = 1$, we define

$$g_1(\alpha, \Phi) = \int_{-\infty}^{\infty} f(\Phi + t\alpha)\, dt \quad \text{for } ||\alpha|| = 1. \tag{3.1}$$

Rather than working with the above function it turns out to be convenient to work with a following function.

$$g(\alpha, \Phi) = \int_{-\infty}^{\infty} f(\Phi + t\alpha)\, dt \quad \text{for } \alpha \in \mathbb{R}^3. \tag{3.2}$$

The reader can verify that

$$g(\alpha, \Phi) = \frac{1}{||\alpha||} g_1\left(\frac{\alpha}{||\alpha||}, \Phi\right) \tag{3.3}$$

In words, $g(\alpha, \Phi)$ is the homogeneous extension of the cone-beam data with a degree of -1. Now consider the Fourier transform of $g(\alpha, \Phi)$ for a fixed Φ.

$$G(\beta, \Phi) = \int_{\mathbb{R}^3} g(\alpha, \Phi)\, e^{-j\alpha\cdot\beta}\, d\alpha. \tag{3.4}$$

[Aside: Since g only decreases as $1/||\alpha||$ as $\alpha \to \infty$, the integral on the right hand side of (3.4) does not exist. The mathematically proper way to define G involves "generalized Fourier transformations" as discussed in [5], [6], and [8]. For a fixed Φ, define G as the generalized Fourier transform of g. The practical importance of the nonexistence of the integral in (3.4) is that if (3.4) was used in a practical method to compute G from g, the best one could hope for is an approximation to G. An equation which could be used to compute G exactly (at least in theory) is for

$$\alpha_{\theta',\varphi'} = (\cos\varphi' \sin\theta', \sin\varphi' \sin\theta', \cos\theta')^T,$$

$$G(\beta, \Phi) = \frac{2}{||\beta||^2} \lim_{\epsilon \to 0} \left\{ \int_0^\pi \int_0^\pi g(\alpha_{\theta',\varphi'}, \Phi)\, F_\epsilon\left(\alpha_{\theta',\varphi'} \cdot \frac{\beta}{||\beta||}\right) \cdot \sin\theta'\, d\theta'\, d\varphi' \right\}.]$$

Using (3.2) and exchanging integrals we obtain from (3.4)

$$G(\beta, \Phi) = \int_{-\infty}^{\infty} \int_{\mathbb{R}^3} f(\Phi + t\alpha)\, e^{-j\alpha\cdot\beta}\, d\alpha\, dt. \tag{3.5}$$

Perform the change in variables defined by $v = \Phi + t\alpha$ to obtain

$$G(\beta, \Phi) = \int_{-\infty}^{\infty} \int_{\mathbb{R}^3} f(v)\, e^{-j[(v-\Phi)/t]\cdot\beta} \frac{1}{|t^3|}\, dv\, dt. \tag{3.6}$$

Now let $t = 1/\tau$

$$G(\beta, \Phi) = \int_{-\infty}^{\infty} \int_{\mathbb{R}^3} f(v)\, e^{-j(v-\Phi)\cdot\beta\tau}\, dv\, |\tau|\, d\tau. \tag{3.7}$$

By considering the inner integral as a Fourier transformation we obtain

$$G(\beta, \Phi) = \int_{-\infty}^{\infty} \tilde{f}(\beta\tau)\, |\tau|\, e^{j\Phi\cdot\beta\tau}\, d\tau. \tag{3.8}$$

Comparing (3.8) with the definition F (2.1) we obtain

$$G(\beta_{\theta,\varphi}, \Phi) = 2\pi F(\beta_{\theta,\varphi}, \Phi \cdot \beta_{\theta,\varphi}). \tag{3.9}$$

for each source point Φ.

Now, to determine a sufficient condition for reconstruction, first observe that given g we can determine G [via (3.4)], and given F (on its whole domain) we can determine f [via (2.5)]. We represent this symbolically as

$$g \to G, \qquad F \to f.$$

If we can determine F on its whole domain, from using (3.9)

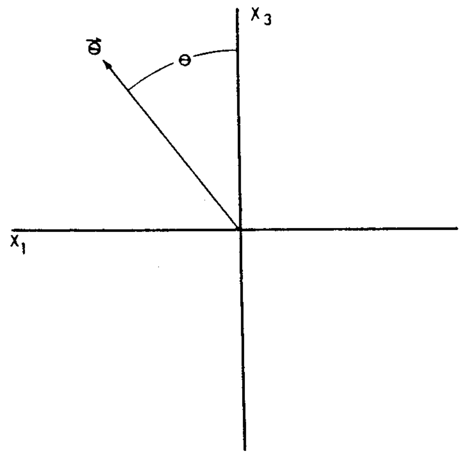

Fig. 2. The coordinate system that defines the projection data of the "object" $f_1(x_1, x_3)$.

at various source points in the configuration, then we have a method of reconstructing the object from that particular configuration of source points. Symbolically, we would have

$$g \to G \to F \to f.$$

F can be obtained from g if, for each direction $\beta_{\theta,\varphi}$ there exists a source point Φ such that

$$\beta_{\theta,\varphi} \cdot \Phi = l \tag{3.10}$$

for all l's $-\infty < l < \infty$. Since for a fixed $\beta_{\theta,\varphi}$ and a fixed l (3.10) is an equation for a plane, we have the following statement.

Statement 1: If on every plane there exists at least one cone-beam source point, one can reconstruct the object.

Statement 1 is (essentially) part of the sufficient condition stated in [1].

IV. Better Sufficient Conditions

In this section, two sufficient conditions are developed that are better (less stringent) then Statement 1. As a first step, we need to develop a relationship between the integrals of f along lines parallel to a given direction vector (let us call the direction vector τ) and the integrals of f over planes parallel to τ. For notational simplicity, assume τ points in the direction of the x_2 axis. Furthermore, assume that the following integration is known.

$$f_1(x_1, x_3) = \int_{-\infty}^{\infty} f(x_1, x_2 + s, x_3)\, ds. \tag{4.1}$$

This integration has defined the function $f_1(x_1, x_3)$, which is supported by a disk of radius R. We consider for the time being $f_1(x_1, x_3)$ to be "a whole new object" independent of the original object $f(x_1, x_2, x_3)$. Since $f_1(x_1, x_3)$ is a two-dimensional object we can take a two-dimensional Radon transform of it. To do this we define

$$\vec{\theta} = (\cos o, \sin\theta)^\tau \quad \text{and} \quad \vec{\theta}_\perp = (\sin\theta, -\cos\theta)^\tau$$

as shown in Fig. 2. The Radon transform is

$$P_1(\theta, l) = \int_{-\infty}^{\infty} f_1(l\vec{\theta} + s\vec{\theta}_\perp)\, ds. \tag{4.2}$$

Now, interpret what we have just done from the three-dimensional point of view. Integrating $f_1(x_1, x_3)$ along a straight line corresponds to integrating $f(x_1, x_2, x_3)$ over a plane. Explicitly,

$$P_1(\theta, l) = \check{f}(\beta_{\theta,0}, l). \tag{4.3}$$

Although (4.3) was developed for $\tau = (0, 1, 0)^\tau$, it is geometrically clear that a similar equation holds for any other τ (for, where was the x_2 axis located in the first place?). Thus, we can state the following.

Statement 2: The two-dimensional Radon transform of the integrals of f over lines parallel to τ are integrals of f over planes parallel to τ.

Now, the following statement should be of no surprise.

Statement 3: The two-dimensional inverse Radon transform of the integrals of f over planes parallel to τ are integrals of f over lines parallel to τ.

To elaborate on this point, consider again $\tau = (0, 1, 0)^\tau$. In this case Radon's two-dimensional inversion formula can be written as

$$f_1(x_1, x_2) = \frac{-1}{2\pi^2} \int_0^{\pi} \int_{-R}^{R} \frac{1}{(X \cdot \beta_{\theta,0} - l)^2} \check{f}(\beta_{\theta,0}, l)\, dl\, d\theta. \tag{4.5}$$

Although (4.5) was developed for $\tau = (0, 1, 0)^\tau$, it is geometrically clear that a similar equation holds for any other τ. (In fact, in Section V an equation for a more general τ is presented.) Thus, we see Statement 3 is in fact true.

Now we are in position to clearly see a scheme that will invert the three-dimensional Radon transform, which is an alternative to (1.9). The first step of this scheme is for each τ perpendicular to the x_3 axis determine the integral of f over the lines parallel to τ. Then, on each plane perpendicular to the x_3 axis, perform a two-dimensional inverse Radon transform using the line integrals obtained in the first step. In fact, this scheme has appeared before in the literature in [7] and [10].

A slight modification of the above scheme yields a method of obtaining f from F [which is an alternative to (2.5)]. To see this, use (2.2) in (4.5) to obtain

$$f_1(x_1, x_3) = \frac{1}{2\pi} \int_0^{\pi} F(\beta_{\theta,0}, \beta_{\theta,0} \cdot x)\, d\theta. \tag{4.6}$$

Using (4.6) rather than (4.5) as the first step of the scheme results in a method of obtaining f from F. It is of most importance to more than F_R is only needed, rather than F. Combining (4.6) with the method of obtaining G from g discussed in Section III, we have

$$g \to G, \quad F_R \to f.$$

Now we have our first sufficient condition.

Statement 4: One can reconstruct the object from a given configuration of source points, if one can obtain F_R from G.

Note that Statement 4 does not make any assumptions that the configuration of source points. The source points could be isolated source points as are obtained in a pin-hole collimator, a curve of source points as discussed in [1], a surface of source points as considered in [11], or even a "volume" of source points; just as long as Statement 4 is satisified.

Recalling the geometric interpretation of (3.9), we can write a second sufficient condition.

Statement 5: If on every plane that intersects the object there exists at least one cone-beam source point, then one can reconstruct the object.

Observe that Statement 5 is always as stringent and in certain cases (take a spherical symmetric object for instance) more stringent then Statement 4.

V. Reconstructing from an Arbitrary Configuration of Source Points

This section will elaborate upon the formulas motivated in the last section. These formulas and others will be case in a form that may enhance their numerical implementation. We start by elaborating on (4.1). For

$$\vec{\varphi} = (\cos\varphi, \sin\varphi, 0)^{\tau} \quad \text{and} \quad \vec{\varphi}_{\perp} = (\sin\varphi, -\cos\varphi, 0)^{\tau}$$

let $P(x, \varphi)$ denote the integral of f along the line that intersects x and is parallel to $\vec{\varphi}_{\perp}$. That is,

$$P(x, \varphi) = \int_{-\infty}^{\infty} f(x + s\vec{\varphi}_{\perp})\, ds. \tag{5.1}$$

One such $\vec{\varphi}$ is shown in Fig. 1. Using this notation the generalization of (4.6) is

$$P(x, \varphi) = \frac{1}{2\pi} \int_0^{\pi} F(\beta_{\theta,\varphi}, x \cdot \beta_{\theta,\varphi})\, d\theta. \tag{5.2}$$

For any $L_1 \geqslant 0$ and $L_2 \geqslant 0$, it follows from the two-dimensinal inverse Radan transform (1.6) that

$$f(x) = \frac{-1}{2\pi^2} \int_0^{\pi} \int_{-\sqrt{R^2 - X_3^2} - X\cdot\vec{\varphi} - L_2}^{\sqrt{R^2 - X_3^2} - X\cdot\vec{\varphi} + L_1} P(X + \vec{\varphi}\epsilon, \varphi) \frac{1}{\epsilon^2}\, d\epsilon\, d\varphi. \tag{5.3}$$

Now combine (5.2) and (5.3) to obtain

$$f(x) = \frac{-1}{4\pi^3} \int_0^{\pi} \int_0^{\pi} \int_{-\sqrt{R^2 - X_3^2} - X\cdot\vec{\varphi} - L_2}^{\sqrt{R^2 - X_3^2} - X\cdot\vec{\varphi} + L_1} \cdot F(\beta_{\theta,\varphi}, \omega)\Big|_{\omega = x\cdot\beta_{\theta,\varphi} + \epsilon \sin\theta} d\theta \frac{1}{\epsilon^2}\, d\epsilon\, d\varphi. \tag{5.4}$$

By exchanging integrals and making the change in variables defined by

$$l = x \cdot \beta_{\theta,\varphi} + \epsilon \sin\theta$$

we obtain

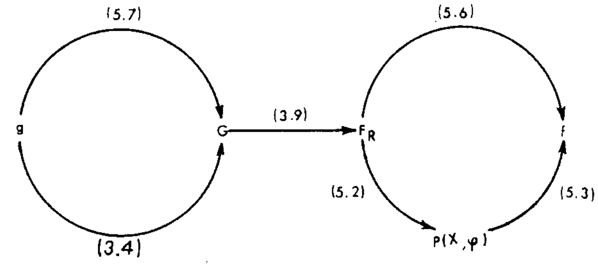

Fig. 3. Several methods of reconstructing when the configuration of source points satisfies Statement 5. The numbers above the arrows indicate the equation which describes that step in the method.

$$f(x) = \frac{-1}{4\pi^3} \int_0^{\pi} \int_0^{\pi} \int F(\beta_{\theta,\varphi}, l) \frac{1}{(x \cdot \beta_{\theta,\varphi} - l)^2}\, dl \cdot \sin\theta\, d\theta\, d\varphi \tag{5.5}$$

where the upper and lower limits of the inner integral are

$$[L_1 + \sqrt{R^2 - x_3^2}]\, \sin\theta + x_3 \cos\theta$$

and

$$[-L_2 - \sqrt{R^2 - x_3^2}]\, \sin\theta + x_3 \cos\theta$$

respectively. We let

$$L_1 = \frac{R - x_3 \cos\theta}{\sin\theta} - \sqrt{R^2 - x_3^2}$$

$$L_2 = \frac{R + x_3 \cos\theta}{\sin\theta} - \sqrt{R^2 - x_3^2}.$$

The reader can verify that for these choices, L_1 and L_2 are both greater than or equal to zero for all $-R \leqslant x_3 \leqslant R$ and $0 \leqslant \theta \leqslant \pi$. Now we have

$$f(x) = \frac{-1}{4\pi^3} \int_0^{k} \int_0^{\pi} \int_{-R}^{R} F(\beta_{\theta,\varphi}, l) \frac{1}{(x \cdot \beta_{\theta,\varphi} - l)^2}\, dl \cdot \sin\theta\, d\theta\, d\varphi. \tag{5.6}$$

It is informative to compare (5.6) with (2.5).

Implementing (5.6) rather than (5.2) and (5.3) to obtain f from F_R may save computation and storage. To possibly save computation in obtaining G from g we give the following results. For

$$\alpha_{\theta',\varphi'} \triangleq (\cos\varphi' \sin\theta', \sin\varphi' \sin\theta', \cos\theta')^{\tau},$$

$$G(\beta, \Phi) = \frac{-2}{\|\beta\|^2} \int_0^{\pi} \int_0^{\pi} \frac{g_1(\alpha_{\theta',\varphi'}, \Phi)}{\left(\frac{\beta}{\|\beta\|} \cdot \alpha_{\theta',\varphi'}\right)^2} \sin\theta'\, d\theta'\, d\varphi'. \tag{5.7}$$

We leave it to the reader to verify (5.7) by first writing (3.4) in polar coordinates and then considering the inner integral as a one-dimensional Fourier transform.

This section has provided several methods of reconstructing when Statement 5 is satisfied. These methods are summarized in Fig. 3.

VI. A Necessary Condition

When considering sufficient conditions, the question of necessity naturally arises. Clearly, Statement 5 cannot be necessary, since a spherical symmetric object requires only one cone-beam source point. [Aside: Although theoretically Statement 5 is not necessary, reconstruction when Statement 5 is violated may be subject to numerical difficulty.] In addition, there are practical situations (for example, electron microscopy [15], [22]) where the symmetry of the object is used in reconstruction. Now then, is Statement 4 necessary? Is it possible to obtain f from g when it is not possible to obtain F_R from G? The following argument shows that this is not possible.

As a first step, observe that one can obtain F_R from f via (2.2) and g from G via an inverse Fourier transform. Symbolically we have

$$G \to g, \quad f \to F_R.$$

Now, assume that there exists a method (which may or may not involve G and F) of obtaining f from g. Then we would have

$$G \to g \to f \to F_R.$$

In words, if there exists a method of obtaining f from g, then there exists a method of obtaining F_R from G. In yet other words, if there does not exist a method of obtaining F_R from G, then there does not exists a method of obtaining f from g. We combine this with Statement 4 by stating the following.

Statement 6: To reconstruct the object from a arbitrary configuration of source points, obtaining F_R from G is not only sufficient but is necessary too.

Part II—A Reconstruction Method for a Curve of Source Points

VII. Assumptions on the Curve

When the configuration of source points is restricted to a curve, a reconstruction method can be obtained which is most likely more efficient then the methods discussed in the previous sections. In this part of the paper we develop this reconstruction method. We represent the curve of source points parametrically by a vector-valued function $\Phi(\lambda) = (\Phi_1(\lambda), \Phi_2(\lambda), \Phi_3(\lambda))^T$ with a domain (a, b) and adopt the notation

$$\Phi'(\lambda) = \left(\frac{d\Phi_1}{d\lambda}, \frac{d\Phi_2}{d\lambda}, \frac{d\Phi_3}{d\lambda}\right)^T.$$

In addition, we assume that

C1: The curve lies outside the object. That is,

$$|\Phi(\lambda)| > R \qquad \text{for all} \quad \lambda \in (a, b).$$

C2: For each pair (θ, φ), $\beta_{\theta,\varphi} \cdot \Phi'(\lambda)$ is piecewise continuous with respect to λ on (a, b). (See [9] for a precise definition of piecewise continuity.)

C3: For each pair (θ, φ), $\beta_{\theta,\varphi} \cdot \Phi'|\lambda) = 0$ has at most a finite number of roots. Each root has a neighborhood that does not contain another root or another point of discontinuity.

The last assumption we make is more substantial than the previous ones.

C4: There is an integer M (which remains constant for a fixed curve) such that each plane that intersects the object intersects the curve exactly M times.

[Aside: Actually the fourth condition can be slightly weakened to:

There exists an integer M (which remains constant for a fixed curve) such that each plane that intersects the object intersects the curve exactly M times except for a "set of planes" with measure zero.

(It is informative to compare this statement with the "geometric condition" stated in [2].) A straight line of source points does satisfy the weakened condition but does not satisfy the stronger condition.]

The above conditions have been stated in rather "geometric terms" to help in the selection of a curve from which one can reconstruct. However, the "geometric terms" of the above conditions are not useful in the derivation that follows; rather, several somewhat "analytic" conditions which stem from the above "geometric conditions" are. In Appendix A, it is shown that if the curve satisfies C1–C4 then the curve also satisfies the following "analytic conditions." For each pair (θ, φ) there exist M sets contained in (a, b)—call them $I(\theta, \varphi, i)$ for $i = 1, 2, 3, \cdots, M$—such that the following is true.

A1: $I(\theta, \varphi, i)$ is a union of closed intervals.

A2: For $i \neq j I(\theta, \varphi, i)$ and $I(\theta, \varphi, j)$ contains at most one point in common.

A3: For each pair (θ, φ), and for a continue function K with support $[-R, R]$, the following change in variables, defined by $l = \Phi(\lambda) \cdot \beta_{\theta,\varphi}$, can be performed.

$$\int_{-R}^{R} k(l)\, dl = \int_{I(\theta,\varphi,i)} k(\beta_{\theta,\varphi} \cdot \Phi(\lambda)\, |\Phi'(\lambda) \cdot \beta_{\theta,\varphi}|\, d\lambda.$$

A4: For each λ such that $|\beta_{\theta,\varphi} \cdot \Phi(\lambda)| \leqslant R$, there exists an i such that $\lambda \in I(\theta, \varphi, i)$ (with a possible exception of a set of measure zero).

The $I(\theta, \varphi, i)$'s are used in the following.

VIII. Derivation of the Reconstruction Method

As the first step in the derivation, we use A3 in (5.6) to obtain

$$f(x) = \frac{-1}{4\pi^3} \int_0^{\pi} \int_0^{\pi} \int_{I(\theta,\varphi,i)} F(\beta_{\theta,\varphi}, \Phi(\lambda) \cdot \beta_{\theta,\varphi}) \cdot \frac{|\Phi'(\lambda) \cdot \beta_{\theta,\varphi}|}{((x - \Phi(\lambda)) \cdot \beta_{\theta,\varphi})^2}\, d\lambda \sin\theta\, d\theta\, d\varphi. \tag{8.1}$$

Since (8.1) is true for $i = 1, 2, 3, \cdots, M$ and since A2 is also true, the following can be done. For $i = 1, 2, 3, \cdots, M$ add the sides of (8.1), respectively, and then divide the sums by M to obtain

$$f(x) = \frac{-1}{4M\pi^3} \int_0^\pi \int_0^\pi \int_{\bigcup_{i=1}^{M} I(\theta,\varphi,i)} F(\beta_{\theta,\varphi}, \Phi(\lambda)\cdot\beta_{\theta,\varphi}) \cdot \frac{|\Phi'(\lambda)\cdot\beta_{\theta,\varphi}|}{((x-\Phi(\lambda))\cdot\beta_{\theta,\varphi})^2}\, d\lambda \sin\theta\, d\theta\, d\varphi. \tag{8.2}$$

Using A4 and Appendix A, we obtain

$$0 = \int_0^\pi \int_0^\pi \int_{(a,b)-\bigcup_{i=1}^{M} I(\theta,\varphi,i)} F(\beta_{\theta,\varphi}, \Phi(\lambda)\cdot\beta_{\theta,\varphi}) \cdot \frac{|\Phi'(\lambda)\cdot\beta_{\theta,\varphi}|}{((X-\Phi(\lambda))\cdot\beta_{\theta,\varphi})^2}\, d\lambda \sin\theta\, d\theta\, d\varphi.$$

Therefore, using (3.9) and exchanging integrals, (8.2) can be written as

$$f(x) = \frac{-1}{8M\pi^4} \int_{(a,b)} \int_0^\pi \int_0^\pi G(\beta_{\theta,\varphi}, \Phi(\lambda)) \cdot \frac{|\Phi'(\lambda)\cdot\beta_{\theta,\varphi}|}{((x-\Phi(\lambda))\cdot\beta_{\theta,\varphi})^2} \sin\theta\, d\theta\, d\varphi\, d\lambda. \tag{8.3}$$

(The above steps in this derivation were also used in [2].) Equation (8.3) can be written as

$$f(x) = \frac{1}{2M\pi} \int_{(a,b)} {}_\alpha\mathcal{F}_\beta^{-1} \cdot \left\{ G(\beta, \Phi(\lambda)) \left| \beta \cdot \frac{\Phi'(\lambda)}{\|\Phi'(\lambda)\|} \right| \right\} \Bigg|_{\alpha = X-\Phi(\lambda)} |\Phi'(\lambda)|\, d\lambda. \tag{8.4}$$

Equations (8.4) and (8.3) are seen to be equivalent by writing the Fourier transform in (8.4) in polar coordinations and then considering the inner integral as a one-dimensional Fourier transform. Now, by observing

$$\left| \frac{\Phi'(\lambda)}{\|\Phi'(\lambda)\|} \cdot \beta \right| = \left(\frac{\Phi'(\lambda)}{\|\Phi'(\lambda)\|} \cdot \beta \right) \operatorname{sgn}\left(\frac{\Phi'(\lambda)}{\|\Phi'(\lambda)\|} \cdot \beta \right)$$

and employing familiar properties of the Fourier transform, we obtain

$$f(x) = \frac{1}{2j\pi M} \int_{(a,b)} {}_\alpha\mathcal{F}_\beta^{-1} \cdot \left\{ \operatorname{sgn}\left(\frac{\Phi'(\lambda)}{|\Phi'(\lambda)|} \cdot \beta \right) \right\} * \left(\nabla_\alpha \cdot \frac{\Phi'(\lambda)}{\|\Phi'(\lambda)\|} \right) \cdot g(\alpha, \Phi(\lambda))\, |\Phi'(\lambda)|\, d\lambda. \tag{8.5}$$

where $*$ denotes a three-dimensional convolution. In Appendix C it is verified that for an orthonormal set of vectors

$$\frac{\Phi'(\lambda)}{\|\Phi'(\lambda)\|}, \Phi'_{\perp 1}(\lambda), \Phi'_{\perp 2}(\lambda)$$

it is true that

$${}_\alpha\mathcal{F}_\beta^{-1} \left\{ \beta \cdot \frac{\Phi'(\lambda)}{\|\Phi'(\lambda)\|} \right\} = \frac{j}{\pi} \frac{1}{\frac{\Phi'(\lambda)}{\|\Phi'(\lambda)\|} \cdot \alpha} \delta(\Phi'_{\perp 1}(\lambda)\cdot\alpha)\, \delta(\Phi'_{\perp 2}(\lambda)\cdot\alpha).$$

Thus, (8.5) can be written as

$$f(x) = \frac{1}{2M\pi^2} \int_{(a,b)} \frac{1}{\alpha \cdot \frac{\Phi'(\lambda)}{|\Phi'(\lambda)|}} \cdot \delta(\Phi'_{\perp 1}(\lambda)\cdot\alpha)\, \delta(\Phi'_{\perp 2}(\lambda)\cdot\alpha) * \left(\nabla_\alpha \cdot \frac{\Phi'(\lambda)}{\|\Phi'(\lambda)\|} \right) g(\alpha, \Phi(\lambda)) \Bigg|_{\alpha = X-\Phi(\lambda)} |\Phi'(\lambda)|\, d\lambda. \tag{8.6}$$

This equation was presented in [1], although no derivation was given there. For a full explanation of (8.6) and how it could be implemented, see [1].

To rewrite (8.6) in a slightly different form we use the following relationship. For a notation matrix A

$$h(AZ) * k(Z) = h(Y) * k(A^T Y)\big|_{Y=AZ}. \tag{8.7}$$

Now we take

$$h(\alpha) = \frac{1}{\alpha_1} \delta(\alpha_2)\, \delta(\alpha_3)$$

and define A by taking the columns of A^T to be the vectors

$$\frac{\Phi'(\lambda)}{\|\Phi'(\lambda)\|}, \Phi'_{\perp 1}(\lambda), \Phi'_{\perp 2}(\lambda).$$

Furthermore, we define

$$g_R(\alpha, \Phi(\lambda)) = g(A^T\alpha, \Phi(\lambda)).$$

Now (8.6) can be written as

$$f(x) = \frac{1}{2M\pi^2} \int_{(a,b)} \int_{-\infty}^{\infty} \frac{1}{(\alpha_1 - l)^2} \cdot g_R((l, \alpha_2, \alpha_3)^T, \Phi(\lambda))\, dl \Bigg|_{\alpha = A(X-\Phi(\lambda))} \|\Phi'(\lambda)\|\, d\lambda. \tag{8.8}$$

It is of interest to note the geometric interpretation of $g_R(\alpha, \Phi(\lambda))$. $g_R(\alpha, \Phi(\lambda))$ describes the cone-beam data in terms of a locally defined coordinate system at each source point as illustrated in Fig. 4. The x_1 axis of the local coordinate system is colinear with the tangent to the curve $\Phi'(\lambda)$. As $\Phi'(\lambda)$ changes so does the coordinate system.

IX. An Improvement in the Reconstruction Method

As explained in [1], the operations to be performed at a given souce point to reconstruct a particular point x via (8.6) or (8.8) involves only the cone-beam data which lie on the

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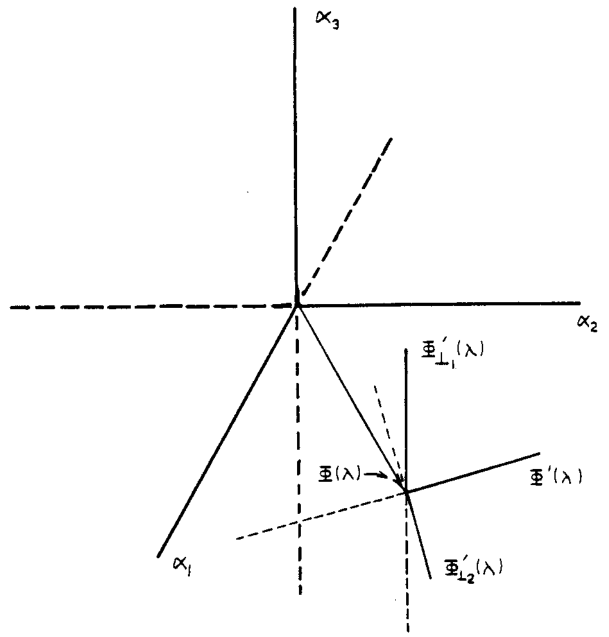

Fig. 4. The local coordinate system defined by $g_R(\alpha, \Phi(\lambda))$. At $\Phi(\lambda)$, the vectors $\Phi'(\lambda)/|\Phi'(\lambda)|$, $\Phi'_{\perp_1}(\lambda)$, $\Phi'_{\perp_2}(\lambda)$ form a three-dimensional coordinate system. $\Phi'_{\perp_1}$ and $\Phi'_{\perp_2}$ are picked such that the vectors $\Phi'(\lambda)/|\Phi'(\lambda)|$, $\Phi'_{\perp_1}(\lambda)$, $\Phi'_{\perp_2}(\lambda)$ form a right-handed coordinate system.

plane that contains $\Phi'(\lambda)$ and x. This cone-beam data, which has been restricted to a plane, can be referred to as fan-beam data. In considering reconstruction of the entire object, it is seen that this restriction defines a one-dimensional family of fan-beams. (In fact, later we define a function that describes this family of fan-beams.) Furthermore, reconstructing the whole object via (8.6) or (8.8) would require multiple ("directional") convolutions of each fan-beam in the family. To reduce computation it is desirable to rewrite (8.6) or (8.8) such that only one convolution is needed per fan-beam. The purpose of this section is to do this.

We need a function that describes the fan-beam data on the plane that contains the reconstruction point x and $\Phi'(\lambda)$—the tangent to the curve. Towards this end, in Fig. 5 we define ψ to be the angle between $\Phi_{\perp_1}(\lambda)$ and

$$X - \Phi(\lambda) - \frac{((X - \Phi(\lambda)) \cdot \Phi'(\lambda))\, \Phi'(\lambda)}{\|\Phi'(\lambda)\|^2}.$$

We describe the fan-beam data on the plane that contains the tangent and x as

$$g_\psi(\alpha_1, \alpha_2, \Phi(\lambda)) = g_R((\alpha_1, \alpha_2 \cos\psi, \alpha_2 \sin\psi)^T, \Phi(\lambda)). \tag{9.1}$$

As in (3.3), $g_\psi(\alpha_1, \alpha_2, \Phi(\lambda))$ is the homogeneous extension with $-1°$ of the projection data; that is

$$g_\psi(\alpha_1, \alpha_2, \Phi(\lambda)) = \frac{1}{\|\alpha\|} g_{\psi_1}(\alpha_1, \alpha_2, \Phi(\lambda)). \tag{9.2}$$

By defining

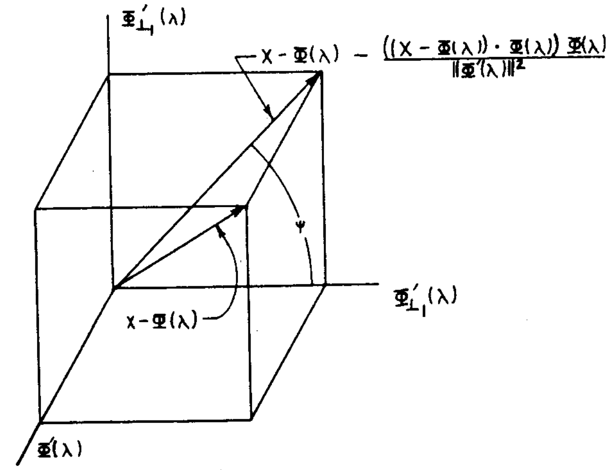

Fig. 5. The angle ψ which defines the fan-beam data on the plane that contains the reconstruction point x and $\Phi'(\lambda)$. Since $\Phi'_{\perp_1}(\lambda)$ is somewhat arbitrary, so is ψ.

$$B = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\psi & \sin\psi \\ 0 & -\sin\psi & \cos\psi \end{bmatrix}$$

and using (9.1), (8.8) can be written as

$$f(x) = \frac{-1}{2M\pi^2} \int_{(a,b)} \frac{\delta(\alpha_2)}{(\alpha_1)^2} * g_\psi(\alpha_1, \alpha_2, \Phi(\lambda)) \Big|_{\alpha = BA(X - \Phi(\lambda))} |\Phi'(\lambda)| \, d\lambda \tag{9.3}$$

where $*$ denotes a two-dimensional convolution. [Note only the first two components of $BA(x - \Phi(\lambda))$ are needed in (9.3).] In Appendix D we verify the following. If

$$\vec{\sigma} = (\sigma_1, \sigma_2)^T = (\cos\sigma, \sin\sigma)^T$$

and

$$\vec{\sigma}_\perp = (\sigma_2, -\sigma_1)$$

then

$$\frac{\delta(\alpha_2)}{\alpha_1^2} * g_\psi(\alpha_1, \alpha_2, \Phi(\lambda)) = \frac{1}{\|\alpha\|^2} \int_0^\pi \frac{\sin\sigma}{\left(\frac{\alpha}{|\alpha|} \cdot \sigma_\perp\right)^2} g_{\psi_1}(\cos\sigma, \sin\sigma, \Phi(\lambda)) \, d\sigma. \tag{9.4}$$

Thus (9.3) becomes

$$f(x) = \frac{-1}{2M\pi^2} \int_{(a,b)} \frac{1}{|X - \Phi(\lambda)|^2} \int_0^\pi \frac{\sin\sigma}{\left(\frac{\alpha}{|\alpha|} \cdot \sigma_\perp\right)^2} \cdot g_{\psi_1}(\cos\sigma, \sin\sigma, \Phi(\lambda)) \Big|_{\alpha = BA(X - \Phi(\lambda))} d\sigma \, |\Phi'(\lambda)| \, d\lambda. \tag{9.5}$$

Note that this equation requires only convolution per fan-beam. [Aside: In more detail (9.5) is

$$f(x) = \frac{1}{2M\pi^2} \int_{(a,b)} \frac{1}{|X - \Phi(\lambda)|^2}$$

$$\cdot \lim_{\epsilon \to 0} \int_{-\infty}^{\infty} F_\epsilon \left(\frac{\alpha}{|\alpha|} \cdot \sigma_\perp \right) \bigg|_{\alpha = BA(X - \Phi(\lambda))} g_{\psi_1}(\cos\sigma, \sin\sigma, \Phi(\lambda))$$

$$\cdot \sin\sigma \, d\sigma \, |\Phi'(\lambda)| \, d\lambda.]$$

In comparing (9.5) to the "extended fan-beam convolution formula"–(5.4 in [2]–one sees a great similarity. The principal difference between these formulas is that the operation performed on the inner integral of (9.5) is on the plane that contains x and $\Phi'(\lambda)$ and that this plane changes as $\Phi'(\lambda)$ or x changes, whereas the plane on which the inner integral of (5.4) in [2] takes place remains fixed.

PART III–Suggestions for the Circle of Source Points

X. The Circle of Source Points

At the present time there are two scanners being developed that reconstruct from a circle of source points–the cardiovascular computed tomographic (CVCT) scanner at the University of California, San Francisco, and the dynamic spatial reconstructor (DSR) at the Mayo Clinic, Rochester, NY. Because these scanners are presently being developed, in this part of the paper we apply the results we have developed to a circle of source points. Although the discussion is confined to a circle of source points, much of what is said applies equally well to other configurations of source points.

Unfortunately, the circle of source points does not satisfy Statement 4. To see this, we introduce the following mechanism to represent planes. Let the order pair $(\beta_{\theta,\varphi}, l)$ represent a plane where $\beta_{\theta,\varphi}$ is the direction of the perpendicular to the plane from the origin and l is its length (see Fig. 6). From Fig. 6 we conclude that the set of planes that intersect the object, but not the circle, is the set

$$\{(\beta_{\theta,\varphi}, l):\ 0 < \theta < \pi, 0 < \varphi < \pi, D \sin\theta \leqslant |l| \leqslant R\}.$$

Thus the circle does not satisfy Statement 4. It therefore cannot satisfy the fourth condition stated in Section VII either. Hence, without making additional assumptions concerning the object (e.g., spherical symmetry) none of the reconstruction methods presented so far can be used. In spite of this, we are still able to make several suggestions for the circle.

XI. Suggestion One: A Change in an Approximate Reconstruction Method

The first suggestion involves a change in the approximate reconstruction method that is presently used in the DSR. The approximate method used in the DSR is explained with the help of Fig. 7. The X-ray source at $\Phi(\lambda)$ projects the cross section of object indicated by broken lines in Fig. 7 onto the line "T" on the detector screen. One observes that as the source and detector rotates about the object, different cross sections of the object will be projected onto the line "T".

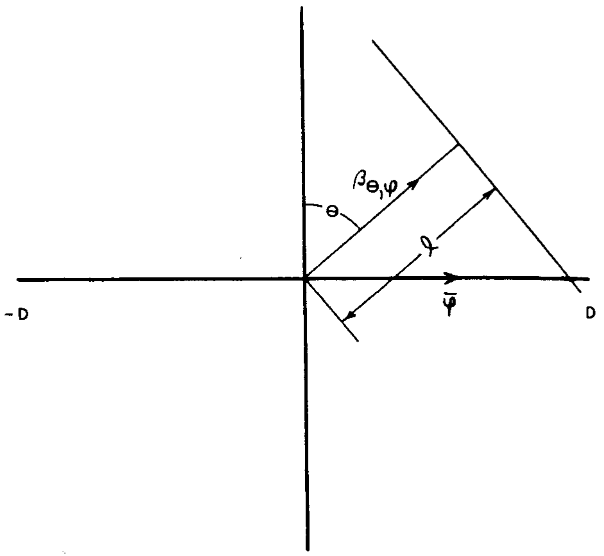

Fig. 6. The plane $(\beta_{\theta,\varphi}, l)$ intersecting the circle of source points with radius D. From the vantage point of this figure, the plane $(\beta_{\theta,\varphi}, l)$ appears as a straight line and the circle appears as a line segment from $-D$ to D. The plane will intersect the circle if and only if $|l| \leqslant D \sin\theta$. Because of the symmetry around the x_3 axis this argument is true for all φ.

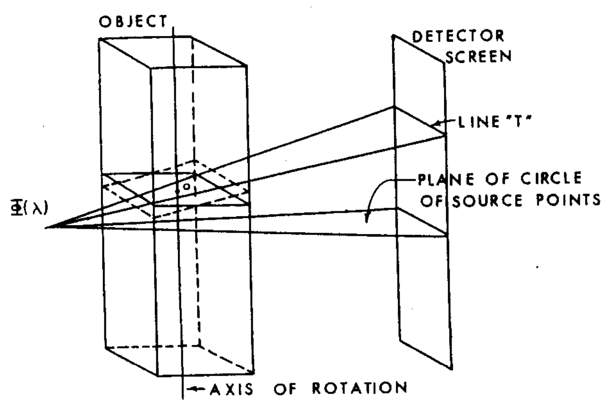

Fig. 7. The approximation used in the DSR and a suggested change.

The only point in common with all these cross sections is the point "o," which is the point of intersection of the cross section with the axis of rotation of the X-ray source points. The approximation used in the DSR is that the data obtained on line "T" from these different cross sections are from one cross section of the object; the cross section (indicated by solid lines in Fig. 7) that contains the point "o" and is parallel to the circle [13]. The change in the approximate reconstruction method that results when one applies (9.5) to the circle rather than using the DSR approximation boils down to the following. At each source point backproject the convoluted fan-beam data of line "T" onto the nonparallel cross section indicated by the broken lines in Fig. 7 rather than backprojecting it on the parallel plane indicated by the solid lines in Fig. 7.

XII. Suggestion Two: Extrapolation

There are several methods that can be proposed that involve extrapolation. First, from the portion of F_R which can be determined via (3.9), extrapolate the remaining unknown portion. If this extrapolation can be posed in the Hilbert space setting discussed in [12], then this extrapolation can be done via the method proposed in [12], iterating back and forth between f and F_R. To save computation observe that actually only $\check{f}$ rather then f needs to be solved for iteratively since 1) the *a priori* information about the support of the object can be imposed upon $\check{f}$, and 2) F_R can be recalculated from f via (2.2).

The second extrapolation method proposed involves extrapolating $P(x, \varphi)$. Once the largest portion of the function $P(x, \varphi)$ has been obtained via (5.2) one can define on each plane perpendicular to the x_3 axis a limited angle problem. Then, on each plane one can use any of the methods previously stated in the literature to solve the limited angle problem.

The third extrapolation method proposed also involves extrapolation $P(x, \varphi)$. Since $P(x, \varphi)$ is constant along the lines perpendicular to φ, that is

$$P(x, \varphi) = P(x + s\varphi_\perp, \varphi) \qquad \text{for all} \quad s,$$

we can do the following. Define

$$x = x - (x \cdot \varphi_\perp)\, \varphi_\perp$$

then for a fixed φ consider $P(x_\varphi, \varphi)$ as a two-dimensional function. Now, for a fixed value of φ extrapolate the unknown portion of the two-dimensional function $P(x_\varphi, \varphi)$ from its known portion. If for various values the two-dimensional function $P(x_\varphi, \varphi)$ cannot be extrapolated, then extrapolate $P(x_\varphi, \varphi)$ as a three-dimensional function of the variables x_{φ_1}, x_{φ_2} and φ. This latter extrapolation can be casted in the frequency domain setting by observing that for a fixed φ the two-dimensional Fourier transform of $P(x_\varphi, \varphi)$ yields a section of the Fourier transform of the object on the plane that is perpendicular to φ and intersects the origin. That is

$$\tilde{f}(X_\varphi) = x_\varphi \mathcal{F}_{x_\varphi} \{P(x_\varphi, \varphi)\}.$$

XIII. Suggestion Three: Add More Source Points

The final suggestion made for the circle is to add more source points. One method of adding more source points which does not require extensive redesign of the CVCT and DSR would be to obtain two circle of source points that are perpendicular to each other. How this could be done is shown in Fig. 8. This configuration of source points was suggested in [23], [14]. Since this configuration satisfies Statement 4, the reconstruction methods discussed earlier can be applied.

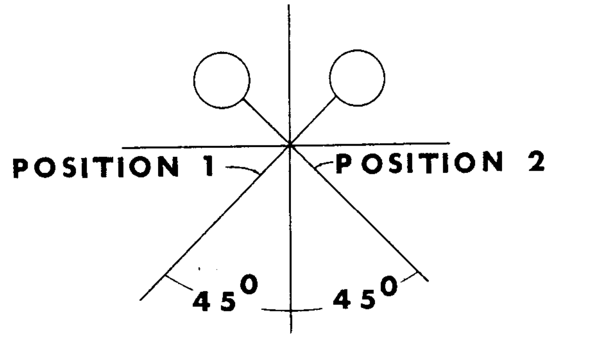

Fig. 8. A schematic representation of how two circle of source points can be obtained without substantial modification of the CVCT or the DSR. From the vantage point of this figure, the circle of source points appears as a horizontal line segment. The line segments with a circle on top represents the patient. First, the patient is placed at a 45° angle on one side of the vertical and then swiveled to a 45° angle on the other side of the vertical.

Conclusion

In Part I a new function F_R was introduced. The relationships of F_R with the object and the object's three-dimensional Radon transform was developed. By defining G as the Fourier transform of the homogeneous extension of the cone-beam data, we verified the following statement.

Statement 6: One can reconstruct the object from a arbitrary configuration of source points if and only if one can obtain F_R from G.

Statement 6 is both necessary and sufficient, at least from the theoretical point of view. More suggestive is Statement 5.

Statement 5: If on every plane that intersects the object, there exists at least one cone-beam source point, then one can reconstruct the object.

Theoretically speaking, Statement 5 is more stringent than Statement 6. However, when numerical stability is considered it may turn out to be that Statement 5 will be more practical.

The reconstruction methods that require only Statement 5 would most likely require a large amount of computation and storage. In Part II, a more efficient reconstruction method was derivated, but it was at the cost of additional constraints on the configuration of source points. The principal constraints were 1) the configuration of source points was a curve, and 2) on every plane that intersects the object there exists exactly M (an integer fixed for a given curve) source points.

In Part III, the ideas developed in this paper were applied to a circle of source points. Unfortunately, we saw that a circle does not satisfy Statement 5. Thus, strictly speaking none of the previously mentioned reconstruction methods can be applied. Nonetheless, several suggestions were made for the circle. 1) A change in the existing approximate reconstruction method used in the DSR. 2) Extrapolation of missing data. 3) Addition of more source points. Many of the suggestions made for the circle apply equally well to any other configuration of source points that violate Statement 5.

This paper has not answered all the questions concerning cone-beam reconstruction. First, there are questions concerning the reconstruction methods developed in Part I (the methods that only require Statements 5 or 6). Is the amount of computation and memory required for implementation reasonable? How ill-conditioned are these methods? Experience from two-dimensional tomography provides ample evidence that the operator $1/l^2 *$ (or equivalently $1/l * \partial/\partial l$) is ill-conditioned. This experience suggests that obtaining f from F_R is ill-conditioned since it involves the some operator [see (5.3) or (5.6)]. Unlike the two-dimensional case we now have another step that may be ill-conditioned. Equation (5.7) suggests that obtaining G from g may well be ill-conditioned too. Also, note that obtaining G from g via (3.4) or (5.7) assumes that the data are known for the whole cone, which, in practice may require X-raying the whole body. Not having the whole cone of data would result in a problem somewhat analogous to the "limited angle problem." Other, somewhat related questions concern sampling. What is the best sampling of the cone of data itself? In the light of (3.9), what configuration of source points results in the best sampling of F_R? What is the best discrete version of (5.6), (3.4), (5.7), (5.2), and (5.3)?

There are also questions concerning the reconstruction method developed in Part II (the method that assumes the source points form a curve). What curves other than a straight line satisfy the requirements of the method? Can the requirements on the curve be reduced? Can acceptable reconstruction be obtained from curves that only approximately satisfy the requirements? Questions like those of Part I concerning sampling should be asked here too. Fortunately, however, we do not necessarily need the whole cone of data at each source point. This can be seen when the curve is a straight line. Does any configuration of source points besides the straight line have this property? Addressing these questions will be the subject of future research.

Appendix A

This Appendix shows that if a curve satisfies C1-C4, then A1-A4 are true. First, for notational convenience, let $h(\lambda) = \beta_{\theta,\varphi} \cdot \Phi(\lambda)$. Note that the functional dependence of h on θ and φ has been suppressed. Conditions C1-C4 imply that for each pair (θ, φ) the following "algorithm" can be used to define $I(\theta, \varphi, i)$, which results in Al-A4 being satisfied.

Let

a_i = ith largest root or discontinuity of $dh/d\lambda$ on (a, b); $i = 1, 2, 3, \cdots$

γ = the largest root or discontinuity of $dh/d\lambda$ on (a, b)

$A_0 = (\gamma, b)$

$A_1 = (a, a_1)$

$A_i = (a_{i-1}, a_i), i = 2, 3, 4, \cdots$.

For each A_i the following two step "algorithm" places A_i in a $I(\theta, \varphi, j)$ or discards it. Initially $L = 0$.

For $i = 0, 1, 2, 3, \cdots$ until all intervals have been considered

Step 1: If the set $\{\lambda \in A_i: |h(\lambda)| \leq R\}$ has measure zero, then discard A_i and go to next i.

Step 2: $L = L + 1$. If the set $[-R, R] - h(A_i)$ has measure zero, then define $I(\theta, \varphi, L) = A_i$; otherwise, there exists a sequence of intervals—call them $A_{N(k)}$ for $k = 1, 2, 3, \cdots$—such that $N(k) \geq i$, $A_{N(1)} = A_i$, and $A_{N(k)}$ satisfy the following two properties.

1) The set $h(A_{N(k)}) \cap h(A_{N(j)}) \cap [-R, R]$ has measure zero for $k \neq j$.

2) The set $[-R, R] - \cup_k h(A_{N(k)})$ has measure zero.

Define $I(\theta, \varphi, L) = \cup_k \bar{A}_{N(k)}$. Remove the sequence $A_{N(k)}$ from the remaining set of intervals to be considered and relabel the remaining intervals sequentially. Go to next i.

Take M to be the final value of L.

Appendix B

Let E be a (measurable) set such that each element of E is either greater then R or less then $-R$. In set notation that is

$$E \subset ((-\infty, -R) \cup (R, \infty)). \tag{B.1}$$

In this appendix we verify that for any E that satisfies (B.1) it is true that

$$0 = \int_0^{\pi} \int_0^{\pi} \int_E F(\beta_{\theta,\varphi}, l) \frac{1}{(X \cdot \beta_{\theta,\varphi} - l)^2} \, dl \sin\theta \, d\theta \, d\varphi. \tag{B.2}$$

[Compare (B.2) with (5.6).] Using the values of L_1 and L_2 chosen in Section V, (B.1) is equivalent to

$$0 = \int_0^{\pi} \int_{E'} P(X + \vec{\varphi}\epsilon, \varphi) \frac{1}{\epsilon^2} \, d\epsilon \, d\varphi \tag{B.3}$$

where E' is a (measurable) set such that

$$E' \subset ((-\infty, -L - X \cdot \vec{\varphi} - \sqrt{R^2 - X_3^2}) \cup (L_1 - X \cdot \vec{\varphi} + \sqrt{R^2 - X_3^2}, \infty))$$

[Compare (B.2) with (5.6).] By observing for all $\epsilon \in E'$

$$P(X + \vec{\varphi}\epsilon, \varphi) = 0$$

we have verified (B.2).

Appendix C

In this Appendix we verify the following. If the vectors X, Y, and Z form a orthonormal set, then

$${}_\alpha\mathcal{F}_\beta^{-1}\{\text{sgn}(X \cdot \beta)\} = \frac{j}{\pi} \frac{1}{X \cdot \alpha} \delta(Y \cdot \alpha)\, \delta(Z \cdot \alpha). \tag{C.1}$$

First, note that

$${}_\alpha\mathcal{F}_\beta^{-1}\{\tilde{f}(\beta_1)\, \tilde{h}(\beta_2)\, \tilde{k}(\beta_3)\} = f(\alpha_1)\, h(\alpha_2)\, k(\alpha_3). \tag{C.2}$$

It is shown in [5] and [6] that

$${}_l\mathcal{F}_\omega^{-1}\{\text{sgn}\,\omega\} = \frac{j}{\pi} \frac{1}{l}. \tag{C.3}$$

Thus, we have

$${}_\alpha\mathcal{F}_\beta^{-1}\{\text{sgn}\,\beta_1\} = \frac{j}{\pi\alpha_1} \delta(\alpha_2)\, \delta(\alpha_3). \tag{C.4}$$

We leave it to the reader to verify that for any rotation matrix A

$${}_\alpha\mathcal{F}_\beta^{-1}\{\tilde{f}(A\beta)\} = f(A\alpha). \tag{C.5}$$

By taking the columns of A^T to be the vectors X, Y, and Z, that is

$$a_{1j} = X_j, \quad a_{2j} = Y_j, \quad a_{3j} = Z_j$$

for $j = 1, 2, 3$, (C.1) then follows.

Appendix D

In this Appendix we verify that

$$\frac{\delta(\alpha_2)}{\alpha_1^2} * g_\psi(\alpha_1, \alpha_2, \Phi(\lambda)) = \frac{1}{\|\alpha\|^2} \int_0^\pi \frac{\sin\sigma}{\left(\frac{\alpha}{|\alpha|}\cdot\sigma_\perp\right)^2} g_{\psi_1}(\cos\sigma, \sin\sigma, \Phi(\lambda))\, d\sigma. \tag{D.1}$$

First, explicitly write out the left-hand side (LHS) of (D.1) and use (9.2) to obtain

$$\mathrm{LHS} = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \cdot \frac{\delta(\alpha_2 - Z_2)}{(\alpha_1 - Z_1)^2} g_{\psi_1}(Z_1, Z_2, \Phi(\lambda)) \frac{1}{|Z|} dZ_1\, dZ_2. \tag{D.2}$$

Perform a change in variables defined by

$$\vec{\sigma} t = Z$$

to obtain

$$\mathrm{LHS} = \int_0^\pi \int_{-\infty}^{\infty} \frac{\delta(\alpha_2 - \sigma_2 t)}{(\alpha_1 - \sigma_1 t)^2} dt\, g_{\psi_1}(\cos\sigma, \sin\sigma, \Phi(\lambda))\, d\sigma. \tag{D.3}$$

With simplification (D.3) becomes

$$\mathrm{LHS} = \frac{1}{\|\alpha\|^2}\int_0^\pi \frac{|\sigma_2|}{\left(\frac{\alpha_1}{\|\alpha\|}\sigma_2 - \frac{\alpha_2}{\|\alpha\|}\sigma_1\right)^2} \cdot g_{\psi_1}(\cos\sigma, \sin\sigma, \Phi(\lambda))\, d\sigma. \tag{D.4}$$

We see that the right-hand side of (D.4) is equivalent to the right-hand side of (D.1); thus, we have verified (D.1).

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