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# Exact and Approximate Cone-Beam X-ray Microtomography

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# 1 Introduction

Nondestructive analysis and visualization of three-dimensional microstructures of opaque specimens are important in biomedical and material sciences and engineering. Due to its penetration ability and contrast mechanism, X-ray microtomography is a powerful tool in this type of applications [Johnson et al., 1986, Russ, 1988, Kinney et al., 1989, Cheng et al., 1991, Wang et al., 1991c, Kinney et al., 1990]. An X-ray shadow projection microscope with a microtomography capability is being developed at the State University of New York at Buffalo (SUNY/Buffalo) [Cheng et al., 1991, Wang et al., 1994a]. As illustrated in Figure 1-1, the X-ray microtomographic system at SUNY/Buffalo uses an X-ray point source generated by a microfocused e-beam. This point source can be electro-magnetically steered across the target window in a precisely controllable fashion. A specimen is mounted on a mechanical stage, which can be translated and/or rotated under a programmable control unit. Projection data can be recorded on a cooled CCD camera, transferred into a high-performance computing unit and reconstructed for cross-sectional or volumetric images as well as stereo-image pairs. As X-rays from a point source and through a spheric specimen form a cone, this approach is commonly called cone-beam tomography.

Both transmission and emission tomography problems involve cone-beam geometry. The primary advantages for use of divergent cone-beams include reduced data acquisition time, improved image resolution, and optimized photon utilization. Examples in X-ray medical imaging are the Dynamic Spatial Reconstructor developed at the Mayo Clinic to investigate the heart and lungs [Robb, 1985], the TRIDIMOS project to measure the bone mineral content of lumbar vertebrae [Grangeat, 1989], the MORPHOMETRE project to image vessel trees and bone structures [Saint-Felix et al., 1990], a microtomographic imaging system to study small objects like biopsies [Morton et al., 1990], and a cone-beam imaging system for angiography [Saint-Félix et al., 1994]. Examples in nuclear medical imaging include development of cone-beam collimators used with large gamma cameras to image a small region of interest like the brain [Jaszczak et al., 1988] and the heart [Gullberg et al., 1991]. In industrial imaging, cone-beam scanners are used for nondestructive evaluation of metal parts and ceramic materials [Feldkamp et al., 1984, Vickers et al., 1989].

There are unique challenges that must be met in cone-beam X-ray microtomography. First, the conventional assumption of the spheric reconstruction region may not be true, because spheric, rod-shaped and planar specimens are all typical in reality. Second, specimen free rotation and projection complete detection may not be always physically implementable due to the hardware limitations. Third, the mechanical errors in the specimen rotation is substantial compared to the image resolution in the order of 1 micron. The accurate measurement and calibration of the rotation axis position is complicated and time-consuming.

Due to the three-dimensional divergency, reconstruction with cone-beam data is much more intricate than in parallel-beam or fan-beam geometry. Having been studied for many years, it is still a major topic in computed tomography (CT). As far as the general principles of CT is concerned, an excellent description can be found in [Russ, 1995], practical algorithms with detailed derivations in [Herman, 1980, Kak and Slaney, 1987], and a rigorous mathematical treatment in [Natterer, 1986]. Cone-beam reconstruction algorithms may be classified from various viewpoints, such as exactness, efficiency, scanning pattern, reconstruction region, and so on. Cone-beam tomography was reviewed by Grangeat [Grangeat, 1990], Smith [Smith, 1990] and Gullberg, *et al.* [Gullberg et al., 1992], respectively. In this text, we classify existing cone-beam algorithms in two categories: exact and approximate. With an emphasis on cone-beam X-ray microtomography, the current state of development in exact and approximate cone-beam algorithms is reviewed, and connections between exact and approximate reconstruction discussed.

# 2 Exact Reconstruction

Kirillov developed a formula for reconstruction of a complex valued $n$-dimensional function from complex valued cone-beam projection data [Kirillov, 1961]. It was shown that a sufficient condition for reconstruction in the Schwarts space is that an unbounded source point locus intersects almost every hyperplane. The complex valued cone-beam formulation cannot be directly used in reality. Consequently, the real valued cone-beam tomography theory was developed based on Kirillov's original work.

## 2.1 Sufficient Condition

It seems that Smith first realized the practical importance of Kirillov's work. He rewrote Kirillov's work for one-dimensional line integrals in the $n$-dimensional real space and developed an inversion formula for an infinitely long source point scanning line [Smith, 1983]. Tuy [Tuy, 1983] derived a formula for reconstruction of a real function with a compact support under the condition that almost every hyperplane through the function support meets a source locus transversely. Tuy established a cone-beam reconstruction formula for two intersecting source circles. His formula requires a gradient be computed at each vertex. Thorough theoretical analyses on cone-beam reconstruction were done by Smith and Grangeat [Smith, 1987, Smith, 1985b, Grangeat, 1987, Grangeat, 1990]. Due to their foundmental work, we have the following sufficient condition for exact cone-beam reconstruction: "*if on every plane that intersects the object there exists at least one cone-beam source point, then one can reconstruct the object*" [Smith, 1985b] (Figure 1-2). It is interesting to recall that *if on every straight line that intersects the object there exists at least one fan-beam source point, then one can reconstruct the object.*

Grangeat's derivation of the above sufficient condition gave a clearer geometrical picture. His work was reformulated by Danielsson and Axelsson [Danielsson, 1992, Axelsson, 1994]. The key idea for obtaining the sufficient condition may be simply explained with help of Figure 1-3. In Figure 1-3, a Radon shell is defined with $SO$ as the diameter, where S and O denote a source position and the reconstruction system origin, respectively. Consider a point $p\vec{n}$ on the Radon shell, Radon and X-ray transforms of the plane perpendicular to the shown great circle and through the point $p\vec{n}$ can be expressed as follows:

$$R(p\vec{n}) = \int\int f(p\vec{n}, r, \beta) r dr d\beta, \tag{1}$$

and

$$X(p\vec{n}, \beta) = \int f(p\vec{n}, r, \beta) dr. \tag{2}$$

If $R(p\vec{n}, \beta)$ is known everywhere, a Radon inversion can be done. However, it is $X(p\vec{n}, \beta)$ that is directly measurable. The essential difference between $R(p\vec{n})$ and $X(p\vec{n}, \beta)$ is the multiplicative factor $r$ in $R(p\vec{n})$. If the undesired $r$ is somehow removed, $X(p\vec{n}, \beta)$ will be linked to $R(p\vec{n})$. Fortunately, this can be achieved as follows:

$$\begin{aligned} \frac{d}{dp} R(p\vec{n}) &= \int\int \frac{d}{dp} f(p\vec{n}, r, \beta) r dr d\beta \\ &= \int\int \frac{d}{d\alpha} \frac{f(p\vec{n}, r, \beta)}{\cos\beta} dr d\beta \\ &= \frac{d}{d\alpha} \int \frac{X(p\vec{n}, \beta)}{\cos\beta} d\beta. \end{aligned} \tag{3}$$

Note that the differential relationship $dp = r\cos\beta d\alpha$ plays a critical role in relating $R(p\vec{n})$ to $X(p\vec{n}, \beta)$. Therefore, there must be at least one source position on the plane through $p\vec{n}$ to compute the radial derivative of $R(p\vec{n})$ in the above manner. This immediately leads to the sufficient condition for exact cone-beam reconstruction.

Chen proposed a theoretical framework for regional cone-beam reconstruction by extending Smith's work [Chen, 1992]. He defined the concept of the planar region of an object as intersection of a set of planes and the object support, proved that *if a source curve is connected and compact, and if its convex hull contains a planar region, then the cone-beam data from the source curve is complete with respect to the planar region*, and established an estimation formula for regional reconstruction and a convergence condition for the formula. However, his formula requires "full angle cone-beam" data, instead of truncated cone-beam data.

## 2.2 Exact Algorithms

Various exact cone-beam reconstruction algorithms have been implemented according to Grangeat's framework [Grangeat, 1987, Grangeat, 1990, Sire et al., 1990, Defrise and Clack, 1994, Kudo and Saito, 1994, Clack et al., 1991, Danielsson, 1992, Axelsson, 1994, Axelsson and Danielsson, 1994, Hu, 1995], Smith's theory [Smith and Chen, 1992, Weng et al., 1993, Zheng and Gullberg, 1992] and Tuy's method [Zeng et al., 1994], respectively. Among these implementations, Axelsson and Danielsson's direct Fourier transform method is computationally the most efficient for a sufficiently large amount of data. The Axelsson and Danielsson method may be regarded as a major refinement of the Grangeat method.

The Grangeat method consists of two parts, which are illustrated in Figure 1-4. In the first part, the radial derivative of planar integrals are computed, according to the relationship between the radial derivative of Radon data and the line integral of cone-beam data. The results are distributed on "Radon shells" determined by a scanning locus. If the scanning locus is complete, the Radon space can be completely filled. In the second part, these Radon data are inverted. Although the direct filtered backprojection formula may be applied with the three-dimensional Radon data, the computational complexity is $O(N^5)$. With the Marr method [Marr et al., 1981], the three-dimensional Radon inversion is decomposed into two steps. First, three-dimensional Radon data are interpolated to vertical planes, and two-dimensional reconstruction is done for each vertical plane. As a result, three-dimensional Radon data are transformed into two-dimensional Radon data associated with the vertical planes. Data in the vertical planes are then grouped into data in horizontal planes, and two-dimensional reconstruction is performed for each of the horizontal planes for volumetric image reconstruction. Line integral and backprojection in the Grangeat method are most time-consuming. The computational complexity of the Grangeat method is $O(N^4)$.

The Axelsson and Danielsson method reduces the complexity of the Grangeat method to $O(N^3 \log N)$. The reduction was made through adapting the linogram method [Edholm and Herman, 1987].

The idea of the linogram method and its inverse version can be explained as follows. The motivation was to apply the Fourier transform with one-dimensional projection profiles to produce equidistant samples along concentric squares in the two-dimensional Fourier domain so that these samples can be one-dimensionally interpolated into samples on a squared grid. Fourier spectrum on the regular grid can be directly inverted to recover the image. The desired linogram sampling pattern in the Fourier domain requires that the projection profile sampling step and the projection angular increment vary appropriately. In the direct application, linogram sampled projection data are the input, and an image reconstructed on a square grid is the output. On the other hand, the linogram method can be employed in reverse to input an image discretized on a square grid, and output linogram-sampled projection data. Actually, the one-dimensional interpolation can be eliminated by using the chirp z-transform.

Axelsson and Danielsson modified the conventional linogram method and applied it in the Grangeat method, as illustrated in Figure 1-5. To obtain linogram-sampled two-dimensional Radon data in the horizontal planes, the reconstructed two-dimensional Radon data in the vertical planes should be distributed on appropriate rectangular grids, and the angular increment between adjacent vertical planes properly varied. The desired Radon data on the rectangular grids can be produced from the correspondingly linogram-sampled Fourier spectrum of three-dimensional Radon data in the same vertical planes. The linogram-sampled Fourier spectrum should be radially computed. These requirements determine sampling patterns in the three-dimensional Radon space. Alternatively, instead of the reconstruction of horizontal and vertical planes, a direct 3D linogram method can be used to produce the same image quality at less computational time.

Exact cone-beam reconstruction with spheric scanning is also interesting. The traditional convolution-backprojection approach was employed for direct reconstruction in the $4\pi$ geometry [Nalcioglu and Cho, 1978, Denton et al., 1979, Smith et al., 1980, Smith, 1982, Imiya and Ogawa, 1984, Peyrin, 1985]. Recently, Cho *et al.* proposed a weighted backprojection algorithm for truncated spherical source coverage [Cho et al., 1994]. The key issue is how to weight cone-beam data appropriately so that three-dimensional backprojection of them produces a blurred image that can be modeled as a convolution of the ideal image and a kernel. The blurred image is then filtered to recover the truth. The surface scanning mode may be also useful in cone-beam microtomography.

# 3 Approximate Reconstruction

Feldkamp *et al.* [Feldkamp et al., 1984] adapted the conventional equispatial fan-beam algorithm for cone-beam reconstruction with a circular scanning locus. In the Feldkamp algorithm, cone-beam projection data from different angles are filtered and back-projected along X-rays after voxel-to-source distance and angular differential are properly modified. The value of a voxel is the sum of contributions from all horizontally tilted fan-beams passing through the voxel. Since its publication, the Feldkamp algorithm has been extended in various ways for approximate reconstruction [Gullberg et al., 1991, Wang et al., 1991c, Kudo and Saito, 1991, Gullberg and Zeng, 1992, Smith, 1990, Wang et al., 1993b, Yan and Leahy, 1992]. Because our generalized Feldkamp algorithm [Wang et al., 1991c, Wang et al., 1993b] was specifically designed for cone-beam X-ray microtomography, it is focused on in this chapter.

## 3.1 Why Approximate Reconstruction

Despite elegant results in exact cone-beam reconstruction, approximate cone-beam formulas remain practically important in cone-beam X-ray microtomography. The advantages of approximate cone-beam reconstruction are as follows. *First, incomplete scanning loci are allowed.* The sufficient condition for exact reconstruction requires that there exist at least a source position on any plane intersecting an object. This condition cannot be satisfied in cone-beam X-ray microtomography when polygonal or dashed-line helical scanning loci are used [Wang et al., 1993b]. *Second, partial detection coverage does not cause a problem.* In cone-beam reconstruction discussed in the preceding section, the cone-beam was assumed to cover the entire object from any source position. Unlike emission tomography, complete detection coverage is impossible in cone-beam X-ray microtomography, since specimens are often either rod-shaped or planar instead of spheric. *Third, computational efficiency is high.* Because of the second advantage, approximate reconstruction involves much less raw data, especially in reconstruction of rod-shaped and planar specimens. The computational structure of Feldkamp-type reconstruction is straightforward, highly parallel, and hardware supported. Feldkamp-type formulas are particularly fast in reconstructing a limited number of slices or small regions of interest. The linogram idea used in exact Fourier cone-beam reconstruction might also be adapted to Feldkamp-type reconstruction. *Fourth, image noise and ringing artifacts are less.* It was found that exact cone-beam reconstruction with the direct Fourier method produces more ringing as compared to the Feldkamp method [Axelsson and Danielsson, 1994]. We hypothesize that this might be inherent to all exact cone-beam reconstruction formulas that filter data two-dimensionally. Further evaluation and comparison would be valuable.

### 3.2 Derivative-Free Non-Circular Fan-Beam Algorithm

The circular equispatial fan-beam reconstruction formula is well known [Kak and Slaney, 1987]:

$$g(x,y) = \frac{1}{2}\int_0^{2\pi} \frac{\rho^2}{(\rho - s)^2} \int_{-\infty}^{\infty} X(\beta,p) r\left(\frac{\rho t}{\rho - s} - p\right) \frac{\rho}{\sqrt{\rho^2 + p^2}} dp d\beta, \tag{4}$$

where $\rho$ is a constant source-to-origin distance, $X(\beta,p)$ represents equispatial fan-beam projection data,

$$\begin{cases} t &= x\cos\beta + y\sin\beta, \\ s &= -x\sin\beta + y\cos\beta, \end{cases} \tag{5}$$

where $\beta$ is the rotation angle, and

$$r(p) = \int_{-\infty}^{\infty} \mid w \mid e^{j2\pi wp} dw. \tag{6}$$

In a non-circular scanning case, the source-to-origin distance is a function of the rotation angle, denoted as $\rho(\beta), \beta \in [0, 2\pi)$. Summing up differential contributions from each source position based on the circular equispatial fan-beam reconstruction formula, we *intuitively* obtain a non-circular fan-beam reconstruction formula as follows:

$$g(x,y) = \frac{1}{2}\int_0^{2\pi} \frac{\rho^2(\beta)}{(\rho(\beta) - s)^2} \int_{-\infty}^{\infty} X(\beta,p) r\left(\frac{\rho(\beta) t}{\rho(\beta) - s} - p\right) \frac{\rho(\beta)}{\sqrt{\rho^2(\beta) + p^2}} dp d\beta. \tag{7}$$

The difference between our non-circular fan-beam formula and the circular one lies in the definition of $\rho$. $\rho$ is a constant in the former and a function in the latter.

Under the following regular conditions that

1. $\rho(\beta) = \rho(\beta + \pi)$;
2. $\rho'(\beta)$ exists almost everywhere;
3. $\rho^2(\beta) > \rho'(\beta) p_m$, where $p_m$ is the minimum value such that $X(\beta,p) = 0$, if $\mid p \mid > p_m$.

we proved this non-circular fan-beam reconstruction formula [Wang et al., 1993a]. Actually, it is not difficult to satisfy the third condition in practice, because in general $\rho(\beta)$ is greater than $p_m$ and $\rho'(\beta)$ is not very large.

Compared to other non-circular formulae [Weinstein, 1980, Smith, 1985a, Gullberg and Zeng, 1992], the above fan-beam formula requires no derivative of a scanning locus with respect to the rotation angle. In X-ray microtomography, a scanning locus contains substantial random interferences introduced by the mechanical motion of the specimen stage [Wang et al., 1993b, Wang et al., 1992c, Lin et al., 1992]. As a result, a precise estimation of the derivative of the scanning locus is difficult. On the other hand, the scanning locus in cone-beam X-ray microtomography can be made to meet our three conditions [Wang et al., 1993b, Wang et al., 1992c, Lin et al., 1992]. With our derivative-free non-circular fan-beam formula, reconstruction will not be affected by the error in estimating the derivative.

### 3.3 Generalized Feldkamp Algorithm

For cone-beam X-ray microtomography, the Feldkamp algorithm is limited by circular scanning, spherical specimen reconstruction and longitudinal image blurring. As part of the development of a cone-beam X-ray microtomographic system at SUNY/Buffalo, the Feldkamp cone-beam algorithm was extended to allow flexible scanning loci, reconstruct spheric, rod-shaped and planar specimens, and facilitate near real-time implementation [Wang et al., 1991c, Wang et al., 1991b, Wang et al., 1994a, Wang et al., 1993b, Wang et al., 1993a, Wang et al., 1992b, Wang et al., 1992a]. In this subsection, the generalized Feldkamp algorithm will be derived via appropriately correcting projection data of a point object.

Systems and variables are illustrated In Figure 1-6. A specimen $s(\vec{x})$ is supported in the cylindrical region $x^2 + y^2 \leq 1$. A scanning locus is described as $\vec{\phi}(\beta) = (\rho(\beta)\cos\beta, \rho(\beta)\sin\beta, z(\beta))^t$, $\rho(\beta) > 1$, $\beta$ is the X-ray source rotation angle around the $z$ axis counterclockwise. Cone-beam projection data $X(p, \zeta, \beta)$, or $X(\vec{\alpha}, \beta)$, are recorded on an imaginary detector plane passing through the $z$ axis and facing the X-ray source, where $p$ and $\zeta$ are horizontal and longitudinal coordinates of the detector plane, and $\vec{\alpha} = (\alpha_x, \alpha_y, \alpha_z)^t$ specifies the direction of an X-ray. Evidently,

$$X(\vec{\alpha}, \beta) = \int_{-\infty}^{\infty} s(\vec{\phi}(\beta) + t\frac{\vec{\alpha}}{\| \vec{\alpha} \|})dt. \tag{8}$$

For convenience, define

$$\tilde{X}(\vec{\alpha}, \beta) \equiv \frac{1}{\| \vec{\alpha} \|} X(\vec{\alpha}, \beta) = \int_{-\infty}^{\infty} s(\vec{\phi}(\beta) + t\vec{\alpha})dt. \tag{9}$$

Let $\delta(\vec{x} - \vec{x}_0)$ model a three-dimensional point object located at $\vec{x}_0$, we have

$$\begin{aligned} \tilde{X}_\delta(\vec{\alpha}, \beta) &= \int_{-\infty}^{\infty} \delta(\vec{\phi}(\beta) + t\vec{\alpha} - \vec{x}_0)dt \\ &= \int_{-\infty}^{\infty} \delta(t\vec{\alpha} - \vec{\alpha}_0)dt, \end{aligned} \tag{10}$$

where $\vec{\alpha}_0 = \vec{x}_0 - \vec{\phi}(\beta)$. Using the matrix $M$ that makes a rotation around the $z$ axis such that the X-ray source position is longitudinally projected on the rotated $y$ axis,

$$\tilde{X}_\delta(\vec{\alpha}, \beta) = \int_{-\infty}^{\infty} \delta(M^{-1}(t\hat{\vec{\alpha}} - \hat{\vec{\alpha}}_0))dt, \tag{11}$$

where $\hat{\vec{\alpha}}$ and $\hat{\vec{\alpha}}_0$ are in rotated coordinates. Since $M$ is unitary and $\hat{\alpha}_y \neq 0$ (the source is outside the reconstruction region),

$$\begin{aligned} \tilde{X}_\delta(\vec{\alpha}, \beta) &= \int_{-\infty}^{\infty} \delta(t\hat{\vec{\alpha}} - \hat{\vec{\alpha}}_0)dt \\ &= \int_{-\infty}^{\infty} \delta(t\hat{\alpha}_x - \hat{\alpha}_{0x})\delta(t\hat{\alpha}_y - \hat{\alpha}_{0y})\delta(t\hat{\alpha}_z - \hat{\alpha}_{0z})dt \\ &= \frac{| \hat{\alpha}_y |}{\hat{\alpha}_{0y}^2}\delta(\hat{\alpha}_x - \frac{\hat{\alpha}_{0x}}{\hat{\alpha}_{0y}}\hat{\alpha}_y)\delta(\hat{\alpha}_z - \frac{\hat{\alpha}_{0z}}{\hat{\alpha}_{0y}}\hat{\alpha}_y). \end{aligned} \tag{12}$$

Therefore,

$$\begin{aligned} X_\delta(p, \zeta, \beta) &= \| \hat{\vec{\alpha}} \| \, \tilde{X}_\delta(\hat{\vec{\alpha}}, \beta) \, |_{\hat{\alpha}_x = p, \hat{\alpha}_y = \rho(\beta), \hat{\alpha}_z = \zeta} \\ &= \left[\frac{\rho^2(\beta)}{\hat{\alpha}_{0y}^2}\right]\left[\frac{\sqrt{\rho^2(\beta) + p^2 + \zeta^2}}{\rho(\beta)}\right]\left[\delta(p - \frac{\hat{\alpha}_{0x}}{\hat{\alpha}_{0y}}\rho(\beta))\delta(\zeta - \frac{\hat{\alpha}_{0z}}{\hat{\alpha}_{0y}}\rho(\beta))\right]. \end{aligned} \tag{13}$$

Geometrically, the first factor magnifies the point object, the second factor is due to the angle between the X-ray ray passing through $\vec{x}_0$ and the normal of the $p$-$\zeta$ plane, and offsets in $\delta$ functions describe the projected point object position in the detector plane.

Longitudinally projecting a scanning locus turn $\vec{\phi}(\beta)$, $\beta \in [c, 2\pi + c)$ where $c$ is a constant, onto the $z = z_0$ plane, we have a scanning locus in the $z = z_0$ plane: $\vec{\phi}^*(\beta) = (\rho(\beta)\cos\beta, \rho(\beta)\sin\beta, z_0)^t$. Clearly, fan-beam data of the $\delta$ function in the $z = z_0$ plane are:

$$X^*_\delta(p, 0, \beta) = \left[\frac{\rho^2(\beta)}{\hat{\alpha}^2_{0y}}\right]\left[\frac{\sqrt{\rho^2(\beta) + p^2}}{\rho(\beta)}\right]\left[\delta(p - \frac{\hat{\alpha}_{0x}}{\hat{\alpha}_{0y}}\rho(\beta))\delta(0)\right]. \tag{14}$$

Comparing Equations (13) with (14), we observe that exact fan-beam data $X^*(p, 0, \beta)$ can be obtained by multiplying the horizontal profile of cone-beam projection $X(p, \zeta, \beta)$, where $\zeta$ is the longitudinal coordinate of the projected $\delta(\vec{x} - \vec{x}_0)$ in the detector plane, with the cosine of the X-ray tilting angle, $\frac{\sqrt{\rho^2(\beta)+p^2}}{\sqrt{\rho^2(\beta)+p^2+\zeta^2}}$.

For either a point object or an arbitrary specimen, applying the derivative-free non-circular fan-beam reconstruction formula with $X^*(p, 0, \beta)$ will produce exact reconstruction on the $z = z_0$ plane. However, $X^*(p, 0, \beta)$ cannot be directly measured in general. Alternatively, we can use the same fan-beam formula with approximate in-plane projection data for approximate reconstruction. Approximating the in-plane fan-beam data using the above cosine correction scheme, we obtain our generalized Feldkamp formula:

$$\begin{aligned} g(x, y, z) \quad &= \quad \frac{1}{2}\int_0^{2\pi} \frac{\rho^2(\beta)}{(\rho(\beta) - s)^2}\int_{-\infty}^{\infty} X(p, \zeta, \beta) r(\frac{\rho(\beta)t}{\rho(\beta) - s} - p) \\ &\qquad \frac{\rho(\beta)}{\sqrt{\rho^2(\beta) + p^2 + \zeta^2}} dp d\beta, \end{aligned} \tag{15}$$

where $\zeta = \frac{\rho(\beta)(z-h(\beta))}{\rho(\beta)-s}$.

Clearly, the essential step in our generalized Feldkamp cone-beam reconstruction is to correct cone-beam projection data so as to achieve exact transaxial reconstruction for any $\delta$ function. Correction is done by multiplying cone-beam data with the cosine of the X-ray tilting angle (Figure 1-7). Consequently, this generalized Feldkamp reconstruction is formulated in two steps: (1) cone-beam to fan-beam data conversion and (2) fan-beam reconstruction. It might appear that $\rho'(\beta)$ and $h'(\beta)$ should have been included in the generalized Feldkamp algorithm, similar to what was done by Gullberg *et al.* [Gullberg et al., 1991], Yan and Leahy [Yan and Leahy, 1992]. However, it is not necessarily so. Actually, certain corrective action has been taken by incorporating functions $\rho(\beta)$ and $h(\beta)$ into the generalized Feldkamp algorithm. These two functions of the rotation angle $\beta$ completely describe the source motion. Absence of the derivative of a scanning locus in the generalized Feldkamp algorithm is advantageous in terms of sensitivity to noise [Wang et al., 1995].

Our derivative-free non-circular fan-beam formula utilizes full-scan data, which consist of two complete projection data sets. Actually, fan-beam reconstruction can also be performed with either half-scan or double full-scan projection data. Accordingly, half-scan and double-helix-scan cone-beam algorithms were formulated [Wang et al., 1994b]. The above discussion with one scanning turn can be extended to half- and double-helix-scan cases, respectively. In the half-scan case, the angular range involved in a transaxial slice reconstruction is substantially reduced. As a result, half-scan cone-beam reconstruction may improve longitudinal/temporal resolution. In the double-helix-scan case, a transaxial slice is reconstructed with cosine-corrected and linearly combined projection data from twins of scanning turns. The double-helix-scan cone-beam reconstruction is exact for a specimen with linear longitudinal variation.

In practice, many specimens are rather plate-shaped, such as typical thick film sections. Tomographic reconstruction of a plate-like specimen is an incomplete data problem. Interestingly, the generalized Feldkamp algorithm can be applied to reconstruct plate-like specimens, as illustrated in Figure 1-8. A circular scanning locus of the X-ray source is made by steering the e-beam on the target window. A plate-like specimen or thick film is placed parallel to the scanning plane. The relative position of the window and the specimen is so arranged that the normal at the center of the scanning circle intersects the center of an area of interest of the specimen. This normal is labeled the principal axis. The locus should be made substantially larger than the area. The detector plane is in parallel behind the specimen, and two-dimensional projection data are recorded for each source position. After each frame of the projection data is mapped onto the imaginary detector plane that faces the X-ray source and contains the principal axis, generalized Feldkamp reconstruction can be done [Wang et al., 1991a, Wang et al., 1992a].

# 4 Between Exact and Approximate Reconstruction

In this section, relationship between exact and approximate cone-beam reconstruction algorithms will be studied. We will not only review the aspects that exact methods can be used for approximate reconstruction, but also describe our results that the generalized Feldkamp algorithm can be extended for exact stereo-imaging and exact image reconstruction.

## 4.1 Exact Reconstruction to Approximate Reconstruction

Clearly, both the Smith algorithm and the Grangeat algorithm can be used for approximate reconstruction if missing data in the Radon space are filled via interpolation and extrapolation. The optimal computational strategy remains an important open question. Smith demonstrated the equivalence between the Feldkamp algorithm and the one he suggested [Smith, 1985b]. Applying the Grangeat method, cone-beam data from a circular orbit for which the Feldkamp algorithm was designed can be converted into data in a torus shaped region in the 3D Radon space. It was established that the exact algorithms are equivalent to the Feldkamp algorithm if the redundancy function is set to 2 over the torus region and 0 outside [Grangeat, 1991, Defrise and Clack, 1994, Kudo and Saito, 1994].

Hu reformulated the Grangeat algorithm for circular scanning [Hu, 1994a]. He showed that an actual image $f(\vec{x})$ can be decomposed into three terms:

$$f(\vec{x}) = f_{M_0}(\vec{x}) + f_{M_1}(\vec{x}) + f_N(\vec{x}), \tag{16}$$

where $f_{M_0}(\vec{x})$ corresponds to the Feldkamp reconstruction, $f_{M_1}(\vec{x})$ represents the information derivable from the circular scan but not utilized in the Feldkamp algorithm, and $f_N(\vec{x})$ is due to the incompleteness of the circular geometry.

The additional term $f_{M_1}(\vec{x})$ appears to contradict the conventional wisdom. An explanation was provided as follows [Hu, 1994b]: "*For the circular orbit, the assumption that the redundancy function equals 2 is correct only for the points inside the torus region. It is incorrect for those points on the boundary of the torus region where the redundancy function equals 1. Consequently, Feldkamp's algorithm correctly represents the contribution of the points inside the torus region, but incorrectly represents the contribution from the points on the boundary of the torus region.*"

### 4.2 Approximate Reconstruction to Exact Reconstruction

In practice, a whole reconstructed volumetric image may not be directly useful, several stereo image pairs are often sufficient for extraction of structural information. Usually, orthogonal projections are preferred for stereo observation. Stereo-imaging may be directly achieved via synthesis of stereograms from cone-beam data. Our studies on cone-beam stereo-imaging led to the rediscovery of the sufficient condition for exact reconstruction.

#### 4.2.1 Stereo-Imaging

Stereograms can be approximately computed directly from cone-beam projections based on our generalized Feldkamp algorithm. A formula for approximate stereogram synthesis was derived [Lin et al., 1993]. For brevity, the approximate formula will not be given here. Instead, a geometrical explanation about its computational structure is offered as follows. First, every two-dimensional frame of cone-beam data are independently filtered as required by the generalized Feldkamp algorithm. Then, the filtered data are integrated with a weighting kernel along a straight line in the detector plane for each source position. The straight line is obtained by projecting a parallel-beam ray of interest onto the corresponding detector plane. Finally, all the integral values are summed up to obtain the two-dimensional parallel-beam projection value along the ray of interest. In the following, we will explain exact stereo-imaging from cone-beam data.

Based on the heuristic derivation of the generalized Feldkamp algorithm given in the preceding section, it can be visualized that *the longitudinal integral of the spatially variant point spread function (PSF) of the generalized Feldkamp reconstruction is a spatially invariant $\delta$ function.* This can be appreciated as follows. First, the cosine correction produces exact fan-beam projection data in any transaxial plane where a point object is located. With the cosine-corrected projection data and appropriate fan-beam reconstruction, the exact in-plane reconstruction can be achieved. Second, reconstruction errors occur out of the plane, where contributions from individual projection profiles cannot be cancelled out due to mis-alignment of the tilted fan-beams. Third, cancellation of "off-focus" errors can be implemented through a longitudinal integral.

Although the generalized Feldkamp algorithm is approximate, using the exactness property of the longitudinal integral of the PSF, it can be proven that *the longitudinal integral of a reconstructed volumetric image is equal to that of the actual image, assuming spheric specimen support and complete detection coverage.* Mathematically, a reconstructed image $g(x, y, z)$ can be expressed in terms of the spatially varying PSF $h(x, y, z)$:

$$g(x, y, z) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(u, v, w) h(x, y, z; u, v, w) du dv dw \tag{17}$$

where $f(x, y, z)$ represents the actual image. Therefore,

$$\begin{aligned}
\int_{-\infty}^{\infty} g(x, y, z) dz &= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(u, v, w) \left[ \int_{-\infty}^{\infty} h(x, y, z; u, v, w) dz \right] du dv dw \\
&= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(u, v, w) \delta(x - u) \delta(y - v) du dv dw \\
&= \int_{-\infty}^{\infty} f(x, y, w) dw
\end{aligned}$$

$$= \int_{-\infty}^{\infty} f(x, y, z) dz \tag{18}$$

The same fact was previously established via lengthy derivation [Feldkamp et al., 1984, Wang et al., 1992b]. Note that the longitudinal integral is the two-dimensional parallel-beam projection along the longitudinal axis.

With arguments similar to those used in derivation of the generalized Feldkamp algorithm in the reconstruction system $x$-$y$-$z$, cone-beam reconstruction can also be achieved via correcting cone-beam data to fan-beam data of a transaxial plane in a rotated reconstruction system $x'$-$y'$-$z'$ under the condition that part of the projected scanning locus allows exact fan-beam reconstruction of the projected specimen support, the projection direction being defined by the normal of the $z'$ axis (Figure 1-9). In this setting, it can be proven in the same way that the integral of the generalized Feldkamp reconstruction along the $z'$ axis is exact. That is, an exact two-dimensional parallel projection can be synthesized along the tilted longitudinal axis [Wang and Cheng, 1995a, Wang and Cheng, 1995b]. This finding updated our approximate stereo-imaging formula [Lin et al., 1993].

Elliptical or polygonal scanning turns, which are typical in our applications, essentially remain the shape after projection. In other words, scanning turns of both types will still satisfy the regular conditions in the rotated reconstruction system when projected onto the $x'$-$y'$ plane. Hence, the generalized Feldkamp algorithm can be applied to reconstruct an image in the system $x'$-$y'$-$z'$ if cone-beam data are corrected onto the new imaginary detector plane passing through the $z'$ axis. If a projected scanning locus does not satisfy the regular conditions, Feldkamp-type reconstruction can still be performed using an appropriate fan-beam reconstruction formula as long as the projected scanning locus meets the condition for exact fan-beam reconstruction. Note that exact two-dimensional parallel-beam projections may be directly reconstructed from longitudinally integrated cosine-corrected cone-beam data.

### 4.2.2 Sufficient Condition: Revisited

If a sufficient amount of exact two-dimensional parallel projection data is available, exact three-dimensional image reconstruction can be performed. Therefore, a new sufficient condition for exact cone-beam reconstruction [Wang and Cheng, 1995b] can be stated as: *If for every projection direction, the projected scanning locus is complete for exact reconstruction on the projected object support.* Our sufficient condition is equivalent to the conventional sufficient condition. If our sufficient condition is satisfied, then for any projection direction the projected scanning locus is complete, and there exists the family of all the planes parallel to the projection direction and containing at least one source position. That is, the traditional sufficient condition is also satisfied. If our sufficient condition is not satisfied, there is a projection direction along which the projected scanning locus is incomplete, and there is at least one plane that contains no source points. This plane corresponds to a line that intersects the projected specimen support but meets no projected source points. That is, the traditional sufficient condition is violated. Actually, the requirement of "*every projection direction*" in our sufficient condition may be relaxed to a set of directional vectors spanning a semi-circle.

Our finding is a bridge from approximate to exact cone-beam reconstruction. In other words, with an incomplete scanning locus, only part of exact two-dimensional parallel-beam projections can be synthesized; with a complete scanning locus, all of exact two-dimensional parallel-beam projections can be computed, hence exact three-dimensional reconstruction can be done.

## 5 Discussion and Conclusion

In our applications, both exact and approximate algorithms are useful. Although exact reconstruction is ideal, approximation cannot be avoided given the hardware limitations. Even if both exact and approximate reconstruction algorithms are applicable, a choice among them can only be made case by case. Detailed comparison is beyond the scope of this text. As an example, Figure 1-10 gives typical slices of the three-dimensional Shepp and Logan phantom, corresponding slices reconstructed using the Feldkamp algorithm with two parallel scanning circles, the generalized Feldkamp algorithm with two helical scanning turns, and the direct three-dimensional Fourier algorithm with two orthogonal scanning circles, respectively. As illustrated in Figure 1-10, compared to exact reconstruction, approximate reconstruction may produce better image sharpness but at the same time may suffer from less image uniformity.

Cone-beam X-ray microtomography is still an active area. Promising directions include refinement of exact and approximate algorithms, extension of cone-beam theory and techniques to address incomplete detection coverage.

In the context of cone-beam X-ray microtomography, we have discussed exact and approximate cone-beam reconstruction as well as their relationships. Especially, we have described approximate and exact methods for orthogonal stereo-imaging from cone-beam data, and identified a link from approximate reconstruction to exact reconstruction, which led to rediscovery of the sufficient condition for exact cone-beam reconstruction.

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