Cone-beam image reconstruction by moving frames

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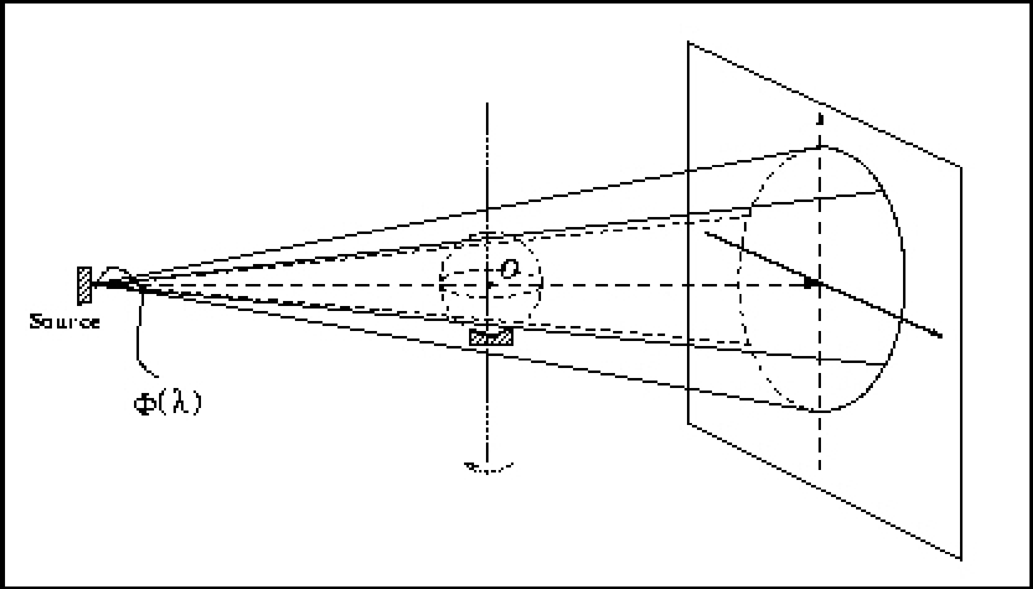
Cone-beam Imaging Apparatus

- Radiation source and area detector
- Source-detector rotate around the object
- Collect integrals of density along straight lines when source travels along a 3D curve



The problem

- Cone-beam reconstruction:

  to recover a 3D density function from its

  integrals along a set of lines emitting from

  a 3D curve

- Among the first problems in integral geometry

Difficulties

- Inverse calculation deals with curved spaces
- Algorithmic implementation encounters:

  Large dataset

  sophisticated geometric mapping

  expensive data interpolations

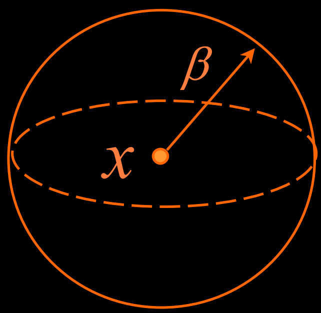
Radon's formula

- Radon transform: integral over planes

$$Rf(l,\beta) := \int_{x\in\{x|x\cdot\beta=l\}} f(x)dx$$

- 3D Radon inverse (1917):

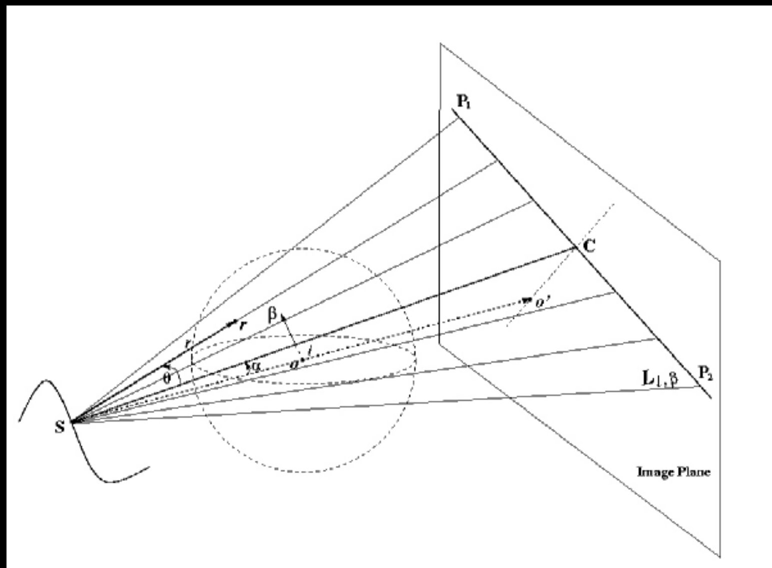
$$f(x) = -\frac{1}{8\pi}\int_{S^2} \left.\frac{\partial^2 Rf(l,\beta)}{\partial l^2}\right|_{l=x\cdot\beta} d\beta$$



Backprojection sphere

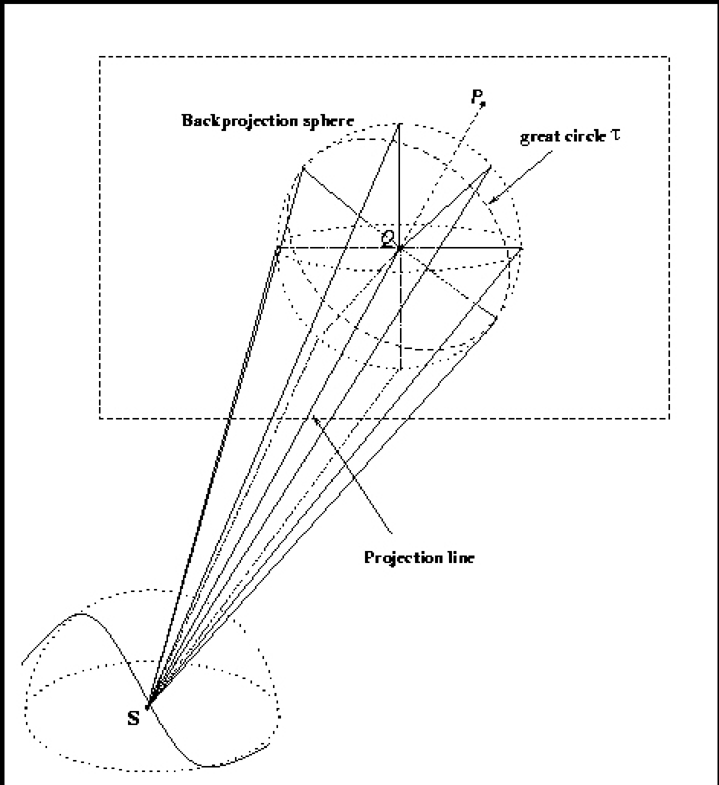
Grangeat's "Fundamental Relation"

- Link cone-beam data to the first-order radial derivative of the Radon transform (1991)

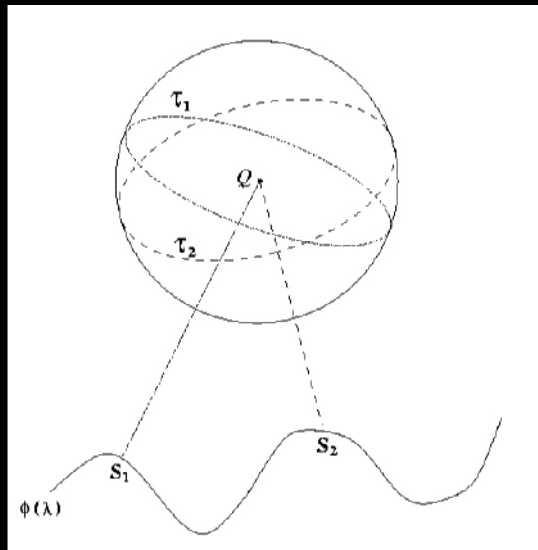


Geometric constraints: within and across projections

- 3D to 2D reduction



- Great circle under a rigid rotation



X. Yang: *Geometry of Cone-beam Reconstruction*, MIT Ph.D. Thesis, 2002

Formulae by Yang and Katsevich

- Yang (2002)

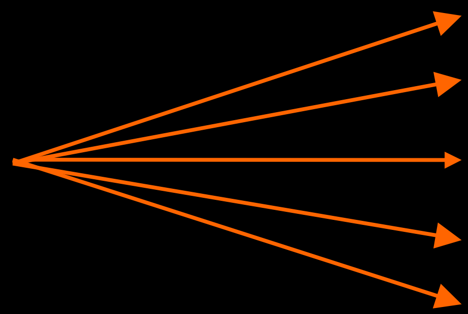
$$f(x) = -\frac{1}{8\pi}\overset{\text{along curve}}{\int_{\Lambda}} \left\{ \overset{\text{within projection}}{\int_{\substack{\beta\in\{x-\Phi(\lambda)\}^{\perp} \\ \beta\in S^2}}} \underbrace{R''}_{\text{across projection}} f(\Phi(\lambda)\cdot\beta,\beta)\frac{|\Phi'(\lambda)\cdot\beta|}{M(\lambda,\beta)}\, d\beta \right\} d\lambda$$

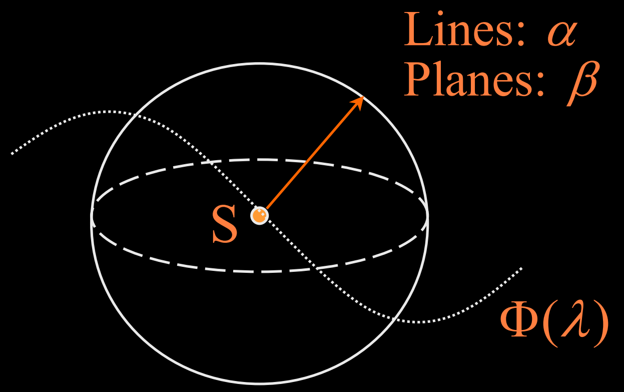
- Katsevich (2003)

$$f(x) = -\frac{1}{8\pi}\underset{\text{along curve}}{\int_{\Lambda}} \frac{\sum c_m(\lambda,x)}{|x-\Phi(\lambda)|}\underset{\text{within projection}}{\int_{0}^{2\pi}} \underbrace{\frac{\partial}{\partial\lambda'}}_{\text{across projection}} Df(\Phi(\lambda'),\cos\gamma\,\alpha(\lambda,x)+\sin\gamma\,\alpha^{\perp}(\lambda,x,\theta_m))\Big|_{\lambda'=\lambda}\frac{d\gamma}{\sin\gamma}\, d\lambda$$

Forward projection and spherical space

- Forward projection maps $\mathbb{E}^3$ to $\mathbb{P}^2$

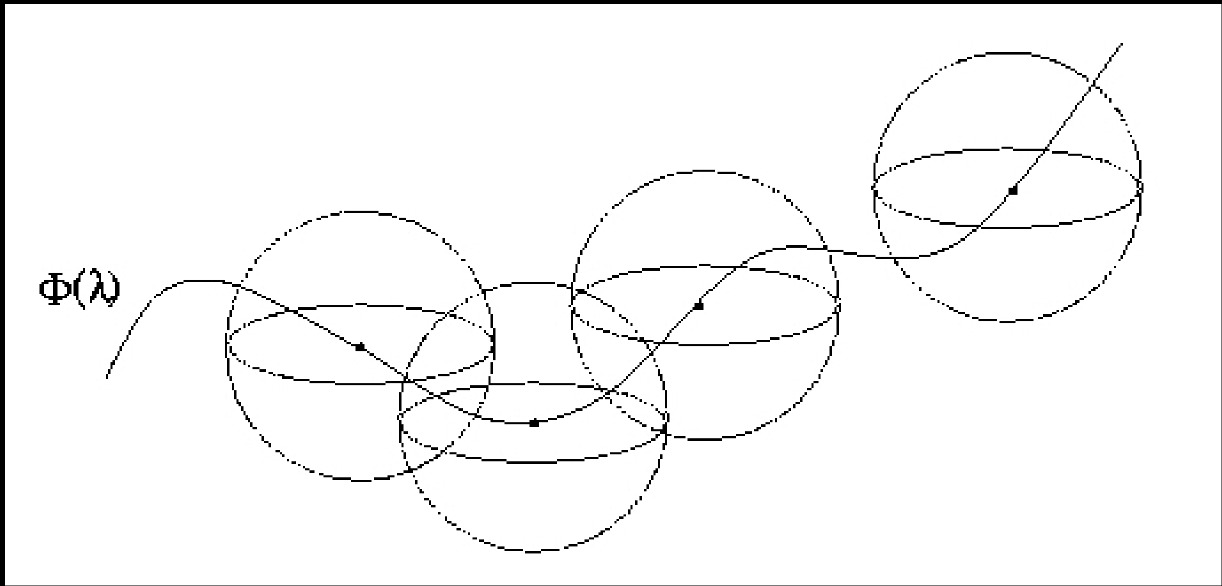




$\Phi(\lambda)$: source orbit

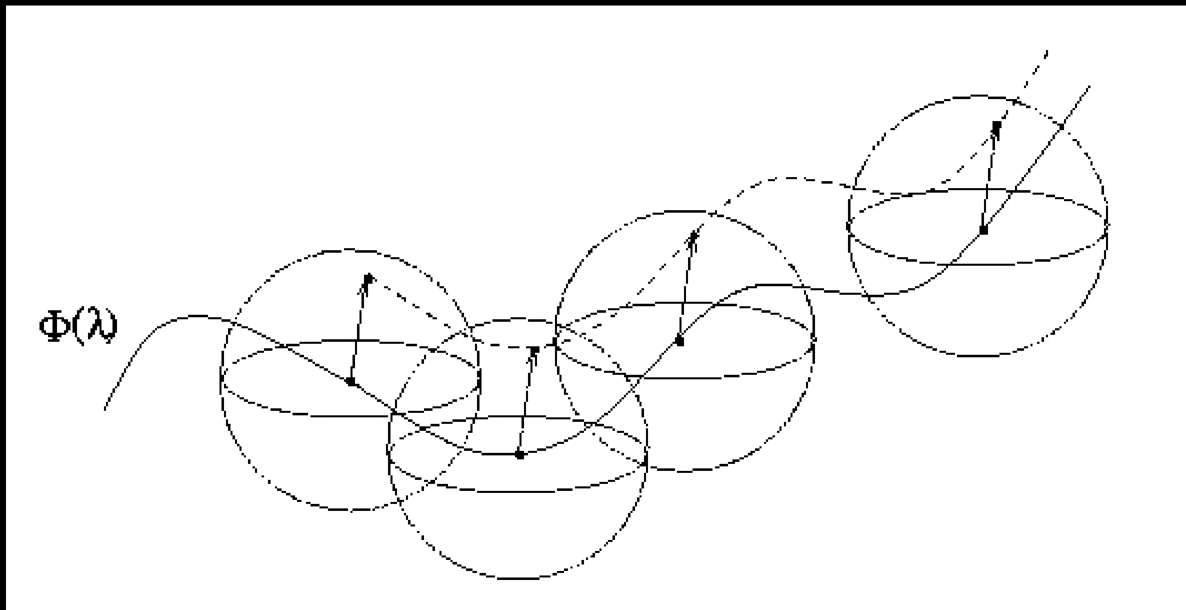
Fiber bundle structure

- Attach to each source point a unit sphere which represents a local fiber
- Fiber bundle: the union of all the spheres



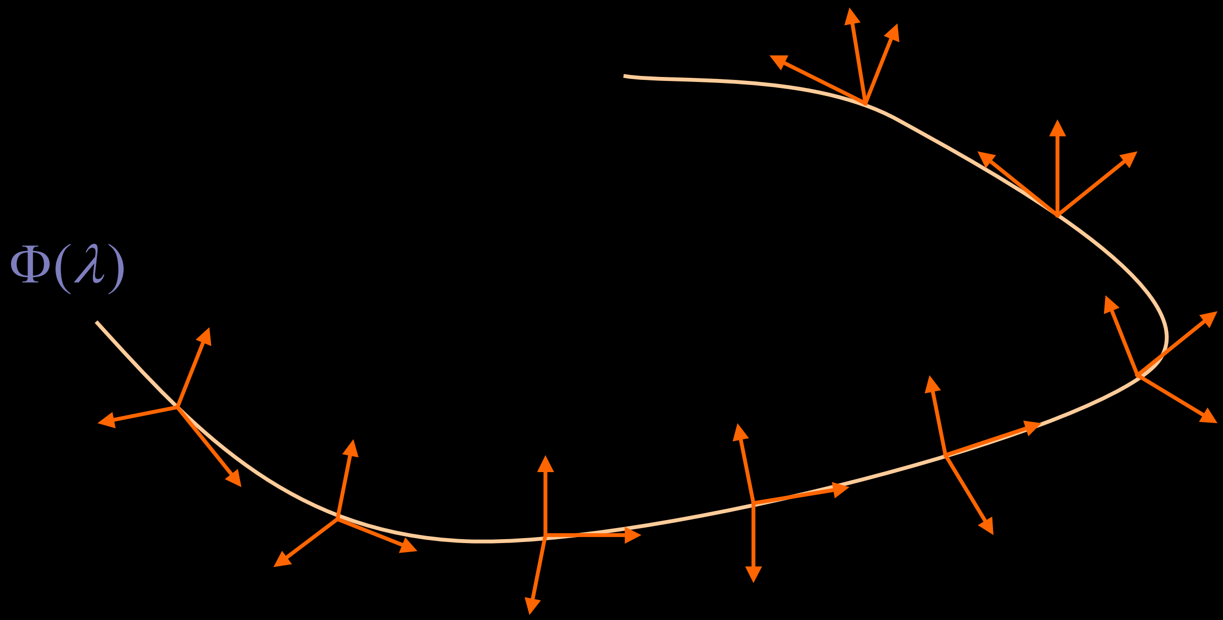
Calculations within and across projections

- Within projection: calculations within each fiber require only local coordinates
- Across projections: differentiation along curves on the fiber bundle requires global coordinates



Euclidean moving frames

- At each source point attach an orthonormal Euclidean basis, i.e.,

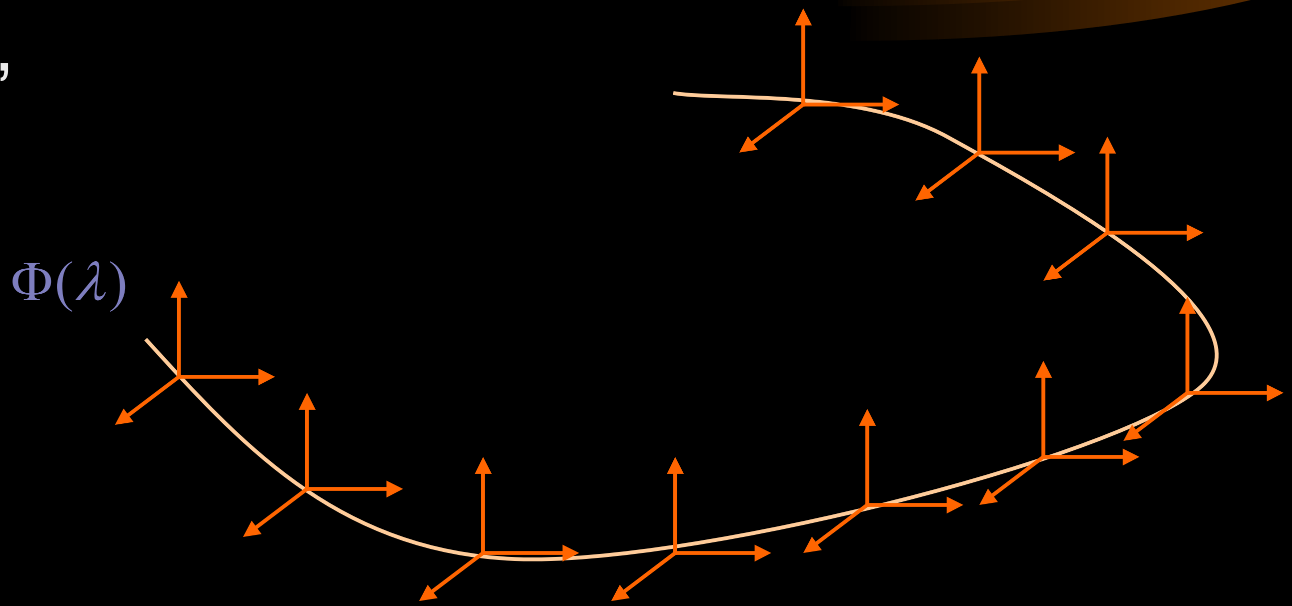


Euclidean moving frames (Cont'd)

- Allows easy exchange between local and global coordinates of points, lines and planes
- Allows treating the non-Euclidean space, "*locally*", as an Euclidean space equipped with Euclidean-like coordinates

Selection of moving frame basis

- Many choices in selecting moving frame bases, i.e.,



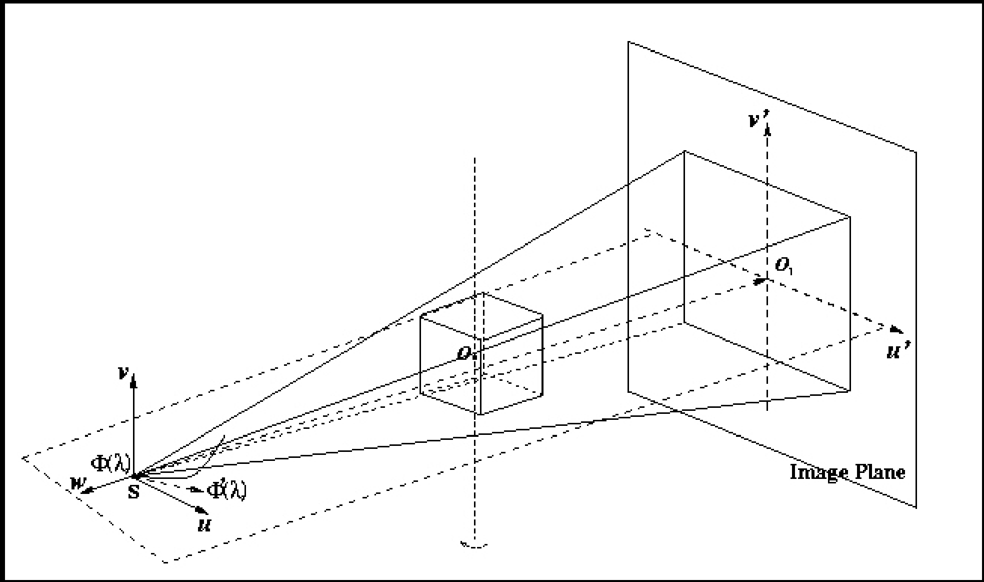
- Main considerations:

  simplify coordinate computation, ease system alignment

Moving frame basis (I)

- Under cylindrical symmetry:

$$\begin{cases} u = \dfrac{\left(-\phi_2(\lambda), \phi_1(\lambda), 0\right)}{\sqrt{\phi_1^2(\lambda) + \phi_2^2(\lambda)}} \\ v = (0, 0, 1) \\ w = \dfrac{\left(\phi_1(\lambda), \phi_2(\lambda), 0\right)}{\sqrt{\phi_1^2(\lambda) + \phi_2^2(\lambda)}} \end{cases}$$

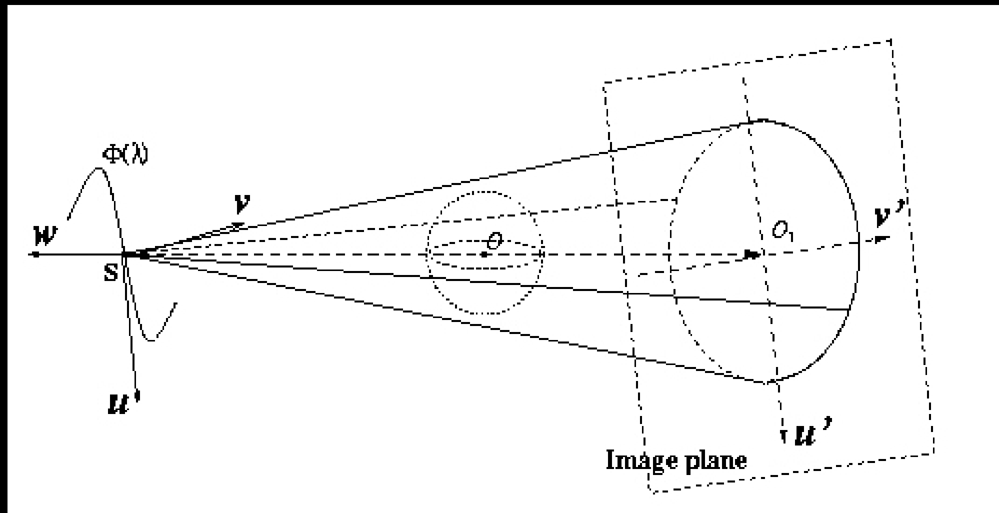


Good selection of moving frames simplifies the geometric computation as well as system alignment.

Moving frame basis (II)

- Under spherical symmetry:

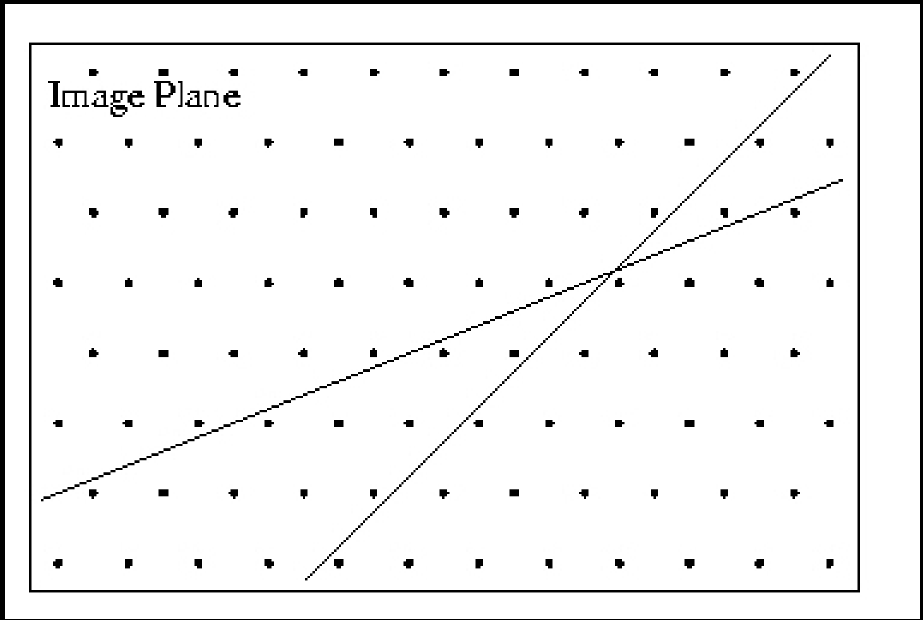
$$\begin{cases} w = \dfrac{\Phi(\lambda)}{|\Phi(\lambda)|} \\[2ex] v = \dfrac{w \times \Phi'(\lambda)}{|w \times \Phi'(\lambda)|} \\[2ex] u = v \times w \end{cases}$$



There is an alignment step in system design to align the axes of the detectors to two of the axes of the moving frames.

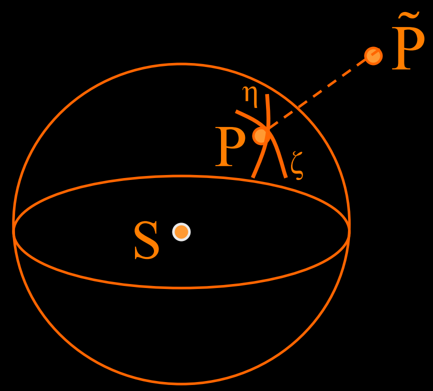
Integral within projection

- Integration in the fiber space
- Local coordinate and discretization
- Irregularity in sampling



Exterior differentiation: differentiation across projections

- Two fibers are disjoint excepts at a zero measure set
- Differentiation across fibers can't be replaced by differentiation within the closed fiber



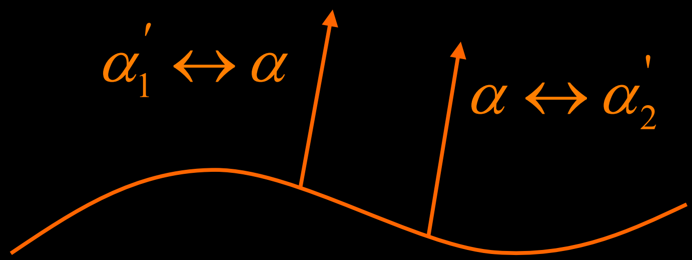
Exterior derivative (Cont'd)

- Parallel lines: having the same α
- The rotational matrix is made up of the three orthonormal basis vectors:

$$R(\lambda) = \left(u(\lambda), v(\lambda), w(\lambda)\right)$$

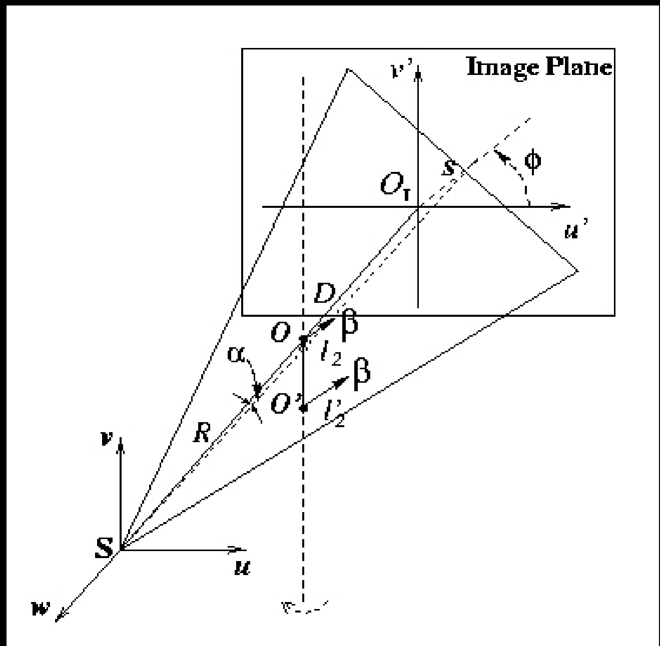
- Local→global→local coordinate transforms:

$$\alpha = R(\lambda_1)\alpha_1'$$
$$\alpha_2' = R(\lambda_2)^{-1}\alpha$$



Exterior derivative (Cont'd)

- Parallel planes: having the same β
- Local→global→local coordinate transforms
- Local coordinate of the intersection line (s, ϕ)



$$\begin{cases} s = D\dfrac{l_2}{\sqrt{R^2 - l_2^2}} \\ \phi = \arctan\left(\dfrac{\beta \cdot v}{\beta \cdot u}\right) \end{cases}$$

Summary

- New geometric representations
- A new "method of moving frames" applied to cone-beam reconstruction (a new computational framework)
- 3D discretization of the curved transform space, applicable to all cone-beam geometries
- Methods to compute the exterior derivatives
- Further study on sampling/interpolation schemes to improve algorithm efficiency