Filtered backprojection formula for exact image reconstruction from cone-beam data along a general scanning curve

Yangbo Ye[a] and Ge Wang[b]
Departments of Radiology and Mathematics, The University of Iowa, 14 MacLean Hall, Iowa City, Iowa 52242-1419

(Received 1 September 2004; revised 8 October 2004; accepted for publication 13 October 2004; published 15 December 2004)

Recently, Katsevich proved a filtered backprojection formula for exact image reconstruction from cone-beam data along a helical scanning locus, which is an important breakthrough since 1991 when the spiral cone-beam scanning mode was proposed. In this paper, we prove a generalized Katsevich's formula for exact image reconstruction from cone-beam data collected along a rather flexible curve. We will also give a general condition on filtering directions. Based on this condition, we suggest a natural choice of filtering directions, which is more convenient than Katsevich's choice and can be applied to general scanning curves. In the derivation, we use analytical techniques instead of geometric arguments. As a result, we do not need the uniqueness of the PI lines. In fact, our formula can be used to reconstruct images on any chord as long as a scanning curve runs from one endpoint of the chord to the other endpoint. This can be considered as a generalization of Orlov's classical theorem. Specifically, our formula can be applied to (i) nonstandard spirals of variable radii and pitches (with PI- or n-PI-windows), and (ii) saddlelike curves. [DOI: 10.1118/1.1828673]

Key words: cone-beam CT, filtered backprojection, image reconstruction, Katsevich formula, general scanning curve

I. INTRODUCTION

The recent development of medical computed tomography (CT) techniques, such as bolus-chasing angiography[1] and electron-beam micro-CT (EBMCT),[2] requires more flexible scanning curves for cone-beam CT. In 1991, Wang *et al.*[3,4] proposed a spiral cone-beam algorithm to solve the long object problem in the case of standard and nonstandard spiral cone-beam scans. However, their algorithms are of the Feldkamp-type and only produce approximate results. Since then, much progress has been made in the area of spiral cone-beam CT. During the past two years, Katsevich developed an exact filtered backprojection formula for standard helical cone-beam scans,[5–7] which is a quantum leap relative to the earlier algorithms.[8–10] Recently, Zou and Pan[11,12] proved an exact backprojected filtration formula as a counterpart of the Katsevich formula. Then, Katsevich *et al.*[13] as well as Zou, Pan, Xia, and Wang[14] extended these formulas to spirals with variable pitches.

A generalization of Zou and Pan's exact reconstruction formula to a smooth scanning curve was proved by Ye *et al.*[15] This generalized formula is still in the backprojected filtration format, and can be applied to nonstandard spirals, saddle-like curves, n-PI-window scans, and other cases. As such, it can be regarded as a generalization of the Orlov theorem[16] to the cone-beam scanning geometry. Another proof of our generalized formula was also given by Zhao *et al.*,[17] which is in the Tuy framework.

The main purpose of this paper is to provide a proof of the Katsevich formula for a general cone-beam scanning geometry, and to give explicitly a filtering direction applicable to arbitrary scanning curves, together with a condition for admissible filtering directions. In the next section, we introduce necessary notations. In the third section, we prove a generalized Katsevich formula, avoiding the use of complicated geometric arguments Katsevich,[5–7] and Zou and Pan[11,12] extensively used. By doing so, we obtain a filtered backprojection formula for exact image reconstruction from cone-beam data along a general scanning curve. Our formula can also be applied to nonstandard spirals (with PI- or n-PI-windows), saddlelike curves, and so on, and may be considered as another generalization of the Orlov theorem.[16] In the last section, we discuss relevant issues and conclude the paper.

II. NOTATIONS AND THE MAIN THEOREM

The main setting for our formula is a general smooth curve $\mathbf{y}(s)$ for $s_b \leq s \leq s_t$, and a chord ℓ connecting the endpoints $\mathbf{y}(s_b)$ and $\mathbf{y}(s_t)$ of the curve. Let $\mathbf{x}$ be an interior point on ℓ. Clearly, this setting covers standard or nonstandard spirals with PI- or n-PI lines, standard or nonstandard saddle curves, and many other cases.

Following Katsevich's convention,[7] denote by $I_{\mathrm{PI}}(\mathbf{x}) = [s_b, s_t]$ the parametric interval, and by

$$D_f(\mathbf{y},\boldsymbol{\Theta}) = \int_0^\infty f(\mathbf{y} + t\boldsymbol{\Theta})dt, \quad \boldsymbol{\Theta} \in S^2, \tag{2.1}$$

the cone-beam data, where S^2 is the unit sphere. Let

$$\boldsymbol{\beta}(s,\mathbf{x}) = \frac{\mathbf{x} - \mathbf{y}(s)}{|\mathbf{x} - \mathbf{y}(s)|}, \quad s \in I_{\mathrm{PI}}(\mathbf{x}), \tag{2.2}$$

be the unit vector pointing toward $\mathbf{x}$ from $\mathbf{y}(s)$, and $\mathbf{e}(s,\mathbf{x})$ a unit vector perpendicular to $\boldsymbol{\beta}(s,\mathbf{x})$. In the work by

Katsevich[5–7] $\mathbf{e}(s,\mathbf{x})$ is selected in an explicit way. However, in the following we first request that $\mathbf{e}(s,\mathbf{x})$ be any unit vector perpendicular to $\boldsymbol{\beta}(s,\mathbf{x})$. Then, we give an admissible condition for $\mathbf{e}(s,\mathbf{x})$ and make a natural choice of $\mathbf{e}(s,\mathbf{x})$. It will become clear later on that Katsevich's $\mathbf{e}(s,\mathbf{x})$ in the standard helical scanning case is a different choice but it indeed satisfies our admissible condition. Hence, a filtering direction can be expressed by

$$\boldsymbol{\Theta}(s,\mathbf{x},\gamma) = \cos\gamma\boldsymbol{\beta}(s,\mathbf{x}) + \sin\gamma\,\mathbf{e}(s,\mathbf{x}). \tag{2.3}$$

Theorem 2.1. Let $f(\mathbf{x})$ be a function of compact support whose fifth partial derivatives are absolutely integrable in $\mathbf{R}^3$. Let $\mathbf{e}(s,\mathbf{x})$ be a unit vector satisfying condition (3.25) for any $s \in (s_b, s_t)$ and $\mathbf{x} \in \mathbf{R}^3$, or simply set $\mathbf{e}(s,\mathbf{x})$ to be a unit vector in the plane determined by ℓ and $\mathbf{x}-\mathbf{y}(s)$ with $\mathbf{e}(s,\mathbf{x})\cdot[\mathbf{y}(s_t)-\mathbf{y}(s_b)]>0$. Then

$$f(\mathbf{x}) = -\frac{1}{2\pi^2}\int_{I_{\mathrm{PI}}(\mathbf{x})}\frac{ds}{|\mathbf{x}-\mathbf{y}(s)|}\mathrm{PV}\int_0^{2\pi}\frac{\partial}{\partial q} \times D_f[\mathbf{y}(q),\boldsymbol{\Theta}(s,\mathbf{x},\gamma)]|_{q=s}\frac{d\gamma}{\sin\gamma}. \tag{2.4}$$

See Sec. IV for a detailed explanation of condition (3.25). The choice of $\mathbf{e}(s,\mathbf{x})$ in Theorem 2.1 means that the filtering plane as defined by Eq. (2.3) contains both ℓ and $\mathbf{x}-\mathbf{y}(s)$ for any source point $\mathbf{y}(s)$.

The assumption of integrable fifth partial derivatives is much weaker than the usual condition requesting $f(\mathbf{x})$ being smooth, and yet strong enough for the convergence of integrals and interchange of integration orders in our proof below. As a result, we can avoid using distribution in our proof. Note that in most medical and industrial applications, an object function $f(\mathbf{x})$ is usually not continuous, which is a main reason for artifacts in CT images. Our ultimate goal is to develop an exact reconstruction formula for discontinuous object functions. Our assumption on $f(\mathbf{x})$ in Theorem 2.1 will become instrumental in our subsequent pursuit towards that goal.

Equation (2.4) was proved by Katsevich[5–7] in the standard helical scanning case with his choice of the filtering direction $\boldsymbol{\Theta}(s,\mathbf{x},\gamma)$. Our Theorem 2.1 is valid for a much more general class of scanning loci and filtering directions or planes.

III. ANALYTICAL PROOF

Denote the right side of Eq. (2.4) by the right-hand side (RHS). We want to show that it equals $f(\mathbf{x})$. As in Ye *et al.*,[15] let $F(\boldsymbol{\nu})$ be the Fourier transform of $f(\mathbf{x})$

$$F(\boldsymbol{\nu}) = \int_{\mathbf{R}^3} f(\mathbf{x})e^{-2\pi i\boldsymbol{\nu}\cdot\mathbf{x}}d\mathbf{x}. \tag{3.1}$$

Then,

$$\begin{aligned}\left(\frac{\partial}{\partial q}D_f[\mathbf{y}(q),\boldsymbol{\Theta}(s,\mathbf{x},\gamma)]\right)\bigg|_{q=s} &= \int_0^\infty \frac{\partial}{\partial q}f[\mathbf{y}(q)+t\boldsymbol{\Theta}(s,\mathbf{x},\gamma)]\bigg|_{q=s}dt \\ &= \int_0^\infty \frac{\partial}{\partial q}\int_{\mathbf{R}^3}F(\boldsymbol{\nu})e^{2\pi i\boldsymbol{\nu}\cdot[\mathbf{y}(q)+t\boldsymbol{\Theta}(s,\mathbf{x},\gamma)]}d\boldsymbol{\nu}\bigg|_{q=s}dt\end{aligned} \tag{3.2}$$

by Fourier's inversion formula. We take the derivative under the inner integral to obtain

$$\left(\frac{\partial}{\partial q}D_f[\mathbf{y}(q),\boldsymbol{\Theta}(s,\mathbf{x},\gamma)]\right)\bigg|_{q=s} = 2\pi i\int_0^\infty dt\int_{\mathbf{R}^3}\left(\boldsymbol{\nu}\cdot\frac{d\mathbf{y}(s)}{ds}\right)F(\boldsymbol{\nu})e^{2\pi i\boldsymbol{\nu}\cdot[\mathbf{y}(s)+t\boldsymbol{\Theta}(s,\mathbf{x},\gamma)]}d\boldsymbol{\nu}. \tag{3.3}$$

Therefore,

$$\begin{aligned}\mathrm{RHS} = \frac{1}{\pi i}\int_{I_{\mathrm{PI}}(\mathbf{x})}\frac{ds}{|\mathbf{x}-\mathbf{y}(s)|}\left(\mathrm{PV}\int_0^{2\pi}\right)\frac{d\gamma}{\sin\gamma}\int_0^\infty dt \\ \times\int_{\mathbf{R}^3}\left(\boldsymbol{\nu}\cdot\frac{d\mathbf{y}(s)}{ds}\right)F(\boldsymbol{\nu})e^{2\pi i\boldsymbol{\nu}\cdot\mathbf{y}(s)} \\ \times e^{2\pi i\boldsymbol{\nu}\cdot[t\cos\gamma\,\boldsymbol{\Theta}(s,\mathbf{x},\gamma)+t\sin\gamma\,\mathbf{e}(s,\mathbf{x})]}d\boldsymbol{\nu}.\end{aligned}$$

If we write

$$\left(\mathrm{PV}\int_0^{2\pi}\right)\frac{d\gamma}{\sin\gamma}\int_0^\infty dt = \int_0^\infty t\,dt\left(\mathrm{PV}\int_0^{2\pi}\right)\frac{d\gamma}{t\sin\gamma},$$

we can use Cartesian coordinates to obtain

$$\begin{aligned}\mathrm{RHS} &= \frac{1}{\pi i}\int_{I_{\mathrm{PI}}(\mathbf{x})}\frac{ds}{|\mathbf{x}-\mathbf{y}(s)|}\int_{\mathbf{R}}du\left(\mathrm{PV}\int_{\mathbf{R}}\right)\frac{dw}{w} \\ &\quad\times\int_{\mathbf{R}^3}\left(\boldsymbol{\nu}\cdot\frac{d\mathbf{y}(s)}{ds}\right)F(\boldsymbol{\nu})e^{2\pi i\boldsymbol{\nu}\cdot\mathbf{y}(s)}e^{2\pi i\boldsymbol{\nu}\cdot[u\boldsymbol{\beta}(s,\mathbf{x})+w\mathbf{e}(s,\mathbf{x})]}d\boldsymbol{\nu} \\ &= \frac{1}{\pi i}\int_{I_{\mathrm{PI}}(\mathbf{x})}ds\int_{\mathbf{R}}du\left(\mathrm{PV}\int_{\mathbf{R}}\right)\frac{dw}{w} \\ &\quad\times\int_{\mathbf{R}^3}\left(\boldsymbol{\nu}\cdot\frac{d\mathbf{y}(s)}{ds}\right)F(\boldsymbol{\nu})e^{2\pi i\boldsymbol{\nu}\cdot\mathbf{y}(s)} \\ &\quad\times e^{2\pi i\boldsymbol{\nu}\cdot\{u[\mathbf{x}-\mathbf{y}(s)]+w|\mathbf{x}-\mathbf{y}(s)|\mathbf{e}(s,\mathbf{x})\}}d\boldsymbol{\nu}.\end{aligned} \tag{3.4}$$

Here we have changed variables from u, w to $u|\mathbf{x}-\mathbf{y}(s)|$ and $w|\mathbf{x}-\mathbf{y}(s)|$ and used the fact that $\boldsymbol{\beta}(s,\mathbf{x})=[\mathbf{x}-\mathbf{y}(s)]/|\mathbf{x}-\mathbf{y}(s)|$.

Recall that $f(\mathbf{x})$ is a function of compact support whose fifth partial derivatives are integrable. Hence, we can apply integration by parts five times to Eq. (3.1) by integrating the exponential function and differentiating $f(\mathbf{x})$. This shows that the absolute value of $F(\boldsymbol{\nu})$ is bounded by $c|\boldsymbol{\nu}|^{-5}$ for a positive constant c. In other words, $|F(\boldsymbol{\nu})|=O(|\boldsymbol{\nu}|^{-5})$ when $|\boldsymbol{\nu}|$ is sufficiently large. Consequently, $[\boldsymbol{\nu}\cdot d\mathbf{y}(s)/ds]F(\boldsymbol{\nu})=O(|\boldsymbol{\nu}|^{-4})$, and the innermost integral on the right side of Eq. (3.4) is absolutely integrable.

Since $f(\mathbf{x})$ is of compact support, we can differentiate Eq. (3.1) under the integral sign. This shows that $F(\boldsymbol{\nu})$ is a smooth function. By the Fourier inversion formula

$$f(\mathbf{x}) = \int_{\mathbf{R}^3} F(\boldsymbol{\nu})e^{2\pi i\boldsymbol{\nu}\cdot\mathbf{x}}d\boldsymbol{\nu},$$

we have

$$\left(c_1\frac{\partial}{\partial x_1} + c_2\frac{\partial}{\partial x_2} + c_3\frac{\partial}{\partial x_3}\right)f(\mathbf{x})$$
$$= 2\pi i\int_{\mathbf{R}^3}(c_1\nu_1 + c_2\nu_2 + c_3\nu_3)F(\boldsymbol{\nu})e^{2\pi i\boldsymbol{\nu}\cdot\mathbf{x}}d\boldsymbol{\nu}, \tag{3.5}$$

where c_1, c_2, c_3 are constants, and $\boldsymbol{\nu}=(\nu_1,\nu_2,\nu_3)$. Here the differentiation under integral sign is legitimate because $|(c_1\nu_1+c_2\nu_2+c_3\nu_3)F(\boldsymbol{\nu})|=O(|\boldsymbol{\nu}|^{-4})$ and the integral on the right side of Eq. (3.5) is absolutely integrable. Consequently,

$$\int_{\mathbf{R}^3}\left(\boldsymbol{\nu}\cdot\frac{d\mathbf{y}(s)}{ds}\right)F(\boldsymbol{\nu})e^{2\pi i\boldsymbol{\nu}\cdot\mathbf{z}}d\boldsymbol{\nu} = \frac{1}{2\pi i}\left(\frac{d\mathbf{y}(s)}{ds}\cdot\frac{\partial}{\partial\mathbf{z}}\right)f(\mathbf{z})$$

is a function of compact support, and the innermost integral on the right side of Eq. (3.4) as a function of u and w is of compact support. Note here that because of our setting, $\mathbf{x}$ is not on $\mathbf{y}(s)$, and $c_4<|\mathbf{x}-\mathbf{y}(s)|<c_5$ for some positive constants c_4 and c_5.

This proves the convergence of integrals with respect to u and w on the right side of Eq. (3.4). We cannot, however, interchange the order of these integrals with that of $\boldsymbol{\nu}$ directly. Hence, we use a convergence factor method as follows. Let $g(t)=e^{-t^2/2}$ and $\varepsilon>0$. Then, the three inner integrals on the right side of Eq. (3.4) equals

$$\lim_{\varepsilon\to 0^+}\int_{\mathbf{R}} g(\varepsilon u)du\left(\mathrm{PV}\int_{\mathbf{R}}\right)g(\varepsilon w)\frac{dw}{w}$$
$$\times\int_{\mathbf{R}^3}\left(\boldsymbol{\nu}\cdot\frac{d\mathbf{y}(s)}{ds}\right)F(\boldsymbol{\nu})e^{2\pi i\boldsymbol{\nu}\cdot\mathbf{y}(s)}$$
$$\times e^{2\pi i\boldsymbol{\nu}\cdot\{u[\mathbf{x}-\mathbf{y}(s)]+w|\mathbf{x}-\mathbf{y}(s)|\mathbf{e}(s,\mathbf{x})\}}d\boldsymbol{\nu},$$

because the innermost integral as a function of u and w is of essentially compact support. Now, we can interchange the order of the integrals to transform the right side of Eq. (3.4) into that

$$\mathrm{RHS} = \frac{1}{\pi i}\lim_{\varepsilon\to 0^+}\int_{I_{\mathrm{PI}}(\mathbf{x})}ds\int_{\mathbf{R}^3}\left(\boldsymbol{\nu}\cdot\frac{d\mathbf{y}(s)}{ds}\right)F(\boldsymbol{\nu})e^{2\pi i\boldsymbol{\nu}\cdot\mathbf{y}(s)}d\boldsymbol{\nu}$$
$$\times\int_{\mathbf{R}} g(\varepsilon u)e^{2\pi i\boldsymbol{\nu}\cdot[\mathbf{x}-\mathbf{y}(s)]u}du\left(\mathrm{PV}\int_{\mathbf{R}}\right)$$
$$\times g(\varepsilon w)e^{2\pi i|\mathbf{x}-\mathbf{y}(s)|[\boldsymbol{\nu}\cdot\mathbf{e}(s,\mathbf{x})]w}\frac{dw}{w}. \tag{3.6}$$

This interchange of integration order is legitimate because the integrals are dominated by $|[\boldsymbol{\nu}\cdot d\mathbf{y}(s)/ds]F(\boldsymbol{\nu})|$, $g(\varepsilon u)$ and $g(\varepsilon w)$, respectively, after rewriting the PV integral as an ordinary integral (see Eq. (3.8) below).

The integral with respect to u in Eq. (3.6) equals

$$\frac{1}{\varepsilon}\int_{\mathbf{R}} g(u)e^{2\pi i\boldsymbol{\nu}\cdot[\mathbf{x}-\mathbf{y}(s)]u/\varepsilon}du = \frac{\sqrt{2\pi}}{\varepsilon}e^{-2\pi^2\{\boldsymbol{\nu}\cdot[\mathbf{x}-\mathbf{y}(s)]\}^2/\varepsilon^2}, \tag{3.7}$$

by a Fourier transform formula [Bateman,[18] p. 15, (11)]. The integral with respect to w becomes

$$\mathrm{PV}\int_{\mathbf{R}} g(w)e^{2\pi i|\mathbf{x}-\mathbf{y}(s)|[\boldsymbol{\nu}\cdot\mathbf{e}(s,\mathbf{x})]w/\varepsilon}\frac{dw}{w}$$
$$= i\int_{\mathbf{R}}\frac{e^{-w^2/2}}{w}\sin\{2\pi|\mathbf{x}-\mathbf{y}(s)|[\boldsymbol{\nu}\cdot\mathbf{e}(s,\mathbf{x})]w/\varepsilon\}dw$$
$$= 2i\,\mathrm{sgn}[\boldsymbol{\nu}\cdot\mathbf{e}(s,\mathbf{x})]\int_0^\infty\frac{e^{-w^2/2}}{w}\sin[2\pi|\mathbf{x}-\mathbf{y}(s)|$$
$$\times|\boldsymbol{\nu}\cdot\mathbf{e}(s,\mathbf{x})|w/\varepsilon]dw. \tag{3.8}$$

Using a formula in Gradshtein and Ryzhik[19] (p. 497, 3.952.7), we can compute the last integral in Eq. (3.8) as follows:

$$\mathrm{PV}\int_{\mathbf{R}} g(w)e^{2\pi i|\mathbf{x}-\mathbf{y}(s)|[\boldsymbol{\nu}\cdot\mathbf{e}(s,\mathbf{x})]w/\varepsilon}\frac{dw}{w}$$
$$= 2\sqrt{\pi}i\,\mathrm{sgn}[\boldsymbol{\nu}\cdot\mathbf{e}(s,\mathbf{x})]\mathrm{Erf}[\sqrt{2}\pi|\mathbf{x}-\mathbf{y}(s)||\boldsymbol{\nu}\cdot\mathbf{e}(s,\mathbf{x})|/\varepsilon], \tag{3.9}$$

where Erf is the error function

$$\mathrm{Erf}(x) = xe^{-x^2}{}_1F_1(1,3/2;x^2)$$
$$= \int_0^x e^{-t^2}dt$$
$$= \int_0^\infty e^{-t^2}dt - \int_x^\infty e^{-t^2}dt$$
$$= \frac{\sqrt{\pi}}{2} - \int_x^\infty e^{-t^2}dt,$$

and

$$\int_x^\infty e^{-t^2}dt =: \mathrm{Erfc}(x) = O\left(\frac{1}{x}e^{-x^2}\right)$$

for large $x>0$. Consequently, for large $x>0$ we have

$$\mathrm{Erf}(x) = \frac{\sqrt{\pi}}{2} + O\left(\frac{1}{x}e^{-x^2}\right). \tag{3.10}$$

Recall $c_4<|\mathbf{x}-\mathbf{y}(s)|<c_5$. Therefore, we may apply Eq. (3.10) to Eq. (3.9) if $|\boldsymbol{\nu}\cdot\mathbf{e}(s,\mathbf{x})|/\varepsilon$ is large. For a fixed $\delta\in(0,1/2)$, we will thus consider two cases: (i) $|\boldsymbol{\nu}\cdot\mathbf{e}(s,\mathbf{x})|\geqslant\varepsilon^{1-\delta}$ and (ii) $|\boldsymbol{\nu}\cdot\mathbf{e}(s,\mathbf{x})|<\varepsilon^{1-\delta}$. Because in case (i) $|\boldsymbol{\nu}\cdot\mathbf{e}(s,\mathbf{x})|/\varepsilon\geqslant\varepsilon^{-\delta}\to\infty$ as $\varepsilon\to 0^+$,

$$\mathrm{Erf}(\sqrt{2}\pi|\mathbf{x}-\mathbf{y}(s)||\boldsymbol{\nu}\cdot\mathbf{e}(s,\mathbf{x})|/\varepsilon)=\frac{\sqrt{\pi}}{2}+O(\varepsilon^{\delta}e^{-c\varepsilon^{-2\delta}}) \tag{3.11}$$

for some $c>0$. Back to Eq. (3.6) using Eqs. (3.7), (3.9), and (3.11), we have

$$\begin{aligned}\mathrm{RHS}=2\sqrt{2}\lim_{\varepsilon\to0^+}\frac{1}{\varepsilon}\int_{I_{\mathrm{PI}}(\mathbf{x})}ds\int_{\substack{\mathbf{R}^3\\|\boldsymbol{\nu}\cdot\mathbf{e}(s,\mathbf{x})|<\varepsilon^{1-\delta}}}\left(\boldsymbol{\nu}\cdot\frac{d\mathbf{y}(s)}{ds}\right)\\ \times F(\boldsymbol{\nu})e^{2\pi i\boldsymbol{\nu}\cdot\mathbf{y}(s)}\\ \times\,\mathrm{sgn}[\boldsymbol{\nu}\cdot\mathbf{e}(s,\mathbf{x})]e^{-2\pi^2\{\boldsymbol{\nu}\cdot[\mathbf{x}-\mathbf{y}(s)]\}^2/\varepsilon^2}\mathrm{Erf}[\sqrt{2}\pi|\mathbf{x}-\mathbf{y}(s)|\\ \times|\boldsymbol{\nu}\cdot\mathbf{e}(s,\mathbf{x})|/\varepsilon]d\boldsymbol{\nu}\end{aligned} \tag{3.12}$$

$$\begin{aligned}+\sqrt{2\pi}\lim_{\varepsilon\to0^+}\frac{1}{\varepsilon}\int_{I_{\mathrm{PI}}(\mathbf{x})}ds\int_{\mathbf{R}^3}\left(\boldsymbol{\nu}\cdot\frac{d\mathbf{y}(s)}{ds}\right)F(\boldsymbol{\nu})e^{2\pi i\boldsymbol{\nu}\cdot\mathbf{y}(s)}\\ \times\mathrm{sgn}[\boldsymbol{\nu}\cdot\mathbf{e}(s,\mathbf{x})]e^{-2\pi^2\{\boldsymbol{\nu}\cdot[\mathbf{x}-\mathbf{y}(s)]\}^2/\varepsilon^2}d\boldsymbol{\nu}\end{aligned} \tag{3.13}$$

$$\begin{aligned}-\sqrt{2\pi}\lim_{\varepsilon\to0^+}\frac{1}{\varepsilon}\int_{I_{\mathrm{PI}}(\mathbf{x})}ds\\ \times\int_{\substack{\mathbf{R}^3\\|\boldsymbol{\nu}\cdot\mathbf{e}(s,\mathbf{x})|<\varepsilon^{1-\delta}}}\left(\boldsymbol{\nu}\cdot\frac{d\mathbf{y}(s)}{ds}\right)F(\boldsymbol{\nu})e^{2\pi i\boldsymbol{\nu}\cdot\mathbf{y}(s)}\\ \times\mathrm{sgn}[\boldsymbol{\nu}\cdot\mathbf{e}(s,\mathbf{x})]e^{-2\pi^2\{\boldsymbol{\nu}\cdot[\mathbf{x}-\mathbf{y}(s)]\}^2/\varepsilon^2}d\boldsymbol{\nu}\end{aligned} \tag{3.14}$$

$$\begin{aligned}+\lim_{\varepsilon\to0^+}O\Bigg\{\varepsilon^{\delta-1}e^{-\varepsilon^{-2\delta}}\int_{I_{\mathrm{PI}}(\mathbf{x})}ds\\ \times\int_{\substack{\mathbf{R}^3\\|\boldsymbol{\nu}\cdot\mathbf{e}(s,\mathbf{x})|\geqslant\varepsilon^{1-\delta}}}\left|\boldsymbol{\nu}\cdot\frac{d\mathbf{y}(s)}{ds}\right|\\ \times|F(\boldsymbol{\nu})|e^{-2\pi^2\{\boldsymbol{\nu}\cdot[\mathbf{x}-\mathbf{y}(s)]\}^2/\varepsilon^2}d\boldsymbol{\nu}\Bigg\}.\end{aligned} \tag{3.15}$$

Actually, Eq. (3.13) is the only significant term. As far as the other terms are concerned, we have

$$\int_{I_{\mathrm{PI}}(\mathbf{x})}ds\int_{\substack{\mathbf{R}^3\\|\boldsymbol{\nu}\cdot\mathbf{e}(s,\mathbf{x})|\geqslant\varepsilon^{1-\delta}}}(\ldots)d\boldsymbol{\nu}=O\left[\int_{\mathbf{R}^3}|\boldsymbol{\nu}||F(\boldsymbol{\nu})|d\boldsymbol{\nu}\right]=O(1).$$

Therefore, Eq. (3.15) $\to 0$ as $\varepsilon\to0^+$. Then, both the expressions (3.12) and (3.14) before taking the limit are bounded by

$$\frac{1}{\varepsilon}\int_{I_{\mathrm{PI}}(\mathbf{x})}ds\int_{\substack{\mathbf{R}^3\\|\boldsymbol{\nu}\cdot\mathbf{e}(s,\mathbf{x})|<\varepsilon^{1-\delta}}}|\boldsymbol{\nu}||F(\boldsymbol{\nu})|e^{-2\pi^2\{\boldsymbol{\nu}\cdot[\mathbf{x}-\mathbf{y}(s)]\}^2/\varepsilon^2}d\boldsymbol{\nu}. \tag{3.16}$$

If $|\boldsymbol{\nu}\cdot[\mathbf{x}-\mathbf{y}(s)]|\geqslant\varepsilon^{1-\delta}$, then $\{\boldsymbol{\nu}\cdot[\mathbf{x}-\mathbf{y}(s)]\}^2/\varepsilon^2\geqslant\varepsilon^{-2\delta}\to+\infty$, and hence

$$\begin{aligned}&\frac{1}{\varepsilon}\int_{I_{\mathrm{PI}}(\mathbf{x})}ds\int_{\substack{\mathbf{R}^3\\|\boldsymbol{\nu}\cdot\mathbf{e}(s,\mathbf{x})|<\varepsilon^{1-\delta}\\|\boldsymbol{\nu}\cdot[\mathbf{x}-\mathbf{y}(s)]|\geqslant\varepsilon^{1-\delta}}}|\boldsymbol{\nu}||F(\boldsymbol{\nu})|e^{-2\pi^2\{\boldsymbol{\nu}\cdot[\mathbf{x}-\mathbf{y}(s)]\}^2/\varepsilon^2}d\boldsymbol{\nu}\\ &=O\left(\frac{1}{\varepsilon}e^{-2\pi^2\varepsilon^{-2\delta}}\int_{I_{\mathrm{PI}}(\mathbf{x})}ds\int_{\mathbf{R}^3}|\boldsymbol{\nu}||F(\boldsymbol{\nu})|d\boldsymbol{\nu}\right)\to0.\end{aligned}$$

The remaining part of Eq. (3.16) is bounded by

$$O\left(\frac{1}{\varepsilon}\int_{I_{\mathrm{PI}}(\mathbf{x})}ds\int_{\substack{\mathbf{R}^3\\|\boldsymbol{\nu}\cdot\mathbf{e}(s,\mathbf{x})|<\varepsilon^{1-\delta}\\|\boldsymbol{\nu}\cdot(\mathbf{x}-\mathbf{y}(s))|<\varepsilon^{1-\delta}}}|\boldsymbol{\nu}||F(\boldsymbol{\nu})|d\boldsymbol{\nu}\right). \tag{3.17}$$

Since $|\boldsymbol{\nu}||F(\boldsymbol{\nu})|$ is decreasing by the order of $|\boldsymbol{\nu}|^{-4}$, it localizes the integral with respect to $\boldsymbol{\nu}$ mainly to a bounded region in $\mathbf{R}^3$. Because $\mathbf{e}(s,\mathbf{x})$ and $\beta(s,\mathbf{x})$ are not in the same direction, Eq. (3.17) is bounded by

$$O\left(\frac{1}{\varepsilon}\int_{I_{\mathrm{PI}}(\mathbf{x})}ds(\varepsilon^{1-\delta})^2\right)=O(\varepsilon^{1-2\delta})\to0,$$

as $\varepsilon\to0^+$. This is equivalent to say that

$$\begin{aligned}\mathrm{RHS}=\sqrt{2\pi}\lim_{\varepsilon\to0^+}\frac{1}{\varepsilon}\int_{I_{\mathrm{PI}}(\mathbf{x})}ds\int_{\mathbf{R}^3}\left(\boldsymbol{\nu}\cdot\frac{d\mathbf{y}(s)}{ds}\right)F(\boldsymbol{\nu})\\ \times e^{2\pi i\boldsymbol{\nu}\cdot\mathbf{y}(s)}\,\mathrm{sgn}[\boldsymbol{\nu}\cdot\mathbf{e}(s,\mathbf{x})]e^{-2\pi^2\{\boldsymbol{\nu}\cdot[\mathbf{x}-\mathbf{y}(s)]\}^2/\varepsilon^2}d\boldsymbol{\nu}.\end{aligned} \tag{3.18}$$

Now we compute Eq. (3.18). Interchanging the orders of integration, we have

$$\begin{aligned}\mathrm{RHS}=\sqrt{2\pi}\lim_{\varepsilon\to0^+}\frac{1}{\varepsilon}\int_{\mathbf{R}^3}F(\boldsymbol{\nu})d\boldsymbol{\nu}\\ \times\int_{I_{\mathrm{PI}}(\mathbf{x})}\left(\boldsymbol{\nu}\cdot\frac{d\mathbf{y}(s)}{ds}\right)e^{2\pi i\boldsymbol{\nu}\cdot\mathbf{y}(s)}\,\mathrm{sgn}[\boldsymbol{\nu}\cdot\mathbf{e}(s,\mathbf{x})]\\ \times e^{-2\pi^2\{\boldsymbol{\nu}\cdot[\mathbf{x}-\mathbf{y}(s)]\}^2/\varepsilon^2}ds.\end{aligned} \tag{3.19}$$

This interchange of integration order is legitimate because the integral with respect to $\boldsymbol{\nu}$ is dominated by the integral of $[\boldsymbol{\nu}\cdot d\mathbf{y}(s)/ds]F(\boldsymbol{\nu})=O(|\boldsymbol{\nu}|^{-4})$, while the integral with respect to s is taken over a finite interval with a bounded integrand.

As before, for a fixed $\boldsymbol{\nu}$, if $|\boldsymbol{\nu}\cdot(\mathbf{x}-\mathbf{y}(s))|\geqslant\varepsilon^{1-\delta}$ for s in a subinterval of $I_{\mathrm{PI}}(\mathbf{x})$, then for such s we have $e^{-2\pi^2\{\boldsymbol{\nu}\cdot[\mathbf{x}-\mathbf{y}(s)]\}^2/\varepsilon^2}<e^{-2\pi^2/\varepsilon^{2\delta}}$. Since $\lim_{\varepsilon\to0^+}e^{-2\pi^2/\varepsilon^{2\delta}}/\varepsilon=0$, the portion of the inner integral in Eq. (3.19) taken over such a subinterval will contribute nothing as $\varepsilon\to0^+$. Therefore, the inner integral in Eq. (3.19) can be evaluated over those $s\in I_{\mathrm{PI}}(\mathbf{x})$ satisfying $|\boldsymbol{\nu}\cdot[\mathbf{x}-\mathbf{y}(s)]|<\varepsilon^{1-\delta}$:

$$\begin{aligned}\mathrm{RHS}=\sqrt{2\pi}\lim_{\varepsilon\to0^+}\frac{1}{\varepsilon}\int_{\mathbf{R}^3}F(\boldsymbol{\nu})d\boldsymbol{\nu}\\ \times\int_{\substack{I_{\mathrm{PI}}(\mathbf{x})\\|\boldsymbol{\nu}\cdot[\mathbf{x}-\mathbf{y}(s)]|<\varepsilon^{1-\delta}}}\left(\boldsymbol{\nu}\cdot\frac{d\mathbf{y}(s)}{ds}\right)\\ \times e^{2\pi i\boldsymbol{\nu}\cdot\mathbf{y}(s)}\,\mathrm{sgn}[\boldsymbol{\nu}\cdot\mathbf{e}(s,\mathbf{x})]e^{-2\pi^2\{\boldsymbol{\nu}\cdot[\mathbf{x}-\mathbf{y}(s)]\}^2/\varepsilon^2}ds.\end{aligned} \tag{3.20}$$

Recall $I_{\mathrm{PI}}(\mathbf{x})=[s_b,s_t]$. Without loss of generality, let us assume that $\boldsymbol{\nu}\cdot[\mathbf{x}-\mathbf{y}(s)]=0$ at $s=s_0,\ldots,s_r$ with $s_b<s_0<\ldots<s_r<s_t$. Note that $s_0,\ldots,s_r$ depend on $\boldsymbol{\nu}$. We will only consider those $\boldsymbol{\nu}\in\mathbf{R}^3$ such that $\boldsymbol{\nu}\cdot[\mathbf{y}(s_t)-\mathbf{y}(s_b)]\neq 0$, since the set of $\boldsymbol{\nu}$ with $\boldsymbol{\nu}\cdot[\mathbf{y}(s_t)-\mathbf{y}(s_b)]=0$ only has a zero measure, and can be ignored in the outer integral in Eq. (3.20). By the same reason, we can ignore those $\boldsymbol{\nu}$ which are parallel to $\mathbf{y}'(s)\times\mathbf{y}''(s)$ for some $s\in[s_b,s_t]$, if we assume that $\mathbf{y}'(s)\times\mathbf{y}''(s)$ never vanishes. We observe that $\boldsymbol{\nu}\cdot[\mathbf{x}-\mathbf{y}(s_b)]$ and $\boldsymbol{\nu}\cdot[\mathbf{x}-\mathbf{y}(s_t)]$ are of opposite signs, because $\mathbf{x}$ is on the PI line ℓ from $\mathbf{y}(s_b)$ to $\mathbf{y}(s_t)$. Thus, there is at least one such s_j. There are four possible cases of $\boldsymbol{\nu}\cdot[\mathbf{x}-\mathbf{y}(s)]$ near s_j, $0\leqslant j\leqslant r$:

(i) $\boldsymbol{\nu}\cdot\mathbf{y}'(s_j)<0$: $\boldsymbol{\nu}\cdot[\mathbf{y}(s)-\mathbf{x}]$ decreases near s_j;
(ii) $\boldsymbol{\nu}\cdot\mathbf{y}'(s_j)>0$: $\boldsymbol{\nu}\cdot[\mathbf{y}(s)-\mathbf{x}]$ increases near s_j;
(iii) $\boldsymbol{\nu}\cdot\mathbf{y}'(s_j)=0$ and $\boldsymbol{\nu}\cdot\mathbf{y}''(s_j)<0$: $\boldsymbol{\nu}\cdot(\mathbf{y}(s)-\mathbf{x})$ has a local maximum at s_j;
(iv) $\boldsymbol{\nu}\cdot\mathbf{y}'(s_j)=0$ and $\boldsymbol{\nu}\cdot\mathbf{y}''(s_j)>0$: $\boldsymbol{\nu}\cdot(\mathbf{y}(s)-\mathbf{x})$ has a local minimum at s_j.

For ε sufficiently small, let us denote by $[s_j-\sigma_j,s_j+\tau_j]$ the neighborhood of s_j satisfying $|\boldsymbol{\nu}\cdot[\mathbf{y}(s)-\mathbf{x}]|\leqslant\varepsilon^{1-\delta}$. More specifically, in each case we set

(i) $\boldsymbol{\nu}\cdot[\mathbf{y}(s_j-\sigma_j)-\mathbf{x}]=\varepsilon^{1-\delta}$, $\boldsymbol{\nu}\cdot[\mathbf{y}(s_j+\tau_j)-\mathbf{x}]=-\varepsilon^{1-\delta}$;
(ii) $\boldsymbol{\nu}\cdot[\mathbf{y}(s_j-\sigma_j)-\mathbf{x}]=-\varepsilon^{1-\delta}$, $\boldsymbol{\nu}\cdot[\mathbf{y}(s_j+\tau_j)-\mathbf{x}]=\varepsilon^{1-\delta}$;
(iii) $\boldsymbol{\nu}\cdot[\mathbf{y}(s_j-\sigma_j)-\mathbf{x}]=\boldsymbol{\nu}\cdot[\mathbf{y}(s_j+\tau_j)-\mathbf{x}]=-\varepsilon^{1-\delta}$;
(iv) $\boldsymbol{\nu}\cdot[\mathbf{y}(s_j-\sigma_j)-\mathbf{x}]=\boldsymbol{\nu}\cdot[\mathbf{y}(s_j+\tau_j)-\mathbf{x}]=\varepsilon^{1-\delta}$.

According to Eq. (3.8), the case of $\boldsymbol{\nu}\cdot\mathbf{e}(s_j,\mathbf{x})=0$ makes no contribution to RHS. Hence, we only analyze the case of $\boldsymbol{\nu}\cdot\mathbf{e}(s_j,\mathbf{x})\neq 0$. Then, for ε sufficiently small we can always assume that $\boldsymbol{\nu}\cdot\mathbf{e}(s,\mathbf{x})$ has the same sign on $[s_j-\sigma_j,s_j+\tau_j]$. Thus,

$$\begin{aligned}&\frac{1}{\varepsilon}\int_{s_j-\sigma_j}^{s_j+\tau_j}\left(\boldsymbol{\nu}\cdot\frac{d\mathbf{y}(s)}{ds}\right)e^{2\pi i\boldsymbol{\nu}\cdot\mathbf{y}(s)}\,\mathrm{sgn}[\boldsymbol{\nu}\cdot\mathbf{e}(s,\mathbf{x})]\\&\quad\times e^{-2\pi^2\{\boldsymbol{\nu}\cdot[\mathbf{x}-\mathbf{y}(s)]\}^2/\varepsilon^2}ds\\&\quad=\mathrm{sgn}[\boldsymbol{\nu}\cdot\mathbf{e}(s_j,\mathbf{x})]\frac{1}{\varepsilon}\int_{s_j-\sigma_j}^{s_j+\tau_j}\left(\boldsymbol{\nu}\cdot\frac{d\mathbf{y}(s)}{ds}\right)e^{2\pi i\boldsymbol{\nu}\cdot\mathbf{y}(s)}\\&\quad\quad\times e^{-2\pi^2\{\boldsymbol{\nu}\cdot[\mathbf{x}-\mathbf{y}(s)]\}^2/\varepsilon^2}ds.\end{aligned}\tag{3.21}$$

Changing variables from s to $t=\boldsymbol{\nu}\cdot\mathbf{y}(s)$, we have

$$\mathrm{sgn}[\boldsymbol{\nu}\cdot\mathbf{e}(s_j,\mathbf{x})]\int_{\boldsymbol{\nu}\cdot\mathbf{y}(s_j-\sigma_j)}^{\boldsymbol{\nu}\cdot\mathbf{y}(s_j+\tau_j)}e^{2\pi it}e^{-2\pi^2(\boldsymbol{\nu}\cdot\mathbf{x}-t)^2/\varepsilon^2}dt$$

using $dt=[\boldsymbol{\nu}\cdot d\mathbf{y}(s)/ds]ds$. Changing variables again from t to $u=(t-\boldsymbol{\nu}\cdot\mathbf{x})/\varepsilon$, we have $t=\boldsymbol{\nu}\cdot\mathbf{x}+\varepsilon u$ and $dt=\varepsilon du$, and Eq. (3.21) becomes

$$\mathrm{sgn}[\boldsymbol{\nu}\cdot\mathbf{e}(s_j,\mathbf{x})]\int_{\boldsymbol{\nu}\cdot[\mathbf{y}(s_j-\sigma_j)-\mathbf{x}]/\varepsilon}^{\boldsymbol{\nu}\cdot[\mathbf{y}(s_j+\tau_j)-\mathbf{x}]/\varepsilon}e^{2\pi i\boldsymbol{\nu}\cdot\mathbf{x}}e^{2\pi i\varepsilon u}e^{-2\pi^2u^2}du.\tag{3.22}$$

We observe that the lower and upper limits of the integral in Eq. (3.22) are the same in cases (iii) and (iv), and the integral vanishes. In case (ii), the integral becomes $\int_{-\infty}^{\infty}$ when $\varepsilon\to 0^+$, while in case (i) it tends to $\int_{\infty}^{-\infty}=-\int_{-\infty}^{\infty}$. Consequently, the integral in Eq. (3.22) has the limit

$$e^{2\pi i\boldsymbol{\nu}\cdot\mathbf{x}}\,\mathrm{sgn}[\boldsymbol{\nu}\cdot\mathbf{y}'(s_j)]\int_{-\infty}^{\infty}e^{-2\pi^2u^2}du,$$

as $\varepsilon\to 0^+$. In fact, in case (ii) we can write the integral in Eq. (3.22) as

$$\int_{\boldsymbol{\nu}\cdot[\mathbf{y}(s_j-\sigma_j)-\mathbf{x}]/\varepsilon}^{-\varepsilon^{-1/2}}+\int_{-\varepsilon^{-1/2}}^{\varepsilon^{-1/2}}+\int_{\varepsilon^{-1/2}}^{\boldsymbol{\nu}\cdot[\mathbf{y}(s_j+\tau_j)-\mathbf{x}]/\varepsilon}.$$

The first and last integrals here tend to 0 as $\varepsilon\to 0^+$, because

$$\int_{\varepsilon^{-1/2}}^{\boldsymbol{\nu}\cdot[\mathbf{y}(s_j+\tau_j)-\mathbf{x}]/\varepsilon}e^{-2\pi^2u^2}du\leqslant\int_{\varepsilon^{-1/2}}^{\infty}e^{-2\pi^2u^2}du=O(\sqrt{\varepsilon}e^{-2\pi^2/\varepsilon}).$$

Inside the interval $[-\varepsilon^{-1/2},\varepsilon^{-1/2}]$, εu is bounded by $\pm\sqrt{\varepsilon}$, and hence $e^{2\pi i\varepsilon u}\to 1$ as $\varepsilon\to 0^+$. This shows that

$$\int_{-\varepsilon^{-1/2}}^{\varepsilon^{-1/2}}e^{2\pi i\varepsilon u}e^{-2\pi^2u^2}du\to\int_{-\infty}^{\infty}e^{-2\pi^2u^2}du.$$

Case (i) can be computed likewise.

Therefore, we have

$$\begin{aligned}\mathrm{RHS}&=\sqrt{2\pi}\int_{\mathbf{R}^3}F(\boldsymbol{\nu})e^{2\pi i\boldsymbol{\nu}\cdot\mathbf{x}}d\boldsymbol{\nu}\sum_{j=0}^{r}\mathrm{sgn}[\boldsymbol{\nu}\cdot\mathbf{y}'(s_j)]\\&\quad\times\mathrm{sgn}[\boldsymbol{\nu}\cdot\mathbf{e}(s_j,\mathbf{x})]\int_{-\infty}^{\infty}e^{-2\pi^2u^2}du.\end{aligned}\tag{3.23}$$

By a classical result

$$\int_{-\infty}^{\infty}e^{-2\pi^2t^2}dt=\frac{1}{\sqrt{2\pi}},$$

we get

$$\begin{aligned}\mathrm{RHS}=\int_{\mathbf{R}^3}F(\boldsymbol{\nu})e^{2\pi i\boldsymbol{\nu}\cdot\mathbf{x}}\Bigg\{&\sum_{j=0}^{r}\mathrm{sgn}[\boldsymbol{\nu}\cdot\mathbf{y}'(s_j)]\\&\times\mathrm{sgn}[\boldsymbol{\nu}\cdot\mathbf{e}(s_j,\mathbf{x})]\Bigg\}d\boldsymbol{\nu}.\end{aligned}\tag{3.24}$$

To prove that $\mathrm{RHS}=f(\mathbf{x})$, we should select $\mathbf{e}(s_j,\mathbf{x})$ so that the following condition is satisfied:

$$\sum_{j=0}^{r}\mathrm{sgn}[\boldsymbol{\nu}\cdot\mathbf{y}'(s_j)]\mathrm{sgn}[\boldsymbol{\nu}\cdot\mathbf{e}(s_j,\mathbf{x})]=1\tag{3.25}$$

for any $\boldsymbol{\nu}\in\mathbf{R}^3$, where s_j depends on $\boldsymbol{\nu}$ as described above.

Equation (3.25) is the general admissible condition for selection of filtering directions. Recall that s_j are solutions of $\boldsymbol{\nu}\cdot[\mathbf{x}-\mathbf{y}(s)]=0$ for a given $\boldsymbol{\nu}\in\mathbf{R}^3$. Thus, $\mathbf{x}$ and $\mathbf{y}(s_j)$, $0\leqslant j\leqslant r$, are on a plane perpendicular to $\boldsymbol{\nu}$.

One natural choice of $\mathbf{e}(s,\mathbf{x})$ satisfying Eq. (3.25) is as follows. Take $\mathbf{e}(s,\mathbf{x})$ in the direction of the projection of the vector $\mathbf{y}(s_t)-\mathbf{y}(s_b)$ on the plane $\mathbf{\Pi}$ perpendicular to $\mathbf{x}-\mathbf{y}(s)$. With this choice of $\mathbf{e}(s,\mathbf{x})$, we have

$$\mathrm{sgn}[\boldsymbol{\nu}\cdot\mathbf{e}(s_j,\mathbf{x})]=\mathrm{sgn}\{\boldsymbol{\nu}\cdot[\mathbf{y}(s_t)-\mathbf{y}(s_b)]\}$$

for all $\boldsymbol{\nu}\in\mathbf{\Pi}$, i.e., for all $\boldsymbol{\nu}\in\mathbf{R}^3$ with $\boldsymbol{\nu}\cdot[\mathbf{x}-\mathbf{y}(s_j)]=0$. Thus,

$$\sum_{j=0}^{r}\mathrm{sgn}[\boldsymbol{\nu}\cdot\mathbf{y}'(s_j)]\mathrm{sgn}[\boldsymbol{\nu}\cdot\mathbf{e}(s_j,\mathbf{x})]$$

$$=\mathrm{sgn}\{\boldsymbol{\nu}\cdot[\mathbf{y}(s_t)-\mathbf{y}(s_b)]\}\sum_{j=0}^{r}\mathrm{sgn}[\boldsymbol{\nu}\cdot\mathbf{y}'(s_j)].$$

Recall that $\boldsymbol{\nu}\cdot[\mathbf{y}(s_t)-\mathbf{x}]$ and $\boldsymbol{\nu}\cdot[\mathbf{y}(s_b)-\mathbf{x}]$ are of opposite signs, and hence

$$\sum_{j=0}^{r}\mathrm{sgn}[\boldsymbol{\nu}\cdot\mathbf{y}'(s_j)]=\mathrm{sgn}\{\boldsymbol{\nu}\cdot[\mathbf{y}(s_t)-\mathbf{y}(s_b)]\}.$$

Under this choice of $\mathbf{e}(s,\mathbf{x})$ we do have Eq. (3.25) and finally

$$\mathrm{RHS}=\int_{\mathbf{R}^3}F(\boldsymbol{\nu})e^{2\pi i\boldsymbol{\nu}\cdot\mathbf{x}}d\boldsymbol{\nu}=f(\mathbf{x}).\tag{3.26}$$

This selection of $\mathbf{e}(s,\mathbf{x})$ can be expressed more explicitly. Note that $\mathbf{e}(s,\mathbf{x})$ is in the direction of the projection of the vector $\mathbf{y}(s_t)-\mathbf{y}(s_b)$ on the plane $\mathbf{\Pi}$ perpendicular to $\mathbf{x}-\mathbf{y}(s)$. Therefore, $\mathbf{e}(s,\mathbf{x})$ is in the plane determined by the line ℓ and the vector $\boldsymbol{\beta}(s,\mathbf{x})=[\mathbf{x}-\mathbf{y}(s)]/|\mathbf{x}-\mathbf{y}(s)|$. Our choice of filtering planes, as given by Eq. (2.3), is therefore always the planes set by ℓ and $\mathbf{x}-\mathbf{y}(s)$.

IV. DISCUSSIONS AND CONCLUSION

Our choice of the filtering direction in the general scanning case is consistent with that of Zou and Pan[20] for standard helical scanning. It has been just proved that filtering modified data along the chord projection onto the detector plane is indeed a way to perform exact cone-beam reconstruction, not only along helical loci but also along other curves, as long as they satisfy certain weak conditions as described above. We emphasize that there are other choices of filtering directions $\mathbf{e}(s,\mathbf{x})$. Let us take a standard helical scanning locus as an example. For a given $\mathbf{x}$ and $s_0\in I_{\mathrm{PI}}(\mathbf{x})$, there is a unique plane passing through $\mathbf{x}$, $\mathbf{y}(s_0)$, $\mathbf{y}(s_2)$, and $\mathbf{y}[(s_0+s_2)/2]$ for some s_2 with $s_0-2\pi<s_2<s_0+2\pi$, according to Katsevich.[5–7] In this case, the filtering direction is defined by the intersection of this plane and the detector plane. Clearly, our choice of filtering directions is different from Katsevich's, and is applicable to general scanning curves. However, we acknowledge that our choice of the filtering direction is not the most efficient, at least in the helical scanning case, because the filtering step is not shift-invariant. Finding better filtering directions for various classes of scanning curves is an important topic for future work. Note that our general formula is also applicable to the case where several chords pass through a given point to be reconstructed, and to the case of multiple filtering directions that are not necessarily associated with chords all the time. Clearly, exact reconstruction based on multiple families of filtering directions can be advantageous in terms of image contrast resolution.

As far as numerical verification is concerned, excellent computer-simulation results have been obtained in our laboratory using our generalized filtered backprojection (FBP) formula in the cases of nonstandard spirals and nonstandard saddle curves. These experimental results and comparative analysis will be reported in a subsequent article. Furthermore, we note that Zou and Pan[20] already reported excellent simulation results using their FBP formula in the helical scanning case, which is a special case of the general FBP formula. This may be considered as independent evidence supporting our formulation.

In the helical cone-beam formulas by Katsevich[5–7] and Katsevich *et al.*,[13] as well as Zou and Pan[11,12] and Zou *et al.*,[14] the uniqueness of PI lines is explicitly or implicitly assumed. Our exact reconstruction for cone-beam scanning along a general curve is performed inside the region of chords, not necessarily inside the region of unique PI lines, as reported by Ye *et al.*[15] Therefore, our formula can be applied to nonstandard spirals of variable radii and pitches, and saddle-like curves. Moreover, let $\mathbf{x}$ be a point on a chord connecting two points $\mathbf{y}(s_b)$ and $\mathbf{y}(s_t)$ on a nonstandard spiral $\mathbf{y}(s)$ with $s_t-s_b>2\pi$. Since the two endpoints are separated by more than one scanning turn, the chord is a generalized PI line. Nevertheless, our generalized FBP formula can be applied to reconstruct any $\mathbf{x}$ on this chord exactly. This is indeed the case of the so-called *n*-PI-window problem studied by Proksa *et al.*,[21] Bontus *et al.*,[22] and Katsevich.[23] In other words, what we have proved is an exact reconstruction formula for cone-beam n-PI-window scanning along a standard or nonstandard spiral. On the other hand, for cardiac imaging Pack *et al.*[24] studied cone-beam reconstruction along saddle curves under Tuy's condition.[25] In this situation, the generalized formula can be used for exact image reconstruction on the so-called *t* lines.

Our work is closely related to Katsevich's[6] general scheme for exact cone-beam reconstruction. Currently, there are two approaches to deriving FBP-type inversion algorithms for general trajectories. One is based on the idea of a piece-wise constant normalized weight. This is the approach in Katsevich.[6] With the simplest choice of a weight function ($n_0=1$) a convolution-based FBP algorithm was obtained for a general complete trajectory (by Katsevich[6] and by Chen[26] with a proposed regularization scheme to handle singularity). The other approach is based on specifying the filtering direction(s) via $\mathbf{e}(s,\mathbf{x})$, such as the one used by the Philips group.[22] They actually used not one family of filtering planes, but several families in the helical scanning case. Our approach is also based on specifying the filtering direction, but has the advantage of being rigorously justified in the general scanning case. Of course, because all the exact reconstruction formulas must be equivalent we can establish a relationship between this general FBP formula and Katsevich's general scheme for exact cone-beam reconstruction. If we denote

$$n(s_j,\mathbf{x},\boldsymbol{\nu})=\mathrm{sgn}[\boldsymbol{\nu}\cdot\mathbf{y}'(s_j)]\mathrm{sgn}[\boldsymbol{\nu}\cdot\mathbf{e}(s_j,\mathbf{x})],$$

our condition (3.25) is the normalization condition on the second line of Eq. (2.4) in Katsevich.[6] Moreover, our reconstruction formula (2.4) follows immediately from Eqs. (2.27)–(2.29) in Katsevich[6] upon substitution of that particular $n(s_j,\mathbf{x},\boldsymbol{\nu})$. In view of the work by Katsevich[6] and Tuy,[25] our main contributions include (1) a proof of the solution to the exact cone-beam reconstruction in the general case, and (2) a specification of filtering directions used in the general solution.

After the acceptance of this paper, we noticed a recently published note by Zou and Pan,[27] in which a revised proof of the backprojection-filtration (BPF) and FBP reconstruction formulas was given in the setting of a standard helical cone-beam scanning. As they pointed out,[27] "the results can also be extended to general, smooth trajectories."

In conclusion, we have presented a general FBP formula of Katsevich-type, and provided an algorithm for exact image reconstruction from cone-beam data along a rather flexible scanning curve. In the derivation, we have used analytic techniques instead of geometric arguments. As a result, our formula can be used to reconstruct images on any chord as long as a scanning curve runs from one endpoint of the segment to the other endpoint. This can be considered as a generalization of Orlov's[16] classical theorem from the parallel-beam case to the cone-beam case. Further research efforts are being devoted to optimization of filtering directions and comparison between filtered backprojection and back-projected filtration methods.

ACKNOWLEDGMENTS

This work was partially supported by NIH/NIBIB Grant Nos. EB002667 and EB004287. The authors would like to express their deep appreciation for Dr. Alexander Katsevich's critical reading of an early draft of this paper, and for his detailed, enlightening comments and suggestions, most of which have been incorporated in this paper. Also, the authors' sincere gratitude is due to Dr. Shiying Zhao for valuable discussions, and to Dr. Hengyong Yu for his quick computer simulation of the general FBP formula. The first author acknowledges the technical support from the Obermann Center for Advanced Studies at the University of Iowa. Finally, the authors thank Ms. Jiehua Zhu for her typing of the manuscript.

[a]Electronic mail: yangbo-ye@uiowa.edu
[b]Electronic mail: ge-wang@uiowa.edu

[1]G. Wang and M. W. Vannier, "Bolus-chasing angiography with adaptive real-time computed tomography" (U.S. Patent 6,535,821, allowed on 11/26/2002, issued on 3/18/2003).
[2]G. Wang and Y. Ye, "Nonstandard spiral cone-beam scanning methods, apparatus, and applications," Patent disclosure filled with University of Iowa, November 2003 (U.S. Provisional patent application pending), 2003.
[3]G. Wang, T. H. Lin, P. C. Cheng, D. M. Shinozaki, and H. G. Kim, "Scanning cone-beam reconstruction algorithms for x-ray microtomography," Proc. SPIE Vol. 1556, Scanning Microscopy Instrumentation, Gordon Kino ed., SPIE Press, Feb. 1992, 99–112.
[4]G. Wang, T. H. Lin, P. C. Cheng, and D. M. Shinozaki, "A general cone-beam reconstruction algorithm," IEEE Trans. Med. Imaging **12**, 486–496 (1993).
[5]A. Katsevich, "Theoretically exact filtered backprojection-type inversion algorithm for Spiral CT," SIAM (Soc. Ind. Appl. Math.) J. Appl. Math. **62**, 2012–2026 (2002).
[6]A. Katsevich, "A general scheme for constructing inversion algorithms for cone beam CT," Int. J. Math. Math. Sci. **21**, 1305–1321 (2003).
[7]A. Katsevich, "An improved exact filtered backprojection algorithm for spiral computed tomography," Adv. Appl. Math. **32**, 681–697 (2004).
[8]K. C. Tam, S. Samarasekera, and F. Sauer, "Exact cone beam CT with a spiral scan," Phys. Med. Biol. **43**, 1015–1024 (1998).
[9]P. E. Danielsson, P. Edholm, J. Eriksson, and S. M. Magnusson, "Towards exact reconstruction for helical cone-beam scanning of long object. A new detector arrangement and a new completeness condition." *1997 International Meeting on Fully three-dimensional Image Reconstruction* in *Radiology and Nuclear Medicine* (Pittsburgh, PA) edited by D. W. Townsend and P. E. Kinahan, Institute of Physics Publishing, 1998, 141–144.
[10]M. Defrise, F. Noo, and H. Kudo, "A solution to the long-object problem in helical cone-beam tomography," Phys. Med. Biol. **45**, 623–643 (2000).
[11]Y. Zou and X. Pan, "Exact image reconstruction on PI lines from minimum data in helical cone-beam CT," Phys. Med. Biol. **49**, 941–959 (2004).
[12]Y. Zou and X. Pan, "Comments on image reconstruction on PI-line segments in helical cone-beam CT," preprint.
[13]A. Katsevich, S. Basu, and J. Hsieh, "Exact filtered backprojection reconstruction for dynamic pitch helical cone beam computed tomography," Phys. Med. Biol. **49**, 3089–3103 (2004).
[14]Y. Zou, X. Pan, D. Xia, and G. Wang, "Exact image reconstruction in helical cone-beam CT with a variable pitch," preprint, 2004.
[15]Y. Ye, S. Zhao, H. Yu, and G. Wang, "Exact reconstruction for cone-beam scanning along nonstandard spirals and other curves," Proc. SPIE Vol. 5535, Developments in X-Ray Tomography IV. Ulrich Bonse ed., SPIE Press, Aug. 2004, 293–300.
[16]S. S. Orlov, "Theory of three dimensional reconstruction, I Conditions for a complete set of projections," Sov. Phys. Crystallogr. **20**, 312–314 (1975); "II The recovery operator," **20**, 429–433 (1975).
[17]S. Zhao, H. Yu, and G. Wang, "A family of analytic algorithms for cone-beam CT," Proc. SPIE Vol. 5535, Developments in X-Ray Tomography IV. Ulrich Bonse ed., SPIE Press, Aug. 2004, 318–328.
[18]H. Bateman, *Table of Integral Transforms*, Vol. 1 (McGraw-Hill, New York, 1954).
[19]I. S. Gradshtein and I. M. Ryzhik, *Table of Integrals, Series, and Products*, 6th ed. (Academic Press, San Diego, 2000).
[20]Y. Zou and X. Pan, "Image reconstruction on PI-lines by use of filtered backprojection in helical cone-beam CT," Phys. Med. Biol. **49**, 2717–2731 (2004).
[21]R. Proksa, Th. Kohler, M. Grass, and J. Timmer, "The n-PI-method for helical cone-beam CT," IEEE Trans. Med. Imaging **19**(9), 848–863 (2000).
[22]C. Bontus, Th. Kohler, and R. Proksa, "A quasiexact reconstruction algorithm for helical CT using a 3-PI acquisition," Med. Phys. **30**(9), 2493–2502 (2003).
[23]A. Katsevich, "On two versions of a 3π algorithm for spiral CT," Phys. Med. Biol. **49**, 2129–2143 (2004).
[24]J. D. Pack, F. Noo, and H. Kudo, "Investigation of saddle trajectories for cardiac CT imaging in cone beam geometry," Phys. Med. Biol. 49, No. 11, Special Isssue: Seventh International Conference on Fully Three-Dimensional Reconstruction in Radiology and Nuclear Medicine, 7 June 2004, 2317–2336.
[25]H. Tuy, "An inversion formula for cone-beam reconstruction," SIAM (Soc. Ind. Appl. Math.) J. Appl. Math. **43**, 546–552 (1983).
[26]G. H. Chen, "An alternative derivation of Katsevich's cone-beam reconstruction formula," Med. Phys. **30**, 3217–3226 (2003).
[27]Y. Zou and X. Pan, "An extended data function and its generalized backprojection for image reconstruction in helical cone-beam CT," Phys. Med. Biol. **49**, N383–N387 (2004).