Method for estimating the phase differences

There are N beams whose complex amplitude is represented by (x_i, y_i) for $i = 1 \ldots N$. The phase of (x_i, y_i) is measured at the origin (center of the imaged area typically) with respect to an arbitrary reference beam.

We cannot directly observe the complex amplitudes of the N beams but have to infere them from the $N(N-1)/2$ peak responses in the transform of the speckle pattern generated by their interference. Suppose that we locate the transform domain peak corresponding to interference between beams i and j and find that the value of the transform there is $(c_{i,j}, s_{i,j})$. Since we are dealing with the transform of a real function we have $c_{j,i} = c_{i,j}$ and $s_{j,i}) = -s_{i,j}$.

The magnitude $(c_{i,j}, s_{i,j})$ should be the product of the magnitudes of the two beams, and the phase should be the phase difference of the two beams. If there were no noise or measurement error we would have

$$c_{i,j} = x_i x_j + y_i y_j$$

$$s_{i,j} = x_i y_j - x_j y_i$$

Our problem then is to estimate (x_i, y_i) for $i = 1 \ldots N$ from the observed $(c_{i,j}, s_{i,j})$ for $i = 1 \ldots N$ and $j = i + 1 \ldots N$. The problem is overdetermined since $N(N-1)/2$ is greater than N.

We can try and minimize the sum of squares of errors

$$E = \sum_{i=1}^{N} \sum_{j=1}^{N} w_{i,j} \left((c_{i,j} - x_i x_j - y_i y_j)^2 + (s_{i,j} - x_i y_j + x_j y_i)^2 \right)$$

where $w_{i,j}$ is a weight that indicates how reliable the measurement of the magnitude of the transform domain peak corresponding to interference between beams i and j is. We let $w_{j,i} = w_{i,j}$. In the above sum we actually count each measurement twice, since the complex amplitude for (j, i) is just the complement of that for (i, j). The duplication is just for convenience of the following derivation.

Differentiating E with respect to x_k, we get

$$2 \sum_{j=1}^{N} w_{k,j} (c_{k,j} - x_k x_j - y_k y_j)(-x_j) + 2 \sum_{i=1}^{N} w_{i,k} (c_{i,k} - x_i x_k - y_i y_k)(-x_i) +$$

$$2 \sum_{j=1}^{N} w_{k,j} (s_{k,j} - x_k y_j + x_j y_k)(-y_j) + 2 \sum_{i=1}^{N} w_{i,k} (s_{i,k} - x_i y_k + x_k y_i)(y_i)$$

the first and second term have the same value, as do the third and fourth term. After some further simplification the above expression becomes

$$\frac{dE}{dx_k} = 4 \sum_{i=1}^{N} w_{i,k} (-c_{i,k} x_i + s_{i,k} y_i + (x_i^2 + y_i^2) x_k)$$

Setting this equal to zero yields:

$$x_k \sum_{i=1}^{N} w_{i,k} (x_i^2 + y_i^2) = \sum_{i=1}^{N} w_{i,k} (c_{i,k} x_i - s_{i,k} y_i)$$

Differentiating E with respect to y_k, we get, after some simplification

$$\frac{dE}{dy_k} = 4 \sum_{i=1}^{N} w_{i,k} (-c_{i,k} y_i - s_{i,k} x_i + (x_i^2 + y_i^2) y_k)$$

Setting this equal to zero yields:

$$y_k \sum_{i=1}^{N} w_{i,k}(x_i^2 + y_i^2) = \sum_{i=1}^{N} w_{i,k}(c_{i,k} y_i + s_{i,k} x_i)$$

We need to solve this set of non-linear equations for the unknown (x_i, y_i) for $i = 1, \ldots N$, given the values of $(c_{i,j}, s_{i,j})$ from the transform of the speckle pattern.

The problem would be easier if knew the value of

$$\sum_{i=1}^{N} w_{i,k}(x_i^2 + y_i^2)$$

This suggest an iterative scheme

$$x_k^{(n+1)} = A \sum_{i=1}^{N} w_{i,k}(c_{i,k} x_i^{(n)} - s_{i,k} y_i^{(n)})$$

$$y_k^{(n+1)} = A \sum_{i=1}^{N} w_{i,k}(c_{i,k} y_i^{(n)} + s_{i,k} x_i^{(n)})$$

for some positive A. The problem is that we don't know A. However, since A is a single overall scale factor on the estimated amplitudes, we can perhaps optimize it in a second step. We need to find the value of A that minimizes

$$\sum_{i=1}^{N} \sum_{j=1}^{N} w_{i,j}(c_{i,j} - A(x_i x_j + y_i y_j))^2 + \sum_{i=1}^{N} \sum_{j=1}^{N} w_{i,j}(s_{i,j} - A(x_i y_j - x_j y_i))^2$$

for fixed (x_i, y_i). Differentiating with respect to A and setting the result equal to zero leads to:

$$\sum_{i=1}^{N} \sum_{j=1}^{N} w_{i,j}(c_{i,j} - A(x_i x_j + y_i y_j))^2 (x_i x_j + y_i y_j) + \sum_{i=1}^{N} \sum_{j=1}^{N} w_{i,j}(s_{i,j} - A(x_i y_j - x_j y_i))^2 (x_i y_j - x_j y_i) = 0$$

Or

$$\sum_{i=1}^{N} \sum_{j=1}^{N} w_{i,j}(c_{i,j}(x_i x_j + y_i y_j) + s_{i,j}(x_i y_j - x_j y_i)) = A \sum_{i=1}^{N} \sum_{j=1}^{N} w_{i,j}((x_i x_j + y_i y_j)^2 + (x_i y_j - x_j y_i)^2)$$

The sum on the right simplifies to

$$\sum_{i=1}^{N} \sum_{j=1}^{N} w_{i,j}(x_i^2 + y_i^2)(x_j^2 + y_j^2)$$

So finally

$$A = \frac{\sum_{i=1}^{N} \sum_{j=1}^{N} w_{i,j}(c_{i,j}(x_i x_j + y_i y_j) + s_{i,j}(x_i y_j - x_j y_i))}{\sum_{i=1}^{N} \sum_{j=1}^{N} w_{i,j}(x_i^2 + y_i^2)(x_j^2 + y_j^2)}$$

We adjust the estimates of (x_i, y_i) by multiplying them by $\sqrt{A}$ before going on to the next iteration. If we are very far from the solution, A as computed above may turn out to be negative. In that case we use its absolute value just to be able to continue the iterative adjustment.