The Speckle Pattern

(Paraphrased from Michael Mermelstein)

The electric field of the plane wave of a single beam at the point $\mathbf{r}$ at time t can be written

$$\mathbf{E}(\mathbf{r}, t) = A\cos(\mathbf{k}\cdot\mathbf{r} - \omega t + \phi(t))\hat{\mathbf{p}}$$

where A is the amplitude, $\mathbf{k}$ is the wave number, ω the frequency, $\phi(t)$ the phase at time t and the unit vector $\hat{\mathbf{p}}$ gives the direction of polarization (which is in the plane of the wave, or equivalently, orthogonal to the direction of the ray). When we add up the fields from n beams we obtain

$$\mathbf{E}_{\text{total}}(\mathbf{r}, t) = \sum_{l=1}^{n} A_l\cos(\mathbf{k}_l\cdot\mathbf{r} - \omega_l t + \phi_l(t))\hat{\mathbf{p}}_l$$

The brightness (power per unit area) then is

$$I(\mathbf{r}, t) = \left\|\mathbf{E}_{\text{total}}(\mathbf{r}, t)\right\|^2 = \mathbf{E}_{\text{total}}(\mathbf{r}, t)\cdot\mathbf{E}_{\text{total}}(\mathbf{r}, t)$$

or

$$\left(\sum_{l=1}^{n} A_l\cos(\mathbf{k}_l\cdot\mathbf{r} - \omega_l t + \phi_l(t))\hat{\mathbf{p}}_l\right)\cdot\left(\sum_{m=1}^{n} A_m\cos(\mathbf{k}_m\cdot\mathbf{r} - \omega_m t + \phi_m(t))\hat{\mathbf{p}}_m\right)$$

or

$$\sum_{l=1}^{n}\sum_{m=1}^{n} A_l A_m\cos(\mathbf{k}_l\cdot\mathbf{r} - \omega_l t + \phi_l(t))\cos(\mathbf{k}_m\cdot\mathbf{r} - \omega_m t + \phi_m(t))\hat{\mathbf{p}}_l\cdot\hat{\mathbf{p}}_m$$

or

$$\begin{aligned}\sum_{l=1}^{n}\sum_{m=1}^{n}\frac{1}{2}A_l A_m\,&(\cos((\mathbf{k}_l + \mathbf{k}_m)\cdot\mathbf{r} - (\omega_l + \omega_m)t + (\phi_l(t) + \phi_m(t)))\\ &+ \cos((\mathbf{k}_l - \mathbf{k}_m)\cdot\mathbf{r} - (\omega_l - \omega_m)t + (\phi_l(t) - \phi_m(t))))\,\hat{\mathbf{p}}_l\cdot\hat{\mathbf{p}}_m\end{aligned}$$

The first cosine term fluctuates at twice the optical frequency and so on average does not contribute to the brightness (that is, the average of this term over one cycle is zero). To simplify the expression even further, let all the frequencies ω_l be the same (if they are not, we can always absorb any difference into a time varying phase difference).

$$I(\mathbf{r}, t) = \sum_{l=1}^{n}\sum_{m=1}^{n}\frac{1}{2}A_l A_m\cos((\mathbf{k}_l - \mathbf{k}_m)\cdot\mathbf{r} + (\phi_l(t) - \phi_m(t))\hat{\mathbf{p}}_l\cdot\hat{\mathbf{p}}_m$$

Separating out the terms for which $l = m$ and exchanging l and m for the terms where $l > m$ gives

$$I(\mathbf{r},t) = \sum_{l=1}^{n} \frac{1}{2} A_l^2 +$$

$$\sum_{l=1}^{n-1} \sum_{m=l+1}^{n} \frac{1}{2} A_l A_m (\cos((\mathbf{k}_l - \mathbf{k}_m) \cdot \mathbf{r} + (\phi_l(t) - \phi_m(t))) +$$

$$\cos((\mathbf{k}_m - \mathbf{k}_l) \cdot \mathbf{r} + (\phi_m(t) - \phi_l(t))))\hat{\mathbf{p}}_l \cdot \hat{\mathbf{p}}_m$$

Finally, simplifying and using the notation $\mathbf{k}_{lm} = (\mathbf{k}_l - \mathbf{k}_m)$, and $\phi_{lm} = (\phi_l - \phi_m)$, we are left with

$$I(\mathbf{r},t) = \frac{1}{2} A + \sum_{l=1}^{n-1} \sum_{m=l+1}^{n} A_l A_m \cos(\mathbf{k}_{kl} \cdot \mathbf{r} + \phi_{lm}(t))\hat{\mathbf{p}}_l \cdot \hat{\mathbf{p}}_m$$

where A is the sum of the A_l^2. If we arrange for the beams to have similar polarization directions $\hat{\mathbf{p}}_l$ then the dot-product of these vectors will be approximately one and can be dropped from the expression. So the speckle brightness pattern is the combination of a constant term and $n(n-1)/2$ spatial fringe interference patterns whose spatial phase is given by the optical phase difference of the pair of beams that produces each pair.

Target Interaction

Now let us introduce a target into the region of space containig the brightness pattern $I(\mathbf{r},t)$. The surface of the target will have some spatially varying contrast or reflectance given by $C(\mathbf{r})$. The signal from a detector positioned to receive light which has interacted with a region of the target is proportional to the integral of the product of the brightness and the contrast:

$$T(t) = \int C(\mathbf{r}) I(\mathbf{r},t)\, d\mathbf{r}$$

Substituting the equation developed above for $I(\mathbf{r},t)$ we obtain

$$T(t) = \int C(\mathbf{r}) \left(\frac{1}{2} A + \sum_{l=1}^{n-1} \sum_{m=l+1}^{n} A_l A_m \cos(\mathbf{k}_{kl} \cdot \mathbf{r} + \phi_{lm}(t))\hat{\mathbf{p}}_l \cdot \hat{\mathbf{p}}_m \right) d\mathbf{r}$$

or

$$\frac{1}{2} A \int C(\mathbf{r})\, d\mathbf{r} + \int C(\mathbf{r}) \sum_{l=1}^{n-1} \sum_{m=l+1}^{n} A_l A_m \cos(\mathbf{k}_{kl} \cdot \mathbf{r} + \phi_{lm}(t))\, d\mathbf{r}$$

or

$$\frac{1}{2}A\int C(\mathbf{r})\,d\mathbf{r}+$$

$$\sum_{l=1}^{n-1}\sum_{m=l+1}^{n} A_l A_m \Big(\cos\phi_{lm}(t)\int C(\mathbf{r})\cos(\mathbf{k}_{lm}\cdot\mathbf{r})\,d\mathbf{r}$$

$$-\sin\phi_{lm}(t)\int C(\mathbf{r})\sin(\mathbf{k}_{lm}\cdot\mathbf{r})\,d\mathbf{r}\Big)$$

or

$$T(t)=\frac{1}{2}AC_0+\sum_{l=1}^{n-1}\sum_{m=l+1}^{n} A_l A_m(\cos\phi_{lm}(t)C_{lm}-\sin\phi_{lm}(t)S_{lm})$$

where C_0, C_{lm} and S_{lm} are the corresponding integrals of the pattern $C(\mathbf{r})$ from the previous equation.

We see that the detector signal is the sum of a constant term and a time-dependent double-sum. The double sum enumerates every pair of laser beams. For each of these pairs, a contribution to the total signal is made by the cosine transform coefficient of $C(\mathbf{r})$ at the vector spatial frequency synthetisized by $\cos(\mathbf{k}_{lm}\cdot\mathbf{r})$ modulated by a time-dependent encoder, and the negative of the sine transform coefficient of $C(\mathbf{r})$ at the spatial frequency synthetsized by $\sin(\mathbf{k}_{lm}\cdot\mathbf{r})$ modulated by a different encoder.

We can think of the "encoders" $\cos\phi_{lm}(t)$ and $\sin\phi_{lm}(t)$ as carrier frequencies, and the detector output as the sum of a number of modulated carrier frequencies, where the modulation contains the information about the cosine and sine transforms C_{lm} and S_{lm} of the pattern $C(\mathbf{r})$.

If we discretize time using intervals of length δ and collect m measurements of the detector output we see that $\mathbf{T}$ equals

$$\begin{pmatrix} 1 & c_{12}(\delta) & -s_{12}(\delta) & c_{13}(\delta) & -s_{13}(\delta) & \dots & -s_{(n-1)n}(\delta) \\ 1 & c_{12}(2\delta) & -s_{12}(2\delta) & c_{13}(2\delta) & -s_{13}(2\delta) & \dots & -s_{(n-1)n}(2\delta) \\ & & & \vdots & & & \\ 1 & c_{12}(m\delta) & -s_{12}(m\delta) & c_{13}(m\delta) & -s_{13}(m\delta) & \dots & -s_{(n-1)n}(m\delta) \end{pmatrix}\begin{pmatrix} C_0 \\ C_{12} \\ S_{12} \\ C_{13} \\ S_{13} \\ \vdots \\ S_{(n-1)n} \end{pmatrix}$$

If $m \geq n(n-1)/2$, then given $\mathbf{T}$, we can solve these equations for the unknown cosine and sine transform coefficients appearing on the right using the pseudo-inverse