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APPENDIX I

CURVE FITTING BY METHOD OF LEAST SQUARES

Suppose we have a function g(x) defined at the n point x_1, $x_2 \ldots x_n$ and which to fit a function f(x) dependent on the m parameters a_1, $a_2 \ldots a_m$ such that the sum of the errors $[g(x_i) - f(x_i)]^2$ is least, ie :

$$\varepsilon = \sum_{i=1}^{n} [g(x_i) - f(x_i)]^2 \qquad \text{is minimal} \tag{1}$$

This is true if all derivatives of ε with respect to the parameters are zero :

$$\frac{\partial\varepsilon}{\partial a_1} = \frac{\partial\varepsilon}{\partial a_2} = \ldots \frac{\partial\varepsilon}{\partial a_m} = 0 \tag{2}$$

i.e.

$$\sum_{i=1}^{n} [g(x_i) - f(x_i)] \frac{\partial f(x_i)}{\partial a_j} = 0 \qquad j = 1, 2 \ldots m \tag{3}$$

or

$$\sum_{i=1}^{n} g(x_i) \frac{\partial f(x_i)}{\partial a_j} = \sum_{i=1}^{n} (x_i) \frac{\partial f(x_i)}{\partial a_j} \qquad j = 1, 2 \ldots m \tag{4}$$

To be able to solve these equations we chose a particularly simple relation for $f(x_i)$:

$$f(x_i) = a_1 f_1(x_i) + a_2 f_2(x_i) + \qquad a_m f_m(x_i) \tag{5}$$

i.e.

$$f(x_i) = \sum_{j=1}^{m} a_j f_j(x_i) \tag{6}$$

$$\therefore \frac{\partial f(x_i)}{\partial a_j} = f_j(x_i) \tag{7}$$

Considering (4), which reads in matrix notation :

$$\left|\begin{matrix} \frac{\partial f(x_1)}{\partial a_1} & \frac{\partial f(x_2)}{\partial a_1} & \cdots & \frac{\partial f(x_n)}{\partial a_1} \\ \frac{\partial f(x_1)}{\partial a_2} & \frac{\partial f(x_2)}{\partial a_2} & \cdots & \frac{\partial f(x_n)}{\partial a_2} \\ \vdots & & & \\ \frac{\partial f(x_1)}{\partial a_m} & \frac{\partial f(x_2)}{\partial a_m} & \cdots & \frac{\partial f(x_n)}{\partial a_m} \end{matrix}\right| \left|\begin{matrix} g(x_1) \\ g(x_2) \\ \\ g(x_n) \end{matrix}\right| = \left|\begin{matrix} \frac{\partial f(x_1)}{\partial a_1} & \frac{\partial f(x_2)}{\partial a_1} & \cdots & \frac{\partial f(x_n)}{\partial a_1} \\ \frac{\partial f(x_1)}{\partial a_2} & \frac{\partial f(x_2)}{\partial a_2} & \cdots & \frac{\partial f(x_n)}{\partial a_2} \\ \vdots & & & \\ \frac{\partial f(x_1)}{\partial a_m} & \frac{\partial f(x_2)}{\partial a_m} & \cdots & \frac{\partial f(x_m)}{\partial a_m} \end{matrix}\right| \left|\begin{matrix} f(x_1) \\ f(x_2) \\ \\ f(x_m) \end{matrix}\right| \tag{8}$$

From (6) :

$$\begin{vmatrix} f(x_1) \\ f(x_2) \\ \\ \\ f(x_n) \end{vmatrix} = \begin{vmatrix} f_1(x_1) & f_2(x_1) & \dots & f_m(x_1) \\ f_1(x_2) & f_2(x_2) & \dots & f_m(x_2) \\ \\ \\ f_1(x_n) & \dots & & f_m(x_n) \end{vmatrix} \begin{vmatrix} a_1 \\ a_2 \\ \\ a_m \end{vmatrix} \tag{9}$$

And part of (8) becomes

$$\begin{vmatrix} f_1(x_1) & f_1(x_2) & .. & f_1(x_n) \\ & & \vdots & \\ f_2(x_1) & f_2(x_2) & .. & \\ \\ \partial f_m(x_1) & f_m(x_2) & & f_m(x_n) \end{vmatrix} = \begin{vmatrix} \dfrac{\partial f(x_1)}{\partial a_1} & \dfrac{\partial f(x_2)}{\partial a_1} & .. & \dfrac{\partial f(x_n)}{\partial a_1} \\ \dfrac{\partial f(x_1)}{\partial a_2} & & & \\ \\ \dfrac{\partial f(x_1)}{\partial a_m} & & & \dfrac{\partial f(x_n)}{\partial a_m} \end{vmatrix} \tag{10}$$

Term the first matrix above F_{ij} or just F, the vector of coefficients A, and the vector of measured points G, then equation (8) becomes

$$FG = F(F'A) \tag{11}$$
$$FG + (FF')A \tag{12}$$

Now FF' is quare and will usually be non-singular :

$$A = (\ (FF')^{-1}\ F)\ .\ G \tag{13}$$

FF' is termed the normal matrix and is m-square and symmetrical. The mcoefficients can thus be found from the n measured points.

Now consider the special case

$$f(x_i) = A_0 + A_1 \cos(s_1x_i) + A_2 \cos(s_2x_i) + \dots A_m \cos(s_mx_i)$$
$$+ B_1 \sin(s_1x_i) + B_2 \sin(s_2x_i) + \dots B_m \sin(s_mx_i) \tag{14}$$

Also $x_1 = 0$ $\quad x_i = i\tau$

The matrix F now is : $\quad (n' = n - 1)$

$$\begin{vmatrix} 1 & 1 & 1 & .. & 1 \\ 1 & \cos(s_1\tau) & \cos(s_1 2\tau) & .. & \cos(s_1n'\tau) \\ 1 & \cos(s_2\tau) & \cos(s_2 2\tau) & .. & \cos(s_2n'\tau) \\ \cdot & & & & \cdot \\ \cdot & & & & \cdot \\ 1 & \cos(s_m\tau) & \cos(s_m 2\tau) & .. & \cos(s_mn'\tau) \\ 0 & \sin(s_1\tau) & \sin(s_1 2\tau) & .. & \sin(s_1n'\tau) \\ \cdot & & & & \\ \cdot & & & & \\ 0 & \sin(s_m\tau) & \sin(s_m 2\tau) & .. & \sin(s_mn'\tau) \end{vmatrix} \tag{15}$$

i.e. it contains n columns 2m + 1 rows. The terms in the product FF' will be :

Let $s_o = 0$

$$\mathcal{S}_{ij} = \sum_{\varkappa=0}^{n-1} \cos(s_i \varkappa\tau) \quad \cos(s_j \varkappa\tau) \qquad \left.\begin{array}{l}\text{For } 0 \leqslant i \leqslant m+1 \\ \text{and } 0 \leqslant j \leqslant m+1\end{array}\right\} \quad (16)$$

$$\mathcal{S}_{ij} = \sum_{\varkappa=0}^{n-1} \cos(s_i \varkappa\tau) \quad \sin(s_{j-m} \varkappa\tau) \qquad \left.\begin{array}{l}\text{For } 0 \leqslant i \leqslant m+1 \\ \text{and } m+1 \leqslant j < 2m+1\end{array}\right\} \quad (17)$$

$$\mathcal{S}_{ij} = \sum_{\varkappa=0}^{n-1} (\sin(s_{i-m} \varkappa\tau) \quad \cos(s_j \varkappa\tau) \qquad \left.\begin{array}{l}\text{For } m+1 \leqslant i < 2m+1 \\ \text{and } 0 \leqslant j < m+1\end{array}\right\} \quad (18)$$

$$\mathcal{S}_{ij} = \sum_{\varkappa=0}^{n-1} (\sin(s_{i-m} \varkappa\tau) \quad \sin(s_{j-m} \varkappa\tau) \qquad \left.\begin{array}{l}\text{For } m+1 \leqslant i < 2m+1 \\ \text{and } m+1 \leqslant j < 2m+1\end{array}\right\} \quad (19)$$

It follows that FF' is symmetrical as it ought to be. Instead of calculating $(2m+1)^2$ term we require only $\frac{(2m+1)(2m+2)/2}{2}$.

Consider (16) :

$$\mathcal{S}_{ij} = \sum_{\varkappa=0}^{n-1} \cos(s_i \varkappa\tau) \quad \cos(s_j \varkappa\tau) \qquad (20)$$

$$= 1/2 \left[\sum_{\varkappa=0}^{n-1} \cos(\varkappa(s_i + s_j)\tau) + \sum_{\varkappa=0}^{n-1} \cos(\varkappa(s_i - s_j)\tau) \right] \qquad (21)$$

Now :

$$\sum_{\varkappa=0}^{n-1} \cos(\varkappa\omega_0\tau) = \operatorname{Re} \sum_{\varkappa=0}^{n-1} e^{j\varkappa\omega_0\tau} \qquad (j = \sqrt{-1}) \quad (22)$$

$$= \operatorname{Re} \frac{1 - e^{jn\omega_0\tau}}{1 - e^{j\omega_0\tau}} = \operatorname{Re} \frac{e^{j\frac{n\omega_0\tau}{2}} \quad e^{j\frac{n\omega_0\tau}{2}} - e^{j\frac{n\omega_0\tau}{2}}}{e^{j\frac{\omega_0\tau}{2}} \quad e^{j\frac{\omega_0\tau}{2}} \quad e^{-j\frac{\omega_0\tau}{2}}} \qquad (23)$$

$$= \operatorname{Re} \, e^{j\frac{\omega_0\tau}{2}(n-1)} \frac{\sin\left(\frac{n\omega_0\tau}{2}\right)}{\sin\left(\frac{\omega_0\tau}{2}\right)} \qquad (24)$$

$$= \cos \quad \frac{\omega_0\tau}{2}(n-1)) \quad \frac{\sin\left(\frac{n\omega_0\tau}{2}\right)}{\sin\left(\frac{\omega_0\tau}{2}\right)} \qquad (25)$$

$$= \frac{1}{2} \quad \frac{\sin\left(\frac{\omega_0\tau}{2}\right) + \sin\left((2n-1)\frac{\omega_0\tau}{2}\right)}{\sin\left(\frac{\omega_0\tau}{2}\right)} \qquad (26)$$

$$\frac{1}{2}\left[1 + \frac{\sin\left((2n-1)\frac{\omega_0\tau}{2}\right)}{\sin\left(\frac{\omega_0\tau}{2}\right)}\right] \qquad (27)$$

$$\therefore \quad \mathcal{S}_{ij} = \frac{1}{4}\left[2 + \frac{\sin((2n-1)(s_i+s_j)\tau/2)}{\sin((s_i+s_j)\tau/2)} + \frac{\sin((2n-1)(s_j-s_i)\tau/2)}{\sin((s_j-s_i)\tau/2)}\right] \qquad (28)$$

Unless $s_j = s_i$

Then :

$$\phi_{ij} = \frac{1}{4}\left[2 + (2n-1) + \frac{\sin((2n-1)(s_i+s_j)\tau/2)}{\sin((s_i+s_j)\tau/2))}\right] \tag{29}$$

Unless $s_i = s_j = 0$

Then :

$$\phi_{ij} = \frac{1}{4}[2 + (2n-1) + (2n-1)] = n \tag{30}$$

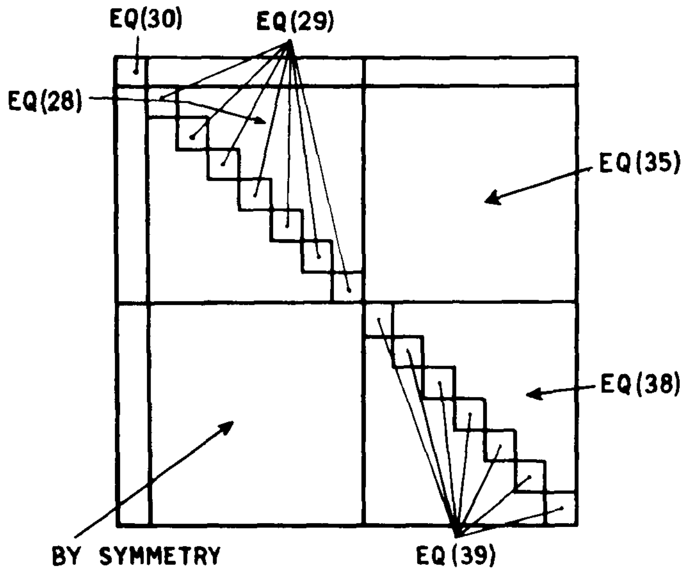

Figure A

Next consider the terms in the right hand top rectangle

$$\phi_{ij} = \sum_{\kappa=0}^{n-1} \cos(s_i\kappa\tau)\ \sin(s_{j-n}\kappa\tau) \tag{31}$$

$$= \frac{1}{2}\left[\sum_{\kappa=0}^{n-1} \sin(\kappa(s_i+s_{j-n})\tau) + \sum_{\kappa=0}^{n-1} \sin(\kappa(s_{j-n}-s_i)\ \tau)\right] \tag{32}$$

Now similarly to (25) :

$$\sum_{\kappa=0}^{n-1} \sin(\kappa\omega_o\tau) = \sin\left(\frac{\omega_o\tau}{2}\right)(n-1)\ .\ \frac{\sin\left(\frac{n\omega_o\tau}{2}\right)}{\sin\left(\frac{\omega_o\tau}{2}\right)} \tag{33}$$

$$\frac{1}{2}\left[\frac{\cos\left(\frac{\omega_o\tau}{2}\right) - \cos\left((2n-1)\ \frac{\omega_o\tau}{2}\right)}{\sin\left(\frac{\omega_o\tau}{2}\right)}\right] \tag{34}$$

$$\phi_{ij} = \frac{1}{4}\left[\frac{\cos((s_i+s_{j-n})\ \tau/2) - \cos((2n-1)\ (s_i+s_{j-n})\ \tau/2)}{\sin((s_i+s_{j-n})\ \tau/2)}\right] \tag{35}$$

$$+ \frac{\cos((s_{j-\blacksquare} - s_i)\ \tau/2)) - \cos((2n-1)(s_{j-\blacksquare} - s_i)\ \tau/2)}{\sin((s_{j-\blacksquare} - s_i\ \tau/2))} \Bigg]$$

Unless $s_{j-\blacksquare} = s_i$

Then :

$$\$_{ij} = \frac{1}{4} \left[\frac{\cos((s_i + s_{j-\blacksquare})\ \tau/2) - \cos((2n-1)\ (s_i + s_{j-\blacksquare})\ \tau/2))}{\sin((s_i + s_{j-\blacksquare})\ \tau/2)} \right]$$

Finally the lower right triangle :

$$\$_{ij} = \sum_{\kappa=0}^{n-1} \sin(s_{i-\blacksquare}\kappa\tau)\ \sin(s_{j-\blacksquare}\ \kappa\tau) \tag{36}$$

$$= \frac{1}{2} \left[\sum_{\kappa=0}^{n-1} \cos(\kappa(s_{i-\blacksquare} - s_{j-\blacksquare})\tau) - \sum_{\kappa=0}^{n-1} \cos(\kappa(s_{i-\blacksquare} + s_{j-\blacksquare})\tau) \right] \tag{37}$$

By using method applied to arrive at (28) :

$$\$_{ij} = \frac{1}{4} \left[\frac{\sin((2n-1)\ (s_{j-\blacksquare} - s_{i-\blacksquare})\ \tau/2)}{\sin((s_{j-\blacksquare} - s_{i-\blacksquare})\ \tau/2)} - \frac{\sin((2n-1)\ (s_{j-\blacksquare} + s_{i-\blacksquare})\ \tau/2)}{\sin((s_{i-\blacksquare} + s_{j-\blacksquare})\ \tau/2)} \right] \tag{38}$$

Unless $s_{j-\blacksquare} = s_{i-\blacksquare}$:

Then

$$\$_{ij} = \frac{1}{4} \left[(2n-1) - \frac{\sin((2n-1)\ (s_{j-\blacksquare} + s_{i-\blacksquare})\ \tau/2)}{\sin((s_{i-\blacksquare} + s_{j-\blacksquare})\ \tau/2)} \right] \tag{39}$$

In addition one can use :

$$\sin((s_i + s_j)\ \tau/2) = \sin(s_i\ \tau/2)\ \cos(s_j\ \tau/2) + \sin(s_j\ \tau/2)\ \cos(s_i\ \tau/2) \tag{40}$$

$$\sin((s_i - s_j)\ \tau/2) = \sin(s_i\ \tau/2)\ \cos(s_j\ \tau/2) - \sin(s_j\ \tau/2)\ \cos(s_i\ \tau/2) \tag{41}$$

$$\cos((s_i + s_j)\ \tau/2) = \cos(s_i\ \tau/2)\ \cos(s_j\ \tau/2) - \sin(s_i\ \tau/2)\ \sin(s_j\ \tau/2) \tag{42}$$

$$\cos((s_i - s_j)\ \tau/2) = \cos(s_i\ \tau/2)\ \cos(s_j\ \tau/2) + \sin(s_i\ \tau/2)\ \sin(s_j\ \tau/2) \tag{43}$$

All terms can thus be derived from four tables :

$$\sin(s_i\ \tau/2),\ \cos(s_i\ \tau/2)$$

$$\sin((2n-1)s_i\ \tau/2),\ \cos((2n-1)s_i\ \tau/2)$$

APPENDIX II
TIDAL ANALYSIS PROGRAMME

The programme given below is written in Manchester Auto-Code (MAC), which is compatible with Extended Mercury Auto-Code (EMA). This programme is therefore acceptable to computers such as ATLAS and ORION II which have EMA compilers.

A few notes on the peculiarities of MAC should make it easier to follow the programme.

As is usual in programming languages, a distinction is made between integers and non-integers. In MAC the former are held in the machine in fixed point form, and are referred to by any of the 12 symbols

I J K L M N O P Q R S T

These are termed indices, and can be used as subscripts in one dimensional arrays.

The remaining letters of the alphabet A - H and U - Z are reserved for numbers held in floating point form to a precision of eight significant figures. It is also permissible to denote floating point numbers by primed letters of the alphabet in the range A' - H' and U' - Z'.

Finally the location with the symbolic address π permanently contains the number 3.14159267.

Brackets may not be used to group multiplications, but can be used to enclose a subscript. Examples are

A3 means A_3

AP means A_p

A(P + 3) means $A_{(p+3)}$

but PA means the integer P multiplied by the floating point number A. The cummutative property of multiplication between integers and non-integers is thus not preserved.

Behind the Immediate Access Core Store is a backing store consisting of one or more drums. Access to and from drum store is by means of the φ6 and φ7 instructions as indicated below :

φ6 (X) AO, N

This transfers the N values A_0, A_1 .... A_{n-1} from consecutive drum store addresses starting with drum store address X to consecutive Immediate Access Store addresses starting with AO. Similarly the instructions

φ7 (X) AO, N

reads N consecutive I.A.S. locations and writes them to drum.

At the head of the programme it is necessary to place dimension statements (directives) specifying the maximum size of the various arrays used, i.e. the directive

$A \longrightarrow 103$

reserves 104 storage locations for the subscripted array A_0, A_1 ... A_{103}.

Finally, as in general a programme is too large to be held in its entirely in the Immediate Access Store, it is divided up into chapters. The programme as a whole is held on the drum, and each chapter is brought down into the core store as required. The chapters are labelled

CHAPTER O, CHAPTER 1, CHAPTER 2, etc.

CHAPTER O is the first chapter to be obeyed, but the last one to be read in.

Instructions are punched one to a card, and comment can be added, provided it is separated by one or more blank columns from the instruction.

Since tide gauge readings are normally rounded off to the nearest tenth of a foot, it is our custom to punch sea heights as an integral number of tenths of a foot, and to restore the decimal point by programme.

MAC PROGRAMME FOR TIDAL ANALYSIS.

```
CHAPTER 1             TIDAL ANALYSIS FOR UP TO 51 CONSTITUENTS
A ⟶ 102
B ⟶ 102
C ⟶ 102
BO = O                SPEED FOR MEAN SEA LEVEL
READ (Q)              NO. OF CONSTITUENTS
READ (K)              NO. OF HOURLY OBSERVATIONS
K = K - 1
M = 2Q
T = Q + 1
P = 1(1) Q
READ (BP)             SPEEDS (DEG./MEAN SOLAR HOUR)
REPEAT
φ7 (A') B1, Q         SPEEDS TO DRUM
P = 1(1) Q
BP = GBP              SPEEDS TO RADIANS
REPEAT
P = O(1) K
READ (A)              HOURLY HEIGHTS
A = 0.1A              RESTORE DECIMAL POINT
AO = AO + A
J = 1(1) Q
B = φCOS(PBJ)
AJ = AJ + AB
REPEAT
J = T(1) M
I = J - Q
B = φSIN(PBI)
AJ = AJ + AB
REPEAT
REPEAT
CLOSE
CHAPTER 2
VARIABLES 1
P = O(1) M
CP = O
N = O(1) M
READ(BN)              ROW OF INVERSE MATRIX
CP = CP + ANBN
REPEAT
REPEAT
CLOSE
CHAPTER 3
VARIABLES 1
P = 1(1) Q
READ (BP)             NODE FACTORS
REPEAT
φ7(D') B1, Q          NODE FACTORS TO DRUM
P = 1(1) Q
READ (BP)             EQUILIBRIUM ARGUMENTS
REPEAT
φ7(E') B1, Q          EQUILIBRIUM ARGUMENTS TO DRUM
READ (X)              HOURS FROM START OF YEAR
```

```
PRINT 'TIDAL CONSTITUENTS'
PRINT 'HOURS FROM START OF YEAR'
PRINT (X) 4, 0
NEWLINE 2
PRINT 'MEAN SEA LEVEL'
PRINT (CO)2, 2
NEWLINE 2
SPACE 8
PRINT 'SPEED AMPLITUDE G NODE FACTOR EQ. ARG.'
NEWLINE 2
CLOSE
CHAPTER 4
VARIABLES 1
P = 1(1) Q
S = P + Q
E = φRADIUS (CP, CS)
F = φARCTAN (CP, - CS)
φ6(D' + P - 1)Z, 1     CALL DOWN P TH. NODE FACTOR
E = E/Z
F = F/G        CONVERT FROM RADIANS TO DEGREES
DOWN 2/5
PRINT (P) 2, O
SPACE
φ6(A' + P - 1)Z, 1     CALL DOWN P TH. SPEED
PRINT (Z)2, 6  PRINT P TH. SPEED
PRINT (E)2, 4  PRINT P TH. AMPLITUDE
SPACE
PRINT (F)3, 2  PRINT P TH. LAG
SPACE 3
φ6(D' + P - 1)Z, 1     CALL DOWN P TH. NODE FACTOR
PRINT (Z)1, 3  PRINT P TH. NODE FACTOR
SPACE 3
φ6(E' + P - 1)Z, 1     CALL DOWN P TH. EQ. ARG.
PRINT (Z)3, 1  PRINT P TH. EQ. ARG.
NEWLINE 2
REPEAT
END
CLOSE
CHAPTER 5
VARIABLES 1
2) φ6(A' + P - 1)V, 1  CALL DOWN P TH. SPEED
φ6(E' + P - 1)Z, 1     CALL DOWN P TH. EQ. ARG.
F = Z + XV - F
3) JUMP 4, F ⩾ 0.0
F = F + 360
JUMP 3
4) JUMP 5, 360.0 > F
F = F - 360
JUMP 4
5) UP
CLOSE
CHAPTER O
VARIABLES 1
G = π/180      TO CONVERT DEG. TO RADIANS
A' = O
D' = 1000
E' = 2000
CLOSE
```

DISCUSSION (Chairman : M. GOUGENHEIM)

LE PRESIDENT remercie Mr. Shipley de son intéressant exposé qui va sans doute soulever un certain nombre de questions de la part des participants.

L'amiral FRANCO fait remarquer qu'une première question se pose qui est liée à la communication que va faire M. Van Ette.

Mr. Shippley a parlé de la conformation des matrices pour la solution du problème de la détermination des constantes harmoniques, et l'Amiral Franco pense qu'il serait intéressant d'entendre l'opinion du Professeur Schoemaker à ce sujet.

D'autre part, Mr. Shipley a dit qu'il employait le système complet, c'est-à-dire non divisé par l'heure centrale, ce qui double le nombre d'inconnues.

Il a dit également que, malgré cela, le temps employé pour obtenir les résultats est le même que si l'on employait l'heure centrale. Or deux personnes sont présentes qui peuvent également intervenir à ce sujet, ce sont MM. Cartwright qui a employé pour sa méthode de Fourier dans l'analyse des marées la formule de Watt pour déterminer les termes connus des équations normales, et M. Horn (un autre M. Horn) qui emploie l'heure centrale et qui pourrait dire aussi quelques mots à ce sujet.

Pr. SCHOEMAKER (Netherlands) said it was rather difficult to dive straight into this matter and suggested they should hold a simultaneous discussion on Mr Shipley's paper and the paper to be presented by Mr van Ette and himself, because he and Mr van Ette had something to add on the special subject of matrices and taking central points of observations. He wondered whether other delegates would prefer to start the discussion now or to wait until his and Mr van Ette's paper had also been presented.

LE PRESIDENT indique que Mr. Horn préfère attendre et demande la position de Mr. Cartwright sur ce point.

Mr. CARTWRIGHT (United Kingdom) had not come prepared to offer any contribution on this subject, partly because he had not himself done any research on harmonic tidal analysis for some years. However, he saw that it was really a question of comparison between the least squares analysis and the Fourier method. There was really little difference because the least squares method was that which gave a minimum to the expected error or expected variance of error, and since it was a minimum one had to differ considerably from the method of least squares to obtain an error which was much greater. It was simply a question of choosing one's filter and deciding whether one had corrected for all the side band effects of that filter. As far as he could see, there was no essential difference between the two methods. Of course, the least squares method had the advantage that it could be very readily adapted to any set of data, even if it had large gaps in it, whereas the Fourier method could only be conveniently applied to a continuous series of data.

LE PRESIDENT pense qu'à vrai dire on ne peut pas parler d'erreur ; on peut parler d'incertitude sur le résultat. La méthode des moindres carrés donne la solution la plus probable ; les méthodes qui en diffèrent légèrement donnent des solutions de probabilité un peu moindres, c'est-à-dire une incertitude légèrement plus grande ; mais le résultat peut être aussi exact en soi-même. C'est simplement l'incertitude qui l'affecte qui peut être plus grande.

LE PRESIDENT estime, avant d'ouvrir la discussion sur la méthode de Mr. Shipley, qu'il serait préférable d'entendre les exposés de MM. Van Ette et Schoemaker. Comme sur ce point vient en numéro 3, le Président suggère de suivre le programme et de demander à Mr. Zetler de bien vouloir présenter sa communication.

RECENT DEVELOPMENTS IN TIDAL ANALYSIS IN SOUTH AFRICA

A. M. SHIPLEY

C. S. I. R. Oceanographic Research Unit, University of Cape Town

INTRODUCTION

Ideally a method of Tidal Analysis should be able to do the following :

1/ Deal with a record of any length, up to a year or more.

2/ Handle a record which has gaps in it.

3/ Use any number of constituents within reason (say up to 64).

The above scheme is rather ambitious unless one has a very large and fast computer and up to now we have refrained from attempting the analysis of records with large gaps. A year's record with a gap of say a week or so can be handled by using a portion of the record, of 3 or 6 months' duration immediately before or after the gap, and from the resulting analysis interpolating the missing hourly heights. One then starts from scratch again and uses the now complete year's record for a major analysis.

MATHEMATICAL MODEL

The mathematical model used was the usual one, i.e. the sea level at any instant was assumed to be given by

$$z_t = A_0 + \sum_{r=1}^{n} f_r H_r \cos(S_r t + u_r - g_r) \qquad (1)$$

where n is the number of constituents used and the rest of the symbols have their usual meanings. By means of the addition formula for the cosine function equation (1) can be written as

$$z_t = A_0 + \sum_{r=1}^{n} A_r \cos S_r t + \sum_{r=1}^{n} B_r \sin S_r t \qquad (2)$$

where

$$\left.\begin{aligned} A_r &= f_r H_r \cos(u_r - g_r) \\ B_r &= - f_r H_r \sin(u_r - g_r) \end{aligned}\right\} \qquad (3)$$

Hence if the A_r and B_r are known the tidal constants are given by

$$\left.\begin{aligned} H_r &= \frac{1}{f_r} \sqrt{A_r^2 + B_r^2} \\ g_r &= u_r - \text{ARCTAN} \left\{ \frac{A_r}{-B_r} \right\} \end{aligned}\right\} \qquad (4)$$

The quadrant in which the inverse tangent lies must be chosen so that the numerator has the same sign as sin g_r and the denominator the same sign as cos g_r. On our I.C.T. 1301 machine this is automatically taken care of by the compiler. (i.e. ARCTAN (X/Y) is regarded as a function of two arguments.)

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Consider now the case where we have m consecutive hourly observations of sea level, taken at hours 0, 1, 2, ... (m-1).

Then taking t = 0, 1, (m-1)

equations (2) become m equations of condition.

Write them in matrix form as CX = Z (5)

Here C is a matrix with m rows and (2n + 1) columns. It is in fact the matrix whose (p + 1)th row is

$1 \quad \cos S_1 p \ \ldots \ \cos S_n p \ \sin S_1 p \ \ldots \ \sin S_n p$

X is the column vector

$$X = \begin{pmatrix} A_0 \\ A_1 \\ A_2 \\ . \\ A_n \\ B_1 \\ . \\ . \\ B_n \end{pmatrix} \tag{6}$$

And Z is the column vector

$$Z = \begin{pmatrix} Z_0 \\ Z_1 \\ . \\ . \\ . \\ Z_m \end{pmatrix} \tag{7}$$

i.e. Z is the column vector of hourly sea levels.

Note that counting starts at the zero-ith sea level (Z_0).

Let C' be the transpose of C. To obtain the normal equations we must pre-multiply equations (5) by C'

$$C'CX = C'Z \tag{8}$$

The matrix CC' is square and in general non-singular.

Denote its inverse by B

$$B = (C'C)^{-} \tag{9}$$

Then the solution to the normal equations (8) is

$$X = BC'Z$$

where X is the solution vector whose transpose is (A_0 .. A_n, B_1 .. B_n

This in conjunction with equations (4) enables us to find the tidal constants and thus solve the problem.

PRACTICAL CONSIDERATIONS

The above theory is quite straightforward, but as is to be expected a number of practical difficulties arose when trying to implement it. One of these is the enormous amount of machine time to generate and invert the normal matrix. For this reason it was decided to choose certain fixed record lengths each with its corresponding set of constituents, and in each case to generate and invert the matrix on a once for all time basis. The inverse matrix could then be kept on punched cards or magnetic tape and used over and over again.

As a beginning we used the set of twelve constituents listed in TABLE N°. 1, and a record length of 697 hourly observations. (i.e. 29 Days). The reason for this choice of constituents is that they are the ones used on the Doodson-Légè machine which the South African Hydrographic Office has used in the past for doing predictions.

Table No. 1

Tidal Constituents Used in 29 day Analysis.

No.	NAME	SPEED
00	M.S.L.	00.000 000 000 0
01	M_2	28.984 104 237 3
02	S_2	30.000 000 000 0
03	N_2	28.439 729 541 5
04	K_2	30.082 137 278 6
05	μ_2	27.968 208 474 6
06	M_4	57.968 208 474 6
07	MS_4	58.984 104 237 3
08	J_1	15.585 443 335 1
09	Q_1	13.398 660 902 2
10	P_1	14.958 931 360 7
11	O_1	13.943 035 598 0
12	K_1	15.041 068 639 3

On using analyses obtained by this method it was found that the results were not very good, in the sense that the agreement between computed and observed sea level left much to be desired. The author must thank Vice-Admiral A. dos Santos Franco for pointing out the reason for this. The noise level in a tidal record is such that the signal to noise ratio prohibits the separation of close constituents such as K_2 and S_2 or P_1 and K_1, unless a record of length considerably longer than 29 days is analysed, or unless certain corrections dependent on equilibrium tide theory are applied. The method of applying these corrections is very lucidly explained in an article by dos Santos Franco in the I.H. Review for July 1964.

On thinking matters over it was decided that the first of the two alternatives would be the better one, for the reasons set out below.

1/ In the case of every harbour for which we have records, these records have an unbroken length far exceeding 29 days. It was felt that if one had a record of say four months' duration one would get better results by analysing the record as a whole rather than doing four separate monthly analyses and taking vectorial means.

2/ It was felt that with a record of a length of three months or longer, it would be realistic to look for far more than 12 constituents in the analysis. Accordingly for 3, 4 and 6 months' records, inverse normal matrices for 25 constituents were prepared. These constituents are listed in TABLE N° 2. For a record extending to 369 days, we felt we could do even better, and in consequence an inverse normal matrix for the 51 constituents, listed in TABLE N° 3., was computed.

Table No. 2

Constituents Used in 3, 4 and 6 month Analysis.

No.	NAME	SPEED
00	M.S.L.	00.000 000 000 0
01	Mm	00.544 374 695 8
02	MSf	01.015 895 762 7
03	Mf	01.098 033 041 3
04	$2Q_1$	12.854 286 206 5
05	Q_1	13.398 660 902 2
06	O_1	13.943 035 598 0
07	P_1	14.958 931 360 7
08	S_1	15.000 000 000 0
09	K_1	15.041 068 639 3
10	J_1	15.585 443 335 1
11	$2N_2$	27.895 354 845 8
12	μ_2	27.968 208 474 6
13	N_2	28.439 729 541 5
14	ν_2	28.512 583 170 4
15	M_2	28.984 104 237 3
16	λ_2	29.455 625 304 2
17	L_2	29.528 478 933 1
18	T_2	29.958 933 322 4
19	S_2	30.000 000 000 0
20	K_2	30.082 137 278 6
21	M_3	43 476 156 356 0
22	SK_3	45.041 068 639 3
23	M_4	57.968 208 474 6
24	MS_4	58.984 104 237 3
25	S_4	60.000 000 000 0

The University of Cape Town has an I.C.T. 1301 computer. This is a fixed word length, binary coded decimal machine, which operates with an 8-digit mantissa when doing arithmetic in the floating point mode. Normal matrices have the reputation of being ill-conditioned, so that it was felt that double precision arithmetic should be used for their generation and inversion. For this reason the Computing Centre of the University of the Witwatersrand (who have a variable word length machine) was commissioned to do the "once for all time" generation and inversion. In the case of a 369 days' record with 51 constituents, an eighteen decimal digit mantissa was used, and in the other cases 16 digits. The final results were in all cases rounded off to eight significant figures.

Table No. 3

Constituents Used in 369 Days Analysis.

No.	NAME	SPEED
00	M. S. L.	00.000 000 000 0
01	Sa	00.041 068 639 3
02	Ssa	00.082 137 278 6
03	Mm	00.544 374 695 8
04	MSf	01.015 895 762 7
05	Mf	01.098 033 041 3
06	$2Q_1$	12.854 286 206 5
07	σ_1	12.927 139 835 3
08	Q_1	13.398 660 902 2
09	ρ_1	13.471 514 531 1
10	O_1	13.943 035 598 0
11	M_1	14.496 693 943 6
12	π_1	14.917 864 683 1
13	P_1	14.958 931 360 7
14	S_1	15.000 000 000 0
15	K_1	15.041 068 639 3
16*	Ψ_1	15.082 135 300 0
17	φ_1	15.123 205 918 0
18*	θ_1	15.512 589 700 0
19	J_1	15.585 443 335 1
20	SO_1	16.056 964 402 0
21	OO_1	16.139 101 680 6
22*	OQ_1	27.341 696 400 0
23	MNS_2	27.423 833 778 9
24	$2N_2$	27.895 354 845 8
25	μ_2	27.968 208 474 6
26	N_2	28.439 729 541 5
27	ν_2	28.512 583 170 4
28	OP_2	28.901 966 958 7
29	M_2	28.984 104 237 3
30	MSK_2	29.066 241 516 0
31	λ_2	29.455 625 304 2
32	L_2	29.528 478 933 1
33	T_2	29.958 933 322 4
34	S_2	30.000 000 000 0
35*	R_2	30.041 066 700 0
36	K_2	30.082 137 278 6

No.	NAME	SPEED
37	MSN_2	30.544 374 695 8
38	KJ_2	30.626 511 974 4
39	$2SM_2$	31.015 895 762 7
40	M_3	43.476 156 356 0
41	MK_3	44.025 172 876 6
42	SK_3	45.041 068 639 3
43	MN_4	57.423 833 778 9
44	M_4	57.968 208 474 6
45	MS_4	58.984 104 237 3
46	MK_4	59.066 241 516 0
47	S_4	60.000 000 000 0
48	SK_4	60.082 137 278 6
49	MSN_6	87.423 833 778 9
50	$2MS_6$	87.968 208 474 6
51	$2MK_6$	88.050 345 753 3

(*) Those constituents marked with asterisks are the ones for which we have been unable to obtain the speeds to 12 significant figures. The last three or four unknown significant figures have been filled in with non-significant zeroes.

The I.C.T. 1301 machine will not accept programmes written in FORTRAN, but uses a source language called "Manchester Auto-Code" (MAC). Moreover, the punching format for "MAC" is not compatible with that for FORTRAN. However, the analysis programme has now been rewritten in FORTRAN IV and the matrices repunched in a FORTRAN format. This has been done so as to run the programme on an IBM 360.

Horn (1960) has pointed out that if an odd number of observations is used and the time origin is taken at the central observation, the matrix can be partitioned into two sub-matrices, one of order (m + 1) and the other of order m. This would save a certain amount of machine time in the generation and inversion of the matrices, but we did not do this for the following two reasons :

1/ The calculation has to be done once only.

2/ Mr. B.K.P. Horn, of the Witwatersrand Computing Centre (unrelated to the Horn mentioned above), found numerous short cuts in the generation of the normal matrix. Mr. Horn's method is given in detail in appendix I. The time origin is zero hour on the first day of the record.

The actual analysis programme written in "MAC" is given in appendix II. Programmes and matrices for both "MAC" and FORTRAN compilers can be supplied on punched cards.

Without entering into the pros and cons of one source language, as opposed to another, it is worth pointing out that, as will be seen from appendix II, in "MAC" it is permissible to use zero as a subscript. This is not the case in any of the "dialects" of FORTRAN, and must be borne in mind if the programme is translated into FORTRAN.

Table No. 4

Inverse normal matrix for 4,177 consecutive hourly observations

1	0.24050624 /-03	-0.10485283 /-04	0.59988397 /-05	0.57578000 /-05	-0.35202393 /-06	-0.22168303 /-06
2	-0.10485283 /-04	0.48703973 /-03	-0.56257361 /-05	-0.75720544 /-05	0.23023566 /-06	0.80887109 /-07
3	0.59988397 /-05	-0.56257361 /-05	0.48353740 /-03	0.25596157 /-04	-0.30134700 /-06	-0.89259623 /-07
4	0.57578000 /-06	-0.75720544 /-05	0.25596157 /-04	0.48105221 /-03	-0.10245732 /-06	0.46599105 /-07
5	-0.35202393 /-06	0.23023566 /-06	-0.30134700 /-06	-0.10245732 /-06	0.47983743 /-03	-0.10729703 /-04
6	-0.22168303 /-06	0.80887109 /-07	-0.89259623 /-07	0.46599105 /-07	-0.10729703 /-04	0.48139603 /-03
7	0.42062626 /-06	-0.36718653 /-06	0.23107446 /-06	-0.41210443 /-07	0.42298378 /-05	-0.10720854 /-04
8	0.10350529 /-06	0.41094046 /-05	0.25861967 /-05	0.34970862 /-05	0.24184905 /-06	0.26316074 /-04
9	-0.77050565 /-06	-0.31879032 /-06	-0.13877581 /-05	-0.11874929 /-05	0.63786680 /-05	0.99040321 /-05
10	-0.31829399 /-07	-0.44171545 /-05	-0.24767345 /-05	-0.34509389 /-05	-0.46045719 /-05	-0.31033438 /-04
11	0.26284416 /-06	-0.49957147 /-06	-0.46560616 /-07	-0.28455816 /-06	0.18812530 /-05	-0.28322517 /-05
12	0.14285933 /-06	-0.15172311 /-06	0.83953402 /-07	-0.10392912 /-07	0.94120805 /-07	-0.51759686 /-06
13	-0.12138092 /-06	-0.62050428 /-07	-0.13202844 /-06	-0.86212884 /-07	-0.45716873 /-06	-0.10194172 /-06
14	0.82060880 /-07	-0.15319561 /-06	-0.22179187 /-07	-0.97303492 /-07	0.31107796 /-06	-0.24650429 /-06
15	0.20164692 /-06	-0.30629903 /-06	0.29867001 /-07	-0.13075421 /-06	0.38793613 /-06	-0.72208196 /-06
16	-0.28035765 /-06	0.32118023 /-07	-0.26593944 /-06	-0.13164638 /-06	-0.56031522 /-06	0.33934050 /-06
17	0.25551643 /-06	-0.22055949 /-06	0.14744653 /-06	-0.15595661 /-07	0.40076908 /-06	-0.67573519 /-06
18	0.55060482 /-07	-0.22335513 /-06	-0.43694594 /-07	-0.11404439 /-06	-0.12178111 /-06	-0.55473379 /-06
19	0.89057523 /-07	0.52844070 /-06	0.44638864 /-06	0.53561169 /-06	-0.92546775 /-07	0.61743332 /-06
20	-0.10160316 /-05	-0.62389875 /-06	-0.14186464 /-05	-0.11054498 /-05	-0.19233919 /-05	0.44630910 /-07
21	-0.42479283 /-06	-0.35834364 /-06	-0.66058714 /-06	-0.55386493 /-06	-0.72202268 /-06	-0.88883758 /-07
22	-0.16144058 /-06	-0.50609091 /-07	-0.18627934 /-06	-0.12348530 /-06	-0.34096820 /-06	0.36188097 /-07
23	-0.20937322 /-07	-0.14728467 /-06	-0.96078027 /-07	-0.11821442 /-06	-0.11954710 /-06	-0.23989263 /-06
24	-0.49980617 /-07	-0.11886801 /-06	-0.11293000 /-06	-0.11615885 /-06	-0.14040511 /-06	-0.15386529 /-06
25	-0.17608917 /-06	-0.40859698 /-07	-0.20124613 /-06	-0.13067910 /-06	-0.31604189 /-06	0.80517924 /-07
26	-0.12595394 /-06	-0.77195263 /-07	-0.17596906 /-06	-0.13718769 /-06	-0.20154737 /-06	0.63909633 /-08
27	-0.15934646 /-04	-0.99422771 /-05	0.27779017 /-04	0.23273529 /-04	-0.55953005 /-06	-0.35139006 /-06
28	-0.48248542 /-05	-0.29633213 /-04	-0.66324865 /-05	-0.30223717 /-05	0.75624535 /-06	0.52325225 /-06
29	-0.62412000 /-05	-0.23838375 /-04	-0.10533228 /-04	-0.66412365 /-05	0.80982602 /-06	0.52681642 /-06
30	-0.12638674 /-06	-0.12304088 /-05	-0.89729443 /-06	-0.11317983 /-05	-0.22101678 /-06	0.15903624 /-04
31	-0.91660547 /-06	-0.61878724 /-06	-0.13340260 /-05	-0.10665987 /-05	-0.16866295 /-04	-0.86888986 /-07
32	-0.45014676 /-06	-0.90009426 /-06	-0.88910702 /-06	-0.87373759 /-06	-0.95385359 /-05	-0.16078892 /-04
33	-0.13932649 /-05	-0.47181517 /-06	-0.19387356 /-05	-0.14591290 /-05	-0.22541526 /-05	-0.76393170 /-05
34	-0.55124146 /-10	-0.68005576 /-05	-0.39279399 /-05	-0.54803828 /-05	-0.32413137 /-05	-0.46220789 /-04
35	-0.42845006 /-06	-0.45166664 /-07	-0.20268587 /-06	0.29170824 /-07	-0.96507669 /-05	-0.43438956 /-05
36	-0.20237570 /-0[illegible]	-0.10300818 /-05	-0.72803636 /-06	-0.85783883 /-06	-0.29355322 /-05	-0.59118387 /-05
37	-0.51510188 /-06	-0.39146959 /-06	-0.73395760 /-06	-0.58699600 /-06	-0.12700412 /-05	-0.26370640 /-06
38	-0.55296018 /-06	-0.34783872 /-06	-0.76940359 /-06	-0.59944094 /-06	-0.11665763 /-05	-0.43115157 /-07
39	-0.26417522 /-07	-0.67932210 /-06	-0.40053274 /-06	-0.53972769 /-06	-0.30199684 /-06	-0.12581460 /-05
40	-0.20682537 /-06	-0.56083795 /-06	-0.50536760 /-06	-0.53409525 /-06	-0.74429265 /-06	-0.92366383 /-06
41	-0.22549112 /-06	-0.39183542 /-06	-0.47218669 /-06	-0.46234031 /-06	-0.39350277 /-06	-0.38624847 /-06
42	-0.38831215 /-06	-0.49884838 /-06	-0.65597944 /-06	-0.58882010 /-06	-0.10765123 /-05	-0.61409217 /-06
43	-0.57561371 /-06	-0.38688845 /-06	-0.79766828 /-06	-0.62319656 /-06	-0.13231154 /-05	-0.16967342 /-06
44	-0.11976265 /-05	-0.70652442 /-06	-0.16356059 /-05	-0.12562176 /-05	-0.24335175 /-05	-0.27091118 /-08
45	-0.10895063 /-12	-0.86101077 /-06	-0.53378166 /-06	-0.73637258 /-06	0.12521629 /-06	-0.13294046 /-05
46	0.63465675 /-07	-0.63877521 /-06	-0.31279044 /-06	-0.48837971 /-06	0.86961786 /-07	-0.11244071 /-05
47	-0.26052400 /-06	-0.19306872 /-06	-0.38160530 /-06	-0.30865764 /-06	-0.46218063 /-06	-0.51741598 /-07
48	-0.28183409 /-06	-0.16689352 /-06	-0.38062138 /-06	-0.29071163 /-06	-0.54643588 /-06	-0.25349204 /-07
49	-0.22769069 /-06	-0.12333613 /-06	-0.30236090 /-06	-0.22740401 /-06	-0.41452329 /-06	0.32243028 /-08
50	-0.14162821 /-06	-0.17320416 /-06	-0.24751054 /-06	-0.22349893 /-06	-0.24436564 /-06	-0.12258909 /-06
51	-0.27012532 /-13	-0.25718532 /-06	-0.14804612 /-06	-0.20641936 /-06	-0.34433389 /-07	-0.37253904 /-06

Table N° 4 is put in purely for illustrative purposes to show the format in which the matrices are punched on cards. This is a floating point format, retaining eight significant figures irrespective of how small the number is.

For example

$$0.73618984 \ / - 09$$

means

$$0.73618984 \times 10^{-9}$$

The first six columns of the inverse matrix for a six months' record are shown.

In conclusion I should like to thank the Hydrographic Office of the South African Navy for their help and collaboration, and for providing the funds necessary for producing the matrices.