On the chain stability of bilateral control model

Liang Wang[a] and Berthold K.P. Horn[b*]

***Abstract*—In this paper, we study the chain stability of vehicles under bilateral control (BCM), and prove that vehicles under bilateral control are chain stable: i.e. *all* input perturbations to the chain decay exponentially (with the length of the chain). Chain stability analysis tells us how vehicles under bilateral control will act in traffic when mixed with cars driven by human drivers. It shows that self-driving cars using bilateral control can reduce traffic flow instabilities in mixed traffic. Indeed, chains of BCM vehicles become perturbation-consuming dampers when inserted in traffic, since they split chains of human-driven vehicles and prevent perturbations from being transmitted from one chain of car-following cars to the next. Thus today's traffic can be improved greatly by the insertion of BCM vehicles. The simulation results validate the theoretical analysis.**

***Index Terms*—bilateral control, car-following model, mixed traffic, chain stability, adaptive cruise control (ACC), self-driving system.**

I. Introduction

As the result of the development of sensors and artificial intelligence, self-driving cars and self-driving assistant systems come closer and closer to reality. One interesting question is "how will self-driving cars impact today's traffic?" For instance, platooning [25]–[29] can be implemented much more easily by self-driving cars than by human drivers. The "stop-and-go" traffic and the so-called "phantom traffic jams" caused by human-driven cars [6]–[19] can then be suppressed. In brief, the platoon controller tries to bind successive cars together and force them to move in lock-step fashion like carriages in a train. A single lead vehicle has global control of the whole chain of cars following and plays the same role as a locomotive in a train. New platoon models, e.g. decentralized platoon, bi-directional platoon, multi-neighbor platoon, are continuing to be proposed to improve the traditional centralized predecessor-following control architecture [30]–[33], [35]–[42]. See [43]–[53] for more theoretical analyses of various platoon models.

One question is "can local control of vehicles solve the traffic flow instability problem *without* such global control?" Such newly designed self-driving cars will run on the road independently, unlike "carriages" in a train. Global-control parameters, e.g. preset desired speed for all cars, preset desired relative position in the traffic, are *not* allowed in the advanced adaptive cruise control (ACC) system of such self-driving cars. Moreover, neither vehicle to road communication (V2R) nor vehicle to vehicle communication (V2V) is required. The ACC system's input can come entirely from the vehicle's on-board sensors — i.e. vehicle control is based totally on the outputs of its own sensors. Independence of vehicles also means that the control system (including control commands) in one car are *not* accessible to ACC systems in other cars. One such new extended ACC system is known as the bilateral control model (BCM) [1]–[5], in which, roughly speaking, the vehicle is controlled to stay as far from the leading car as from the following car. See also [23], [24] for previous efforts involving use of information about the following car in addition to the leading car.

We should mention that BCM focuses on longitudinal control of individual vehicles. The traffic flow is simplified as a single-lane highway vehicular system (i.e. not inner city with traffic lights, for example). Vehicles under BCM do *not* require full autonomy (actually, BCM is just a special ACC system). Nor is V2V or V2R required, and thus vehicles obtain *no* more information about the environment than available from measurements by on-board sensors. Thus, BCM vehicles are far from the "smart cars on smart roads" [27]. They are just "normal cars on normal roads" (with additional pair of backward sensors).

Although traffic purely under car-following control, or purely under bilateral control, may appear to be similar to special cases of platooning, i.e. decentralized platoon with infinite boundaries, here, we should mention that both car-following control and bilateral control apply to single vehicles, and thus there is *no* requirement that all vehicles operate under the same control strategy. Indeed, cars under car-following control and vehicles under bilateral control can operate independently and coexist in traffic[1]. Realistically, not all cars will be converted to bilateral control at once, so the question arises as to what role BCM vehicles will play during a transition period in such a mixture of cars under car following control (CFM) and cars under bilateral control. The existing stability analysis of pure BCM traffic does not answer this question.

For traffic flow under predecessor following control architectures, e.g. car-following control or one directional platooning, there are two important concepts, i.e. stability and string stability [47], [48], [53], that are used to qualify the (traffic) system's performance. Basically, these two concepts try to answer the following two questions:

- Stability: Will the traffic system return to the equilibrium state from an arbitrary initial state?

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[1] Although there are some studies on merge control of platooning [54]–[56], the merged cars become a new lock-step train of "carriages", rather than independent vehicles. Gaps between cars in platoon are designed to be very small, which prevents other vehicles from merging in.

- String stability: What is the impact of continuous perturbation of the first car on the traffic flow? Or, to be more precise, will such perturbations be amplified or suppressed by the following cars?

For bi-directional information flow system, e.g. bidirectional platooning or bilateral control, the stability property has been well studied. Basically, the stability of a platoon can be increased greatly by adding information about the following cars [30]–[32], [49], and traffic flow purely under bilateral control is stable [1]–[5]. However, at least to our knowledge, string-stability analysis for the bi-directional information flow system has *not* been well studied. Due to the particular topological structure of the bi-directional system, the techniques used for string-stability analysis of the one-directional information flow systems can not be directly extended to analyze the string stability of systems with bi-directional information flow. Thus, more generalized definition and new techniques are needed in order to deal with "string stability" analysis for the bi-directional topological structure.

In this paper, we study the (generalized) string stability of systems with bi-directional information flow. We call this "chain stability" to emphasize the special topological structure — different from the one-directional predecessor following control architecture for which the string-stability concept was defined. In this paper, we focus only on bilateral control system (i.e. BCM chains). However, the definition and techniques provided here can be used to analyze other bi-directional systems, e.g. bi-directional platooning, directly. We prove here the chain stability of bilateral control model. The chain stability analysis provides the answer to the question above about the roles that BCM vehicles play in mixed traffic. Namely, BCM chains become perturbation-consuming dampers inserted in the traffic — they split the chains of human-driven vehicles and prevent the perturbations transmitted from one chain of car-following vehicles to the following chains. Thus, today's pure CFM traffic can be improved greatly by inserting BCM chains (in order to split CFM chains).

II. Car-following control and bilateral control

Let $y_n(t)$ be the position of the $n-$th car, and $v_n(t) = \dot{y}_n(t)$ be its velocity[2]. The pair $\{y_n(t), v_n(t)\}$ gives the state of the $n-$th car, which is adjusted through the acceleration $a_n(t) = \ddot{y}_n(t)$ commanded by the control system. First, for the car-following model (CFM),

$$a_n = k_d(d_n - s_n) + k_v(r_n - u_n) \tag{1}$$

where, $d_n = y_{n-1} - y_n - L$ denotes the space between the current car and its leading car (with car length L) and $r_n = v_{n-1} - v_n$ denotes the relative velocity between the current car and its leading car, $k_d > 0$, and $k_v > 0$ are the proportional and derivative gains respectively. For "constant headway" CFM, the desired space s_n is a constant *safe distance* s, and the desired speed difference u_n is simply

[2]Note that y_{n-1} and y_n denote the position of the leading and current cars. The positive direction is chosen as the direction in which cars are moving, thus, $y_{n-1} - y_n > 0$ (see Figure 1).

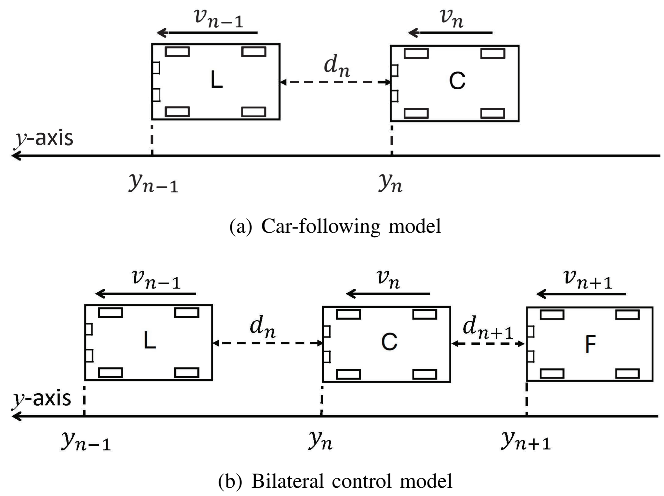


Fig. 1. Illustration of the car-following model and bilateral control model. The blocks with "L", "C" and "F" denote the leading car, current car and following car. (a) Car-following control is based *only* on the state of the leading car "L". (b) Bilateral control uses the states of *both* leading car "L" and following car "F".

chosen to be zero, i.e.

$$s_n = s \qquad \text{and} \qquad u_n = 0 \tag{2}$$

The desired space s_n can instead be set adaptively according to the car's speed, i.e.,

$$s_n = v_n T \qquad \text{and} \qquad u_n = 0 \tag{3}$$

where T is known as the *reaction time*. Such a car-following control strategy with the adaptively controlled desired spacing ($s_n = v_n T$) is known as the "constant time headway" model [20]–[22]. In these car-following models, control of car n is based *only* on the relative position and relative velocity of car $n-1$ immediately ahead.

For human drivers, it would be difficult or distracting to look forward and backward all the time, but this is not a problem for a sensor based system. So a second pair of sensors can be used to measure distance and speed difference between the current car and the following car. These two new measurements d_{n+1} and r_{n+1} can then be used for control. For instance, we can set

$$s_n = d_{n+1} \qquad \text{and} \qquad u_n = r_{n+1} \tag{4}$$

Then, eq. (1) becomes

$$a_n = k_d\big(d_n - d_{n+1}\big) + k_v\big(r_n - r_{n+1}\big) \tag{5}$$

We call this new control strategy the bilateral control model (BCM). Here, the control of car n is based on the relative positions and relative velocities of *both* car $n-1$ ahead and car $n+1$ behind. In words, the control objective of BCM is to stay in the middle between its "front and back" neighbours, and to travel at the average speed of these two neighbours. Figure 1 shows the two models.

One attractive advantagement of BCM is that traffic flow under bilateral control *is* stable for *all* values of $k_d > 0$, and $k_v > 0$ [1], [4], [5]. A physical analog of a line of traffic under bilateral control (5) is a "spring-damper-mass" system shown in Fig. 2. Intuitively, a perturbation will lead to damped waves travelling outward in *both* directions from the point of perturbation. The amplitude of these waves decays as they travel. See [1], [5] for more detailed analysis. Ref. [4] also proves that bilateral control is stable under any and all boundary conditions[3]: infinite line, circular boundaries, fixed-fixed boundaries, free-free boundaries and fixed-free boundaries.

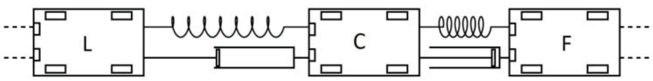


Fig. 2. A physical analog of the traffic flow under bilateral control is a big "spring-damper-mass" system.

The stability property of BCM shows that traffic flow instabilities can be suppressed by automated control systems in individual vehicles. For traffic flow under CFM or one directional platooning (i.e. predecessor following control architecture), another important concept is *string stability* [47], [48], [53]. Is there a corresponding property similar to "string stability" for BCM traffic flow?

III. String stability and chain stability

Stability and string stability are two different concepts used to analyze (single-lane) traffic flow. For a chain (or string) of vehicles, the *equilibrium state* denotes the case in which the cars are equally spaced and all moving at the same speed[4] [4]. Stability analysis relates to the impact of initial condition (i.e. the initial states of all cars) to the traffic system, i.e.

- If there is a small perturbation in the initial states of the cars from the equilibrium state, will the traffic system return to the equilibrium state (**stable**) or will there be increasing departures from the equilibrium (**unstable**) — which ultimately lead to a traffic jam?

String stability instead describes the impact of the first car's behavior on the traffic system, i.e.

- Suppose the traffic initially is in the equilibrium state, but the first car keeps generating perturbations (e.g. sinusoidal oscillations or emergency braking). Will the perturbation generated by the first car be amplified by the following cars (**string-unstable**) or be suppressed by the following cars (**string-stable**) and disappear finally?

Note that the above description of string stability implies that information flow in the system is one directional, that is, car n *only* receives information about the state of car $n-1$ ahead.

[3]The boundaries in platooning are used to control the desired states of all vehicles in the platoon. In contrast, BCM only focuses on the control of a single car. The boundary condition in BCM is just to design the ACC system such that the car can also run alone on the road. For instance, by setting a threshold on the range values, the chain will split into two sub-chains automatically if some car in the BCM chain (say car n) brakes suddenly (perhaps due to some external disturbance). The sub-chain in front of car n keeps moving and the cars behind car n will stop gradually.

[4]Different from platooning, *no* pre-set desired spacing (typically very tight in platooning) is used in car following and in bilateral control. The "equal spacing" in the equilibrium state of a BCM vehicular chain is determined by the boundary conditions. Some thresholds are also used by the BCM controller [1], [5]. For instance, if the space between two successive BCM vehicles is large enough, then the BCM vehicular chain will split into two chains automatically.

In order to deal with the bi-directional (information flow) system, we need to change the above question from one about the relationship between successive cars to the relationship between the first and last car in the traffic system. That is,

- Suppose that traffic is initially in the equilibrium state, and the first car keeps generating perturbations (e.g. sinusoidal oscillations or emergency braking). Will the result in the last car, due to the perturbation in the first car, be bounded, unbounded, or finally disappear (as the number of cars goes to infinity)?

For one-directional information flow system, e.g. CFM traffic or one-directional platooning, the answer to the above question can be obtained directly by analyzing the relationship between two successive cars in the system. However, for two-directional information flow system, the answer can not be obtained directly from the relationship between two successive cars. Based on the above question, we can extend the traditional concept of string stability (for one-directional information flow system) to one for more general system (e.g. bi-directional information flow system).

Now, we give the detailed mathematical description. Let $\overline{y}_n(t) = n(s+L) + Vt$ denote the equilibrium position of car n (with equal gaps of size s and the same speed V), and let $x_n(t)$ denote the perturbation of car n. By the *superposition principle*, the perturbation $x_n(t)$ can be decomposed into a combination of pure-frequency components $X_n(\omega)e^{j\omega t}$, and the system's performance can be described by the coefficients $X_n(\omega)$, i.e. the response to a sinusoid of a single frequency.

Definition 1 (Generalized string stability): Supposed there are a total of $N+1$ cars (with car 0 the first car and car N the last car) in traffic. If for any $\omega \neq 0$, we have:

$$\lim_{N\to\infty} \frac{\|X_N(\omega)\|}{\|X_0(\omega)\|} = 0 \tag{6}$$

then the traffic system is called **(generalized) string stable**[5]. Otherwise, the traffic system is *not* (generalized) string stable.

If the information flow in the traffic system is one directional, e.g. CFM traffic or one-directional platooning, with the same *transfer function* $H(\omega)$ form car $n-1$ to car n, i.e. $X_n(\omega) = H(\omega)X_{n-1}(\omega)$, then we find:

$$\frac{\|X_N(\omega)\|}{\|X_0(\omega)\|} = \left\| \frac{X_N(\omega)}{X_{N-1}(\omega)} \right\| \cdots \left\| \frac{X_1(\omega)}{X_0(\omega)} \right\| = \|H(\omega)\|^N \tag{7}$$

Thus, the generalized string stability condition (6) is the same as the following condition

$$\|H(\omega)\| < 1, \qquad \text{(for all } \omega \neq 0) \tag{8}$$

which is the criterion used to identify the string stability of CFM traffic flow or one-directional platooning.

[5]*No* pre-set desired spacing is used in BCM. That is, there is *no* (pre-set) desired relative position for each car in the BCM vehicular chain. Thus, only the effect of the first vehicle in the BCM chain is considered. In platoon models, pre-set desired space (or equivalently the pre-set desired relative position for each car) is used, in which case the effect of any vehicle with respect to the following ones should also be analyzed. In this paper, we focus mainly on a BCM chain, and thus, only the effect of the first car needs to be considered. Actually, the performance of the intermediate vehicles in the BCM chain is guaranteed by the stability property of BCM [1], [4], [5].

For two directional information flow system, e.g. BCM traffic or bidirectional platooning, the criterion (8) can not be used directly. Instead, the generalized string stability condition (6) should be used to identify the system's performance. In this paper, the condition (6) for two directional information flow system is called the **chain stability condition**.

Definition 2 (chain stability): Supposed there are a total of $N+1$ cars (with car 0 the first car and car N the last car) in the traffic, and the information flow between the middle $N-1$ cars (i.e. car 1 to car $N-1$) is two-directional with two *transfer function* $H_1(\omega)$ (from car $n-1$ to car n) and $H_2(\omega)$ (from car $n+1$ to car n), i.e. $X_n(\omega) = H_1(\omega)X_{n-1}(\omega) + H_2(\omega)X_{n+1}(\omega)$ (for $n = 1, 2, \cdots, N-1$). Car 0 (the source of the perturbations) moves independently and is treated as input to the system. Car N is "connected" to Car $N-1$ with a new transfer function $H(\omega)$, i.e. $X_N(\omega) = H(\omega)X_{N-1}(\omega)$. If for any $\omega \neq 0$, we have:

$$\lim_{N\to\infty} \frac{\|X_N(\omega)\|}{\|X_0(\omega)\|} = 0 \tag{9}$$

then the (bi-directional) traffic system is called **chain stable**[6]. Otherwise, the traffic system is *not* chain stable.

For traffic flow under the constant headway CFM, whose $H(\omega)$ is shown later in eq. (19), it is easy to prove that the traffic system is *not* string stable for all $k_d > 0$ and $k_v > 0$ (by direct calculation from eq. (8)). While for the traffic flow under the constant-time headway CFM, whose $H(\omega)$ is shown later in eq. (20), the string stability condition can be calculated from eq. (8) directly [4], [5], [58]. That is,

$$\frac{1}{2}k_d T^2 + k_v T > 1 \tag{10}$$

Next, we consider the question of the chain stability condition for traffic flow under bilateral control.

IV. Chain stability analysis

In a bi-directional information flow system shown in Fig. 3, the state of each sub-system inside the system (except the two at the ends) depends on *both* the sub-system ahead and the sub-system behind.

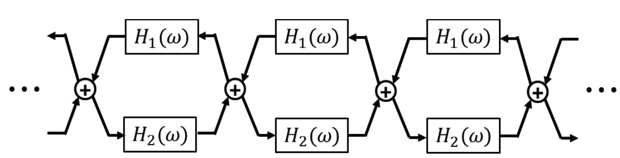


Fig. 3. In a two-directional information flow system, the state of each sub-system depends on *both* the sub-system ahead and the sub-system behind. The cascade of "spring-damper-mass" in Fig. 2 is one typical example of such system.

[6]Some other concepts, e.g. formation coherence [33], [34], are also defined in platoon control. Pre-set desired space (or equivalently pre-set desired relative positions for each car) is used by platoon models to form a rigid lattice. Formation coherence captures the notion of how well the formation resembles a rigid lattice or a "solid object" [34]. For BCM, *no* pre-set desired space is used, and the vehicular chain under BCM performs like a "soft object." Chain stability captures the notion of how well the chain absorbs a (input) perturbation on one side and prevents it from being transmitted to the other side (output). Moreover, chain stability requires the perturbation in the-other-side vehicle disappearing rather than only being bounded as $N \to \infty$.

The states of the N sub-systems in such a bi-directional system are determined by 1). the input $X_0(\omega)$ to the system, i.e. state of the first sub-system; and 2). the following N equations:

$$X_n(\omega) = H_1(\omega)X_{n-1}(\omega) + H_2(\omega)X_{n+1}(\omega) \tag{11}$$

for $n = 1, 2, \cdots N-1$, and

$$X_N(\omega) = H(\omega)X_{N-1}(\omega) \tag{12}$$

The N unknowns

$$\mathbf{X} = \big(X_1(\omega), X_2(\omega), \cdots, X_N(\omega)\big)^T \tag{13}$$

can be found by solving the following linear system:

$$\mathbf{HX} = -H_1(\omega)\mathbf{b}_0 \tag{14}$$

where the $N \times N$ matrix $\mathbf{H}$ is

$$\mathbf{H} = \begin{pmatrix} -1 & H_2(\omega) & & & \\ H_1(\omega) & -1 & H_2(\omega) & & \\ & \ddots & \ddots & \ddots & \\ & & H_1(\omega) & -1 & H_2(\omega) \\ & & & H(\omega) & -1 \end{pmatrix} \tag{15}$$

and the $N \times 1$ vector $\mathbf{b}_0$ is

$$\mathbf{b}_0 = \big(X_0(\omega), 0, \cdots, 0\big)^T \tag{16}$$

Thus, the unknowns $\mathbf{X}$ are exactly the first column in $\mathbf{H}^{-1}$ multiplied by $-H_1(\omega)X_0(\omega)$. The ratio $X_N(\omega)/X_0(\omega)$ is the entry in row N and column 1 of $\mathbf{H}^{-1}$ multiplied by $-H_1(\omega)$. Only one entry in $\mathbf{H}^{-1}$ is used to calculate the ratio $X_N(\omega)/X_0(\omega)$, thus, it is not efficient to calculate $\mathbf{H}^{-1}$ by brute force. Moreover, it is not straight-forward to solve the linear system (14) (especially as N goes to infinity). A simpler approach is needed (and introduced in section IV-B) to implement the chain stability analysis.

A. Linear system of the BCM chain

In this paper, we only focus on the chain stability analysis of BCM traffic. Thus, we specify the corresponding $\mathbf{H}$ first. Note that $y_n(t) = \overline{y}_n(t) + x_n(t)$. Thus, $v_n(t) = \dot{x}_n(t) + V$, and $a_n(t) = \ddot{x}_n(t)$. Moreover, we find $d_n - d_{n+1} = x_{n-1}(t) - 2x_n(t) + x_{n+1}(t)$ and $r_n - r_{n+1} = \dot{x}_{n-1}(t) - 2\dot{x}_n(t) + \dot{x}_{n+1}(t)$. For pure-frequency-response analysis, the perturbation is taken to be a simple sinusoidal oscillation at a fixed frequency, i.e. $x_n(t) = X_n(\omega)e^{j\omega t}$. From eq. (5), we find

$$X_n(\omega) = \frac{k_d + k_v j\omega}{2k_d - \omega^2 + 2k_v j\omega}\big(X_{n-1}(\omega) + X_{n+1}(\omega)\big) \tag{17}$$

for $n = 1, 2, \cdots, N-1$. Thus, we find that

$$H_1(\omega) = H_2(\omega) = \frac{k_d + k_v j\omega}{2k_d - \omega^2 + 2k_v j\omega} \tag{18}$$

For BCM, the boundary, i.e. the last vehicle in the BCM chain, simply implements car-following. Two different cases are studied in this paper:

1) Free BCM chain in the traffic: There is *no* CFM car following behind the BCM chain (at least not in the

range of the distance sensor). The last BCM vehicle in the chain just implements constant headway CFM, i.e. eq. (2). The corresponding transfer function is:

$$H(\omega) = \frac{k_d + k_v j\omega}{k_d - \omega^2 + k_v j\omega} \tag{19}$$

2) BCM chain is followed by human drivers: There *is* a vehicle driven by a human driver behind the last BCM vehicle (within the range of distance sensor). The human-driven car immediately following the BCM chain, i.e. the first vehicle in the following CFM chain, becomes the boundary of the BCM chain ahead. The human driver could implement constant headway CFM in eq. (2), or constant-time headway CFM in eq. (3). The case of constant headway CFM has been discussed above. Thus, we only focus on the case of constant-time headway CFM here. The corresponding transfer function is:

$$H(\omega) = \frac{k_d + k_v j\omega}{k_d - \omega^2 + (k_v + k_d T) j\omega} \tag{20}$$

First, let $p(\omega) = 1/H_1(\omega)$ and $q = 1/H(\omega)$. Now, the linear system (14) is specified as following:

$$\mathbf{S}\mathbf{X} = -\mathbf{b}_0 \tag{21}$$

where $\mathbf{S}$ is the following *symmetric* $N \times N$ matrix:

$$\mathbf{S} = \begin{pmatrix} -p & 1 & & & & \\ 1 & -p & 1 & & & \\ & 1 & -p & 1 & & \\ & & \ddots & \ddots & \ddots & \\ & & & 1 & -p & 1 \\ & & & & 1 & -q \end{pmatrix} \tag{22}$$

and the $N \times 1$ vector $-\mathbf{b}_0 = \big(-X_0(\omega), 0, \cdots, 0\big)^T$.

B. Solving the linear system

Instead of solving the linear system (21) directly by Gaussian elimination (i.e. LU decomposition of $\mathbf{S}$) [59], some tricks are used here to solve the linear system. Note that $\mathbf{S}$ is a tri-diagonal (symmetric) matrix. The linear system (21) will be much easier to solve if $\mathbf{S}$ can be turned into an upper triangular matrix. This can be done by adding a new equation $X_N = X_N$ to the linear system (21), i.e.

$$\begin{pmatrix} 1 & -p & 1 & & \\ & \ddots & \ddots & \ddots & \\ & & 1 & -p & 1 \\ & & & 1 & -q \\ & & & & 1 \end{pmatrix} \begin{pmatrix} X_0 \\ X_1 \\ X_2 \\ \vdots \\ X_N \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ X_N \end{pmatrix} \tag{23}$$

Now, the matrix in the above system becomes a $(N+1) \times (N+1)$ upper-triangular matrix. However, note that X_N in the right-hand side vector of eq. (23) is unknown (rather than some known input such as X_0). The trick is to divide both sides of eq. (23) by X_N. Then, we find:

$$\underbrace{\begin{pmatrix} 1 & -p & 1 & & \\ & \ddots & \ddots & \ddots & \\ & & 1 & -p & 1 \\ & & & 1 & -q \\ & & & & 1 \end{pmatrix}}_{\mathbf{U}} \underbrace{\begin{pmatrix} z_N \\ z_{N-1} \\ \vdots \\ z_1 \\ z_0 \end{pmatrix}}_{\mathbf{z}} = \underbrace{\begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{pmatrix}}_{\mathbf{e}_N} \tag{24}$$

or in "matrix-vector" form: $\mathbf{U}\mathbf{z} = \mathbf{e}_N$, where

$$z_k(\omega) = \frac{X_{N-k}(\omega)}{X_N(\omega)} \tag{25}$$

Now, all the entries in the right-hand side vector of eq. (24) are known constants. The linear system (24) can now be solved directly by *back substitution* [59]. Then, the chain stability can be indicated by z_N (i.e. the first entry in the solution of (24)).

Note also that the first $N-2$ rows of the matrix $\mathbf{U}$ in eq. (24) are of *Toeplitz* form, i.e. have a constant diagonal. Moreover, The last two equations in (24) provides two initial values: $z_0(\omega) = 1$ and $z_1(\omega) = q(\omega)$. This type of linear system (24) is well studied [59], [60]. The solution is:

$$z_k(\omega) = c_1 r_1^k(\omega) + c_2 r_2^k(\omega) \tag{26}$$

with r_1 and r_2 as the two roots of the following quadratic equation:

$$r^2 - p(\omega) r + 1 = 0 \tag{27}$$

Note that

$$p(\omega) = 2 - \frac{\omega^2}{k_d + k_v j\omega} \neq \pm 2 \tag{28}$$

for all $\omega \neq 0$ and $k_v \neq 0$. Thus, we see that the two roots r_1 and r_2 are *different*. Substituting (26), we can check that $z_k - p(\omega) z_{k-1} + z_{k-2} = c_1 r_1^{k-2}\big(r_1^2 - p(\omega) r_1 + 1\big) + c_2 r_2^{k-2}\big(r_2^2 - p(\omega) r_2 + 1\big) = 0 + 0 = 0$ for all $k = 2, 3, \cdots, N$ (see eq.(27)). Thus, the first $N-1$ equations in the linear system (24) are matched. The two coefficients c_1 and c_2 in eq. (26) are determined by the two initial values $z_0 = 1$ and $z_1 = q$, i.e.

$$\begin{cases} c_1 + c_2 = z_0 \\ c_1 r_1 + c_2 r_2 = z_1 \end{cases} \tag{29}$$

Note that $r_1 \neq r_2$ (according to eq. (28)). Thus, we find:

$$c_1 = \frac{r_2 - q}{r_2 - r_1} \quad \text{and} \quad c_2 = \frac{r_1 - q}{r_1 - r_2} \tag{30}$$

Now, the solution $z_k(\omega)$ in (26) becomes:

$$z_k(\omega) = q(\omega) \frac{r_2^k(\omega) - r_1^k(\omega)}{r_2(\omega) - r_1(\omega)} - \frac{r_2^k(\omega) r_1(\omega) - r_1^k(\omega) r_2(\omega)}{r_2(\omega) - r_1(\omega)} \tag{31}$$

which satisfies all of the linear equations in (24). Note that the determinant of the (upper triangular) matrix $\mathbf{U}$ in eq. (24) is 1, i.e. *non-zero*. That is, the matrix $\mathbf{U}$ in eq. (24) is *invertible*, and thus the solution of the linear system (24) is *unique*: $\mathbf{z} = \mathbf{U}^{-1}\mathbf{e}_N$ [59]. Thus, $z_k(\omega)$ in eq. (31) (for $k = 0, 1, \cdots, N$) is the solution to the linear system (24).

C. Chain stability of the BCM system

Note that $z_N(\omega) = X_0(\omega)/X_N(\omega)$. Thus, the chain stability condition (6) for BCM traffic becomes

$$\lim_{N\to\infty} \|z_N(\omega)\| = \infty \tag{32}$$

for all $\omega \neq 0$. Without loss of generality, we suppose that

$$\|r_2\| \geq \|r_1\| \tag{33}$$

First, let's prove the following Lemma:

Lemma 1: For all $\omega \neq 0$, the coefficient $c_2(\omega) \neq 0$ for both cases of free BCM chain in the traffic, i.e. $H(\omega)$ chosen as (19), and BCM chain followed by human drivers, i.e. $H(\omega)$ chosen as (20).

Proof of Lemma 1: First, note that $H(\omega)$ in (19) can be viewed as a special case of $H(\omega)$ in (20) with $T = 0$. Thus, we analyze the $H(\omega)$ in (20) with two different cases: $T = 0$ and $T > 0$. From eq. (20), we find:

$$q(\omega) = 1 - \frac{\omega^2 - k_d T j\omega}{k_d + k_v j\omega} \tag{34}$$

Then, prove by contradiction. Suppose $c_2 = 0$, from eq. (30), we find that $r_1(\omega) = q(\omega)$. Note also that r_1 is one of the two roots: $\frac{1}{2}\left(p \pm \sqrt{p^2 - 4}\right)$ (see eq. (27)). Thus, we find $\big(p(\omega) - 2q(\omega)\big)^2 = p^2(\omega) - 4$. Substituting eq. (28) and (34), we find:

$$\big(\omega^2 - 2k_d T j\omega\big)^2 = \omega^4 - 4\omega^2\big(k_d + k_v j\omega\big) \tag{35}$$

That is,

$$k_d T^2 = 1 \qquad \text{and} \qquad k_v T = 1 \tag{36}$$

Thus we find:

1) When $T = 0$: Eq. (36) can not be satisfied. Thus, the hypothesis $c_2 = 0$ is wrong.
2) When $T > 0$: From eq. (36), we find $k_d = 1/T^2$ and $k_v = 1/T$. Then, eq. (34) becomes $q(\omega) = 1 + Tj\omega$. Thus, $\|q(\omega)\|^2 = 1 + T^2\omega^2 > 1$ for all $\omega \neq 0$. Thus, $\|r_1(\omega)\| > 1$ (note that $r_1(\omega) = q(\omega)$). From (27), we find $r_1(\omega)r_2(\omega) = 1$, and thus $\|r_1(\omega)\|\|r_2(\omega)\| = 1$. Thus, $\|r_2(\omega)\| < 1 < \|r_1(\omega)\|$, which is contradict to the hypothesis (33).

Thus, $c_2 \neq 0$ for both of the cases that $H(\omega)$ is chosen as (19) and (20) . □

Now, we prove the following:

Theorem 1 : The system of BCM vehicles is chain stable for arbitrary $k_d > 0$ and $k_v > 0$.

Proof of theorem 1: From eq. (27), we find $r_1(\omega) + r_2(\omega) = p(\omega)$ and $r_1(\omega)r_2(\omega) = 1$. First, note that

$$\|r_1(\omega)\|\|r_2(\omega)\| = 1 \tag{37}$$

From (28), we find that $p(\omega)$ is *not* a real number for all $k_d > 0$, $k_v > 0$ and $\omega \neq 0$. Thus[7] $\|r_1(\omega)\| \neq \|r_2(\omega)\|$. Thus, $\|r_1(\omega)\| < 1$ and $\|r_2(\omega)\| > 1$ for any $\omega \neq 0$ (according to the convention (33)). Finally, from (26), we find:

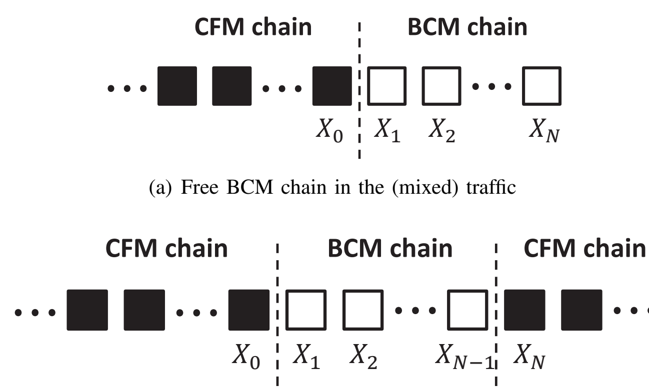
$$\|z_N(\omega)\| \geq \|c_2(\omega)\|\|r_2(\omega)\|^N - \|c_1(\omega)\| \tag{38}$$

Note that $c_2(\omega) \neq 0$ (by Lemma 1). Thus, $\|z_N(\omega)\| \to \infty$ (for all $\omega \neq 0$) as $N \to \infty$. □

In this paper, we focus only on bilateral control. However, the above techniques can be used to analyze other bi-directional information flow systems directly, e.g. bi-directional platooning or asymmetric bi-directional vehicle control.

V. BCM CHAINS IN TRAFFIC

The chain stability analysis provides the performance of BCM chains in mixed traffic. First, let's consider the traffic flow shown in Fig. 4: CFM vehicles followed by BCM cars. The fist car in the chain studied in Definition 2 is: the last vehicle in the CFM chain immediately ahead of the BCM chain. There are two different cases: (1). there is *no* following CFM car behind the BCM chain (in the range of the distance sensor), and there *is* a CFM car following the BCM chain. In case (1), the last BCM vehicle becomes the end of the chain in Definition 2 (as shown in Fig. 4(a)). The last BCM vehicle can just implement constant headway CFM, i.e. free-boundary. In case (2), the CFM car immediately following the last BCM vehicle becomes the end of the chain in Definition 2 (as shown in Fig. 4(b)). The whole BCM chain becomes the set of middle sub-systems of the chain in Definition 2, which "connect" the last CFM vehicle in the CFM chain immediately ahead to the first CFM car in the immediately following CFM chain.



(a) Free BCM chain in the (mixed) traffic

(b) BCM chain followed by human drivers

Fig. 4. In mixed traffic, the first car in the chain studied in Definition 2 is: the CFM vehicle (i.e. the black block) immediately ahead of the BCM chain (i.e. the white blocks). (a) if there is *no* CFM car following the BCM chain, then the last BCM vehicle becomes the end of the chain in Definition 2. (b) Otherwise, the CFM car immediately following the BCM chain becomes the end (or boundary) of the chain discussed in Definition 2.

The "stop-and-go" traffic jams in the CFM chain will cause continuous perturbation $X_0(\omega)$. The chain stability of the following BCM vehicles will guarantee such perturbations will be damped away. That is, $X_N(\omega)$ goes to zero for all $\omega \neq 0$.

Now, the effect of a BCM chain in mixed traffic becomes clear. Without the BCM chain, the perturbation X_0 is transferred to the following vehicles. By adding a BCM chain,

[7] Prove by contradiction. Suppose that $\|r_1(\omega)\| = \|r_2(\omega)\|$, then the condition $r_1(\omega)r_2(\omega) = 1$ implies that $r_1(\omega)$ equals to the conjugate of $r_2(\omega)$, and thus $p(\omega) = r_1(\omega) + r_2(\omega)$ will be a real number. However, $p(\omega)$ is *not* a real number (see (28))

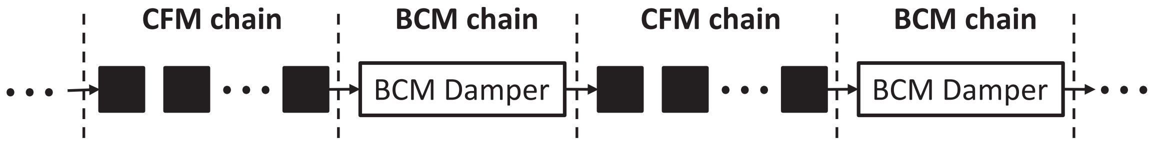


Fig. 5. The BCM chains in traffic play the role of perturbation-consuming dampers (called BCM dampers), which split the CFM chains and prevent the perturbations transferred from one CFM chain to the following CFM chain.

the perturbation transferred to the following human-driven vehicles is X_N, which is guaranteed to be small according to Theorem 1 (chain stability). That is, the BCM chains become perturbation-consuming dampers inserted in the traffic, which split the CFM chains and prevent the transfer of perturbations from one CFM chain to the following CFM chains (See Fig. 5). Thus, traffic flow can be improved greatly by the BCM chains, comparing to today's pure CFM traffic.

Moreover, the input perturbations X_0 to the BCM chain asymptotically decay exponentially (see eq. (38)). Thus, the inserted BCM chains (also called BCM dampers) consume the perturbations (in the CFM chains) very efficiently. The effectiveness of the impact of BCM vehicles depends on the following two parameters:

1) The fraction of BCM vehicles: *some* perturbations are amplified exponentially by the CFM chain (see eq. (7)). *All* input perturbations to the BCM chain decay asymptotically exponentially (see eq.(38)). Thus, the higher the ratio of BCM-chain length to CFM-chain length (i.e. the higher fraction of BCM vehicles), the more effective the BCM-chain will be in stabilizing the whole traffic flow by consuming the perturbations.
2) The length of the BCM chain: according to eq. (38) and Definition 2, the effectiveness of BCM chain stability improves with the length of the chain, so it should not be too small.

VI. Simulations

We build a simulator to demonstrate the perturbation consuming behavior of chains of vehicles under bilateral control. The parameters used in the simulations are listed in Table I. In order to deal with some events that may occur during simulation, we add another rule: if the space between the successive two cars is less than the car-length L, i.e. car collision can occur, then, we let the second car implement "emergency stop" in the next iteration.

TABLE I
The parameters used in the simulation.

distance feedback k_d	0.3 (1/sec^2)
velocity feedback k_v	0.2 (1/sec)
Car length L	5 (meters)
max velocity v_{max}	160 (km/h)
min velocity v_{min}	0 (km/h)
max acceleration a_{max}	5 (m/sec^2)
min deceleration a_{min}	-5 (m/sec^2)
time step Δt	0.1 (sec)

In the beginning, the vehicles are all spaced apart by 25 meters, and all move at the same speed of 25 m/s. The simulation results, i.e. the trajectories of vehicles in space-time domain, are shown in a relative reference system, which moves at the constant speed of 25 m/s. The initial space between the successive vehicle trajectories is 30 meters (i.e. initial vehicle gap plus car length).

First, we demonstrate the chain stability of BCM (Theorem 1). The number of BCM vehicles is $N = 20$. The car ahead of the first BCM vehicle generates continuous "stop-and-go" perturbations. It decelerates at -5 m/s^2 until the speed is 15 m/s (i.e. -10 m/s in the relative reference system), then accelerates at 5 m/s^2 until the speed is 35 m/s (i.e. 10 m/s in the relative reference system), and then decelerates again at -5 m/s^2 until the speed as 15 m/s (or -10 m/s in the relative reference system), and so on. Fig. 6 shows some simulation results at different times: 50 sec., 52.5 sec., 55 sec. and 57.5 sec. The whole simulation result (in 600 sec.) is shown in: http://people.csail.mit.edu/wangliang/#ChainStability.

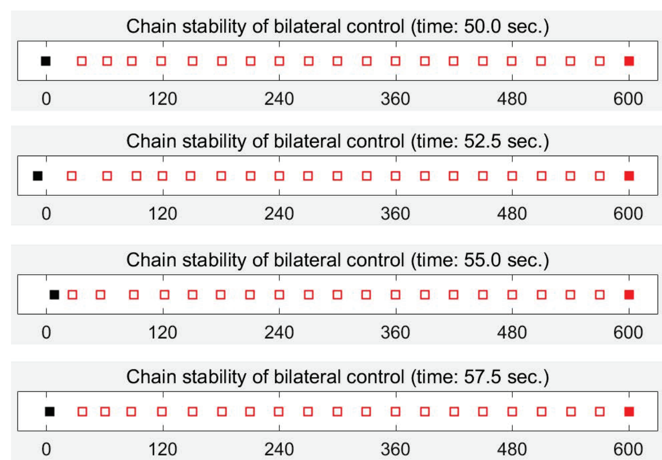


Fig. 6. Some of the simulations results. The red squares indicate the BCM vehicles, and the solid red square indicates the last BCM car in the chain. The solid black square is the input to the BCM chain.

The solid black square in Fig. 6 shows the first car, i.e. the input to the following BCM chain. The perturbations in the input cause the perturbations in its several following (BCM) vehicles. Due to the chain stability of BCM, such perturbations will be consumed by the BCM chain. There is no noticeable perturbations in the last BCM vehicle (marked by the solid red square in Fig. 6).

Fig. 7 shows the trajectories of the totally 21 vehicles (in the relative reference system). The obvious perturbations in the first car (i.e. the red curve in Fig. 7) are consumed by the BCM chain. The last BCM vehicle of the chain (i.e. the thick

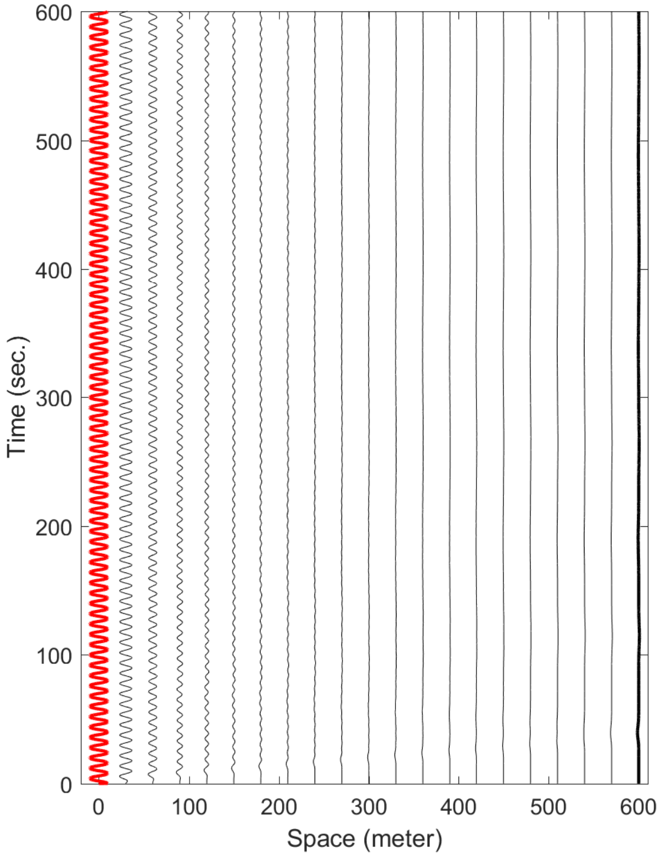


Fig. 7. The trajectories of vehicles in space-time domain of a relative reference system (which moves at the constant speed 25 m/s). The number of BCM vehicles is $N = 20$. The first car introduces continuous "stop-and-go" perturbations (i.e. red curve) to the system. Due to the chain stability, the perturbations introduced are consumed by the BCM chain. The perturbations of the last BCM vehicle (thick black curve) are very small.

black curve) has only very low amplitude fluctuations.

Fig. 8 demonstrates the effect of a BCM chain in mixed traffic. The black curves in Fig. 8(a) are trajectories of CFM vehicles; while the red curves in Fig. 8(a) are trajectories of BCM vehicles. Totally, there are 20 CFM vehicles and another 20 BCM cars. We can suppose that vehicle patten in 8(a) appears periodic in the whole traffic. That is, the ratio of BCM vehicles and CFM cars is one. At time 0 sec., 100 sec. and 200 sec., the first CFM car generate perturbations by sudden break and then speed up. That is, decelerates at -5 m/s^2 for 1.5 sec., then speed up at 5 m/s^2 for another 3 sec. and then decelerates again at -5 m/s^2 for another 1.5 sec. Now, its speed is 25 m/s (or 0 m/s in the relative reference system) and position is 0. The perturbation generated by the emergency brake of the first car is amplified by the successive CFM vehicles, and finally becomes larger perturbations input to the following BCM chain. Still, due to the chain stability of the BCM vehicles, the input perturbations (i.e. the thick black curve in Fig. 8(a)) are consumed by the BCM chain (i.e. the red curve in Fig. 8(a)). The perturbations in the first CFM car immediately following the BCM chain (i.e. the 31-th curve in Fig. 8(a)) is almost negligible. Such tiny perturbations might be amplified by the following CFM vehicles. However, the following BCM chain can then be expected to consume the perturbations before it becomes large enough to generate traffic jams. Thus, traffic flow will be relatively smooth.

For comparison, Fig. 8(b) shows the result of pure CFM traffic (with 20 CFM cars). The first 10 CFM vehicles (i.e. the first 10 black curves in Fig. 8(b)) are exactly the same as the first CFM chain in Fig. 8(a) (i.e. the first 10 curves in Fig. 8(a)). However, the performance of the next 10 CFM vehicles in Fig. 8(b) (i.e. the last 10 curves in Fig. 8(b)) are totally different from the second CMF chain in Fig. 8(a) (i.e. the last 10 curves in Fig. 8(a)). Without a BCM chain, the perturbations can not be consumed promptly, and thus, the continuously amplified perturbations will generate traffic jams (and potentially car collisions) very soon.

Fig. 9 shows some simulation results at 130 sec., 135 sec., 140 sec. and 145 sec. These results visually demonstrate the conclusion from Fig. 8. The last car in the first CFM chain (marked by solid black square) is subject to large perturbations. These perturbations are consumed effectively by the following BCM chain (marked by red squares). Thus, the perturbations in the first CFM vehicle of the second CFM chain (containing 10 CFM cars) is unnoticeable, and thus, the second CFM chain (which is following the BCM chain) moves relatively smoothly. Without the inserted BCM chain, the perturbations in the 10-th CFM car (i.e. the solid black square) are amplified continuously, and then result in today's "stop-and-go" traffic jams that we are all familiar with. The whole simulation result is shown in: http://people.csail.mit.edu/wangliang/#ChainStability.

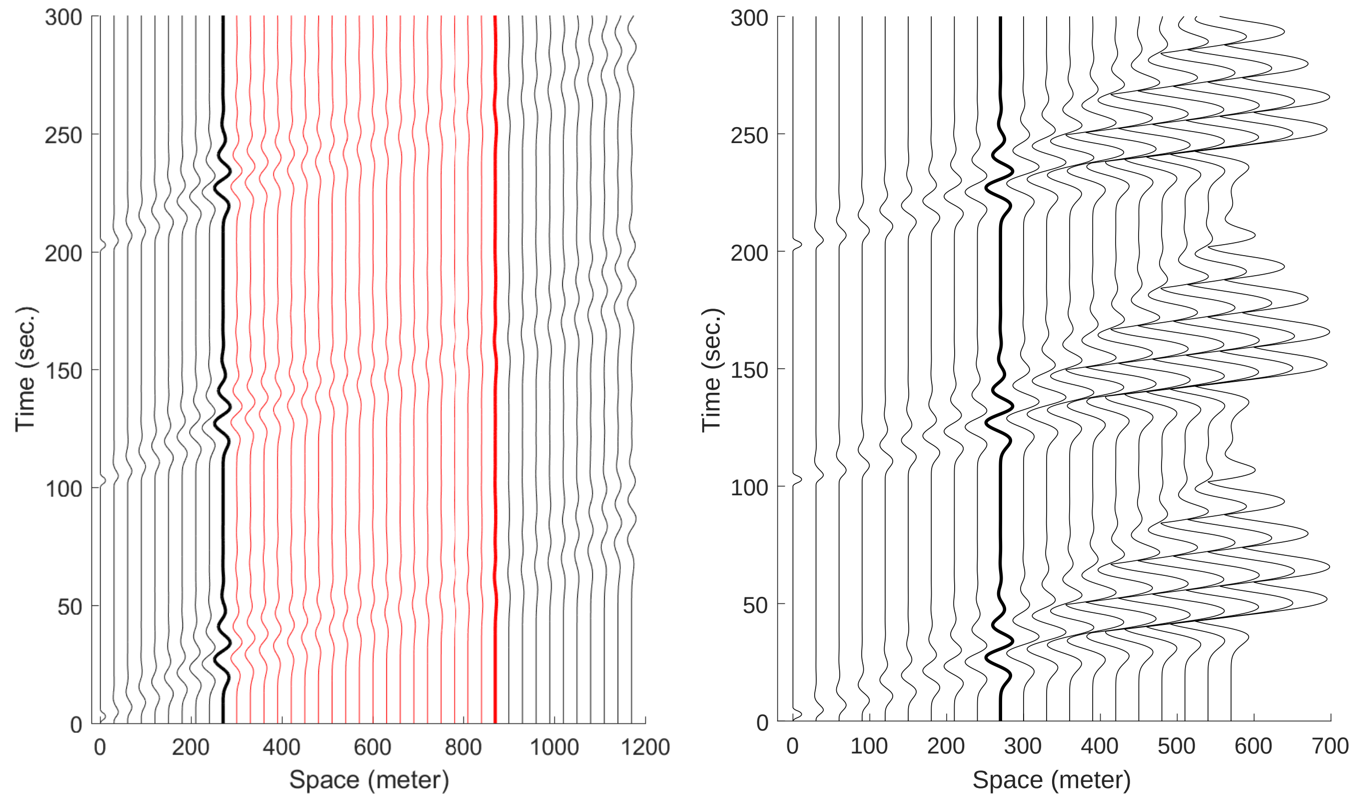
VII. Conclusion

It is well-known that:

- Traffic flow instabilities, including alternating "stop-and-go" driving conditions, are the result of the natural behavior of human-drivers, where control of vehicles generally is based *only* on the state of the car ahead.
- Such traffic flow instabilities can be suppressed effectively if control of *all* of the vehicles also takes into account the state of the following car, using a bi-directional control strategy. This can greatly improve the traffic system's stability.

However, the existing stability analysis of both bi-directional platooning and bilateral control strategy focuses on "pure" traffic system, where all of the cars are purely under platooning control or purely under bilateral control. One interesting question then is "what is the impact of such vehicle chains (purely under bi-directional control strategy) on the whole traffic flow when mixed with cars driven by human drivers?"

To answer this question, another concept is needed to replace the one applicable on one-directional information flow system, e.g. for pure CFM traffic or predecessor following platoon control. That is, a concept similar to that of string stability for the one-directional information flow system is needed. Due to the particular topology of the bi-directional information flow system, the analysis and conclusions for the one-directional predecessor following system can not be directly extended to the bi-directional information flow system. At least to our knowledge, something similar to string stability



(a) Mixed traffic

(b) Pure CFM traffic

Fig. 8. Demonstration of the effect of a BCM chain. (a). The perturbation generated by emergency braking of the first car (at 0 sec., 100 sec. and 200 sec.) is amplified by the successive CFM vehicles to be large perturbations (i.e. the thick black curve). Due to the chain stability of the BCM system, such perturbations are consumed by the following BCM chain (i.e. red curves) before traffic jams are generated. The traffic will be relatively smooth. (b) Without BCM chain, the perturbations in the thick black curve is not consumed promptly, and thus, the continuously amplified perturbations will generate traffic jams (and possible car collisions) very soon. The first CFM chain (i.e. the first 10 curves) in (a) are exactly the same as the first 10 CFM cars in (b). However, the second CFM chain (i.e. the last 10 curves) in (a) move much more smoothly than the last 10 CFM cars in (b).

analysis for such bi-directional control based traffic system has not been well studied before.

In this paper, we first extend string stability for one directional predecessor following system to a generalized concept (in Definition 1): the system's performance of "transferring" perturbations from one end to the other end of a "chain" (or "string"). Such generalized string stability indicates the role that the vehicle "string" (i.e. under one-directional control) or vehicle "chain" (i.e. under bi-directional control) plays in the whole traffic. If the vehicle "string" or "chain" is string stable in the generalized sense, then the perturbations caused by the vehicles ahead of the vehicle "string" (or "chain") will dissipate. Otherwise, such perturbations will be passed to the following vehicles behind the vehicle "string" (or "chain"). That is, if the vehicle "string" (or "chain") is string stable in the generalized sense, then it will have a positive effect on traffic flow by preventing the propagation of perturbations.

The generalization of string stability to vehicle chains under bi-directional-control topology is called chain stability (Definition 2) in this paper to emphasize the special topological structure different from the one-directional predecessor following control architecture for which the string-stability concept was defined. In this paper, we focus only on the bilateral control system (i.e. BCM chains). We analyze such BCM-chain systems, and prove their chain stability (Theorem 1). This result provides a mathematical description of the roles BCM vehicles play in mixed traffic. That is, the BCM chains become perturbation-consuming dampers inserted in the traffic, which split CFM chains and prevent perturbations from being transferred from one CFM chain to the following CFM chains. Thus, BCM vehicles play a very definite "positive" roles in mixed traffic.

As mentioned, in this paper, we only focus on bilateral control model. However, the definition and techniques provided here can be used to analyze other bi-directional systems, e.g. bi-directional platooning or asymmetric bi-directional ACC system, directly. The analysis and techniques used in this paper can also help design optimal parameters for these other bi-directional systems. (Note that, for example, in Theorem 1, we prove bilateral control is strictly chain stable for *all* $k_d > 0$ and $k_v > 0$. But this result will not be true in general for other bi-directional systems.)

The chain stability analysis of BCM shows in part the impact that self-driving cars can have on today's traffic. The BCM vehicles are still "normal cars" with additional pair of backward sensors. Still, traffic flow can be improved by

Fig. 9. Some of the simulations results. The black squares indicate the CFM vehicles, while the red squares indicate the BCM vehicles. The solid black square indicates the last car in the CFM chain, whose perturbations become the input to the following BCM chain. The solid red square indicates the last car in the BCM chain. The following CFM chain moves much more smoothly, because the BCM chain consumes the perturbations from the solid black square.

BCM vehicles inserted in traffic. Under the assumption that more information about the environment (via e.g. V2V and V2R) can be used, cars can become "smarter and smarter" through e.g. reinforcement learning [61]. Unsurprisingly, such "smart" self-driving cars can do more than platooning or bilateral control, and thus potentially can provide even more positive impact on traffic, e.g. multi-lane control [62], route flow estimate and control [63]. These will be some new and interesting topics.

We focus in this paper on theoretical analysis. In real application, the feedback control model (1) is implemented as a discrete-time system by the ACC controller, and the effect of quantization in time should also be considered, as should delay. The corresponding stability/string stability conditions for both CFM and BCM will change somewhat in the discrete case. Study of the stability/string stability under these conditions and the effect of delay will be future work. Moreover, other control strategies, e.g. the linear combination of BCM and CFM, may also be used by self-driving cars. Then, for example, more attention could be paid to the leading car than the following one. Different vehicles may use various gains, and gains for one vehicle may also be determined adaptively according to e.g. the traffic situation. Again, these problems will be addressed in future work.

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