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Motion Fields Are Hardly Ever Ambiguous*

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Abstract

There has been much concern with ambiguity in the recovery of motion and structure from time-varying images. I show here that the class of surfaces leading to ambiguous motion fields is extremely restricted—only certain hyperboloids of one sheet (and some degenerate forms) qualify. Furthermore, the viewer must be on the surface for it to lead to a potentially ambiguous motion field. Thus the motion field over an appreciable image region almost always uniquely defines the instantaneous translational and rotational velocities, as well as the shape of the surface (up to a scale factor).

1 Introduction

An important question in motion vision research is whether a given motion field could have arisen from two different motions with two corresponding surface shapes. I felt originally that the answer to this question should be "No" [1, 2], and so was startled by results showing that, in the planar case, two solutions are possible [3, 4, 5, 6]. I was further taken aback when the two-way ambiguity persisted when quadratic patches were considered [7, 8]. Fortunately, I have been able to restore my faith in the uniqueness of the solution. I show here that an ambiguous motion field can arise only from a hyperboloid of one sheet (or a degenerate form) viewed from a point on its surface. This is a subset of measure zero of all possible smooth surfaces and viewing positions.

Furthermore, a surface has to be in front of the camera to be imaged—so the "depth" must be positive. Surfaces leading to ambiguous motion fields correspond to positive depth values only in certain image regions. A motion field can be ambiguous only if our knowledge of it is confined to an image region in which both "solutions" happen to be positive. There can be no ambiguity if the image region under consideration includes areas where one of the two solutions changes sign.

The ambiguity problem is, however, likely to continue to plague those who use purely local analysis techniques. In a sufficiently small patch, the estimated motion field may not be distinguishable from one resulting from a hyperboloid of one sheet viewed from a point on its surface. That is, while the estimated motion field itself is very unlikely to be ambiguous, it may be equal to an ambiguous motion field corrupted by a small amount of noise. (In any case, the problem of motion estimation becomes ill-conditioned as the size of the image region shrinks [9].)

Why start with the motion field? In practice we deal with image sequences and have to estimate the motion field either at discrete points using some form of matching, or derive an optical flow from the brightness gradients. We have the time derivative of brightness at each picture cell (one variable), not the motion field (two variables). It is nevertheless valuable to study what can be achieved using the motion field, since this puts an upper bound on what one can do with images, and separates the problems occasioned by inadequate surface texture contrast from the purely geometric problems that are common to a variety of approaches to the motion vision problem.

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2 Critical Surface Are Hyperboloids of One Sheet

The question usually posed is whether two different motions—and two corresponding surfaces—could have yielded a given motion field. I turn this question around here to ask instead what surfaces could lead to the same motion field, given two different motions. This new question is easier to answer and, indirectly, helps answer the original one.

I call two surfaces yielding the same motion field for two given motions a *critical surface pair*. I show that each member of such a pair must be a hyperboloid of one sheet or one of its degenerate forms. (It had been noted previously that only quadric surfaces can give rise to potentially ambiguous flow fields [10, 11]). I show further that each of these hyperboloids must be viewed from a point on its surface. Vector notation allows compact derivations [12]—this notation will now be briefly reviewed.

2.1 Review of Notation

Without loss of generality, let the focal length be unity and suppose that the optical axis lies along the Z-axis. The center of projection is at the origin and the image is formed on the plane $Z = 1$. If $\mathbf{r} = (x,y,1)^T$ is the image of the point $\mathbf{R} = (X,Y,Z)^T$, we have the perspective projection relationship[1]

$$\mathbf{r} = \frac{1}{\mathbf{R}\cdot\hat{\mathbf{z}}}\mathbf{R}$$

We are interested in the motion of the image point $\mathbf{r}$ induced by motion of the point $\mathbf{R}$. If we multiply the above equation by $(\mathbf{R}\cdot\hat{\mathbf{z}})$, differentiate with respect to time, and solve for the time derivative of $\mathbf{r}$ we obtain

$$\mathbf{r}_t = \frac{1}{\mathbf{R}\cdot\hat{\mathbf{z}}}(\mathbf{R}_t - (\mathbf{R}_t\cdot\hat{\mathbf{z}})\mathbf{r})$$

Now suppose that the observer is moving with instantaneous translational velocity $\mathbf{t} = (U,V,W)^T$ and instantaneous rotational velocity $\omega = (A,B,C)^T$. Then the velocity of a point in the fixed environment relative to the observer will be

$$\mathbf{R}_t = -t - \mathbf{R}\times\omega$$

or, upon substituting for $\mathbf{R}$ in terms of $\mathbf{r}$,

$$\mathbf{R}_t = -\mathbf{t} - (\mathbf{R}\cdot\hat{\mathbf{z}})\mathbf{r}\times\omega$$

We now substitute this expression for $\mathbf{R}_t$ in the equation above for $\mathbf{r}_t$, and finally arive at[2]

$$\mathbf{r}_t = \frac{1}{\mathbf{R}\cdot\hat{\mathbf{z}}}((\mathbf{t}\cdot\hat{\mathbf{z}})\mathbf{r} - \mathbf{t}) + [\mathbf{r}\ \omega\ \hat{\mathbf{z}}]\mathbf{r} - \mathbf{r}\times\omega$$

This is the equation for the motion field $\mathbf{r}_t = (u, v, 0)^T$ as a function of the image position $\mathbf{r} = (x, y, 1)^T$ and the motion $\{\mathbf{t}, \omega\}$. The first part of the expression is the translational component, which depends on the depth $Z = \mathbf{R}\cdot\hat{\mathbf{z}}$, while the last two terms constitute the rotational component, which does not depend on depth.

2.2 Equality of Motion Fields

Suppose we have a motion $\{\mathbf{t}_1, \omega_1\}$ along with depth $Z_1(x, y)$ that yields the same motion field as does a motion $\{\mathbf{t}_2, \omega_2\}$ along with depth $Z_2(x, y)$. Then

$$\frac{1}{Z_1}((\mathbf{t}_1\cdot\hat{\mathbf{z}})\mathbf{r} - \mathbf{t}_1) + [\mathbf{r}\ \omega_1\ \hat{\mathbf{z}}]\mathbf{r} - \mathbf{r}\times\omega_1$$
$$= \frac{1}{Z_2}((\mathbf{t}_2\cdot\hat{\mathbf{z}})\mathbf{r} - \mathbf{t}_2) + [\mathbf{r}\ \omega_2\ \hat{\mathbf{z}}]\mathbf{r} - \mathbf{r}\times\omega_2$$

or, more compactly,

$$\frac{1}{Z_1}((\mathbf{t}_1\cdot\hat{\mathbf{z}})\mathbf{r} - \mathbf{t}_1) - \frac{1}{Z_2}((\mathbf{t}_2\cdot\hat{\mathbf{z}})\mathbf{r} - \mathbf{t}_2)$$
$$= [\mathbf{r}\ \delta\omega\ \hat{\mathbf{z}}]\mathbf{r} - \mathbf{r}\times\delta\omega$$

where $\delta\omega = \omega_2 - \omega_1$. Note that the two motion fields are changed equally by equal changes in instantaneous rotational velocity, and so only the *difference* of the two rotational velocities is relevant.

The above equation represents a powerful constraint on critical surface pairs and the corresponding motion. Let us suppose that we are

[1]A unit vector in the z-direction is denoted $\hat{\mathbf{z}}$.

[2]Square brackets enclosing three vectors denote the triple product of the three vectors.

given the two motions $\{\mathbf{t}_1, \omega_1\}$ and $\{\mathbf{t}_2, \omega_2\}$ and are to find two surface $Z_1(x, y)$ and $Z_2(x, y)$ such that the resulting motion fields are equal. The constraint equation above can be solved for Z_1 and Z_2 by taking the dot-product with judiciously chosen vectors, as we see later.

2.3 General Case

It is convenient to postpone analysis of a large number of special cases until later. For now we assume that there are no special relationships between the three vectors, $\mathbf{t}_1$, $\mathbf{t}_2$, and $\delta\omega$. Thus

$$\mathbf{t}_1 \cdot \mathbf{t}_2 \neq 0$$

$$\delta\omega \cdot \mathbf{t}_1 \neq 0$$

$$\delta\omega \cdot \mathbf{t}_2 \neq 0$$

as well as

$$\|\mathbf{t}_1 \times \mathbf{t}_2\| \neq 0$$

$$\|\delta\omega \times \mathbf{t}_1\| \neq 0$$

$$\|\delta\omega \times \mathbf{t}_2\| \neq 0$$

This implies that the three vectors have nonzero length and are neither pairwise parallel nor pairwise orthogonal. We further assume, for reasons that will become apparent later, that

$$(\mathbf{t}_2 \times \mathbf{t}_1) \cdot (\mathbf{t}_2 \times \delta\omega) \neq 0$$

and

$$(\mathbf{t}_2 \times \mathbf{t}_1) \cdot (\mathbf{t}_1 \times \delta\omega) \neq 0$$

These conditions ensure that neither the plane containing $\delta\omega$ and $\mathbf{t}_2$, nor the plane containing $\delta\omega$ and $\mathbf{t}_1$, are orthogonal to the plane containing $\mathbf{t}_1$ and $\mathbf{t}_2$.

To find the surface Z_1, we take the dot-product of the constraint equation with $\mathbf{t}_2 \times \mathbf{r}$. Vectors parallel to $\mathbf{t}_2$ or $\mathbf{r}$ yield zero and we are left with just

$$\frac{1}{Z_1} [\mathbf{t}_2\, \mathbf{t}_1\, \mathbf{r}] + (\mathbf{r} \times \delta\omega) \cdot (\mathbf{t}_2 \times \mathbf{r}) = 0$$

Symmetrically, taking the dot-product with $\mathbf{t}_1 \times \mathbf{r}$ we obtain an equation for the other surface

$$\frac{1}{Z_2} [\mathbf{t}_2\, \mathbf{t}_1\, \mathbf{r}] + (\mathbf{r} \times \delta\omega) \cdot (\mathbf{t}_1 \times \mathbf{r}) = 0$$

We can solve these equations for Z_1 and Z_2. The resulting formulas give depth as a function of image position $\mathbf{r} = (x,y,1)^T$. (A formula of this form for the ambiguous surface was apparently first derived by Maybank [11]). We are not accustomed to the hybrid parameterization of surfaces and would most likely not recognize the type of surface we are dealing with from these formulas. It is helpful then to express everything in terms of scene coordinates $\mathbf{R} = (X,Y,Z)^T$ instead. If we substitute $Z_1\mathbf{r} = \mathbf{R}$ into the equation of the first surface, we obtain

$$(\mathbf{R} \times \delta\omega) \cdot (\mathbf{t}_2 \times \mathbf{R}) + [\mathbf{t}_2\, \mathbf{t}_1\, \mathbf{R}] = 0$$

Similarly, substituting $Z_2\mathbf{r} = \mathbf{R}$ into the equation of the second surface, we obtain

$$(\mathbf{R} \times \delta\omega) \cdot (\mathbf{t}_1 \times \mathbf{R}) + [\mathbf{t}_2\, \mathbf{t}_1\, \mathbf{R}] = 0$$

These two equations can also be written in the form

$$(\mathbf{R} \cdot \mathbf{t}_2)(\delta\omega \cdot \mathbf{R}) - (\mathbf{t}_2 \cdot \delta\omega)(\mathbf{R} \cdot \mathbf{R}) + (\mathbf{t}_2 \times \mathbf{t}_1) \cdot \mathbf{R} = 0$$

and

$$(\mathbf{R} \cdot \mathbf{t}_1)(\delta\omega \cdot \mathbf{R}) - (\mathbf{t}_1 \cdot \delta\omega)(\mathbf{R} \cdot \mathbf{R}) + (\mathbf{t}_2 \times \mathbf{t}_1) \cdot \mathbf{R} = 0$$

The expressions on the lefthand sides of these equalities are quadratic in $\mathbf{R}$. That is, written out in terms of the components X, Y, and Z, we obtained a second-order expression. To see this, just note that the dot-products $\mathbf{R} \cdot \mathbf{t}_1$ and $\mathbf{R} \cdot \mathbf{t}_2$ are linear in X, Y, and Z, while $\mathbf{R} \cdot \mathbf{R} = X^2 + Y^2 + Z^2$.

2.3.1 Scaling of Motion Parameters. Before we go on, it may be worthwhile considering how the critical surfaces change if we scale the given motion parameters $\mathbf{t}_1$, $\mathbf{t}_2$, and $\delta\omega$. Replacing $\mathbf{t}_1$ by $k\mathbf{t}_1$ does not change the second surface and scales the first by k (that is, the translational motion field component is not changed if we multiply the translational velocity and the depth by the same factor). Similarly, if we replace $\mathbf{t}_2$ by $k\mathbf{t}_2$, the first surface is unchanged, while the second is scaled by k.

This implies that we could, if wished, treat the translational velocity vectors $\mathbf{t}_1$ and $\mathbf{t}_2$ as unit vectors. Replacing $\delta\omega$ by $k\delta\omega$, on the other hand, leads to a change in the shape of the surface that is more complex than simple scaling.

2.4 Quadric Surfaces

We conclude that both surfaces must be quadrics (for a discussion of the properties of quadrics see references [13] and [14]). There are several categories of quadrics.[3] In the case of a proper central quadric, it is possible to rotate the coordinate system and translate the origin so that the equation for the surface is transformed into standard form, where linear terms and products of different components drop out.[4] The result is something of the form

$$\pm\left(\frac{X'}{A}\right)^2 \pm \left(\frac{Y'}{B}\right)^2 \pm \left(\frac{Z'}{C}\right)^2 = 1$$

The new origin is called the "center" of the quadric surface and the new coordinate axes are the intersections of the planes of symmetry of the quadric surface. The lengths of the semi-major axes are given by the quantities A, B, and C.

If all three signs are positive, we have an ellipsoid. If two signs are positive, we are dealing with a hyperboloid of one sheet. If only one sign is positive, we have a hyperboloid of two sheets. Finally, if all signs are negative, we have an imaginary ellipsoid (no real locus).

It may happen that some of the second-order terms are missing, in which case we are dealing with a degenerate form such as an elliptic cone or a noncentral quadric such as a hyperbolic paraboloid. A degenerate noncentral surface of particular interest is that formed by two intersecting planes, as we note later.

2.5 Hyperboloids of One Sheet

If we substitute $\mathbf{R} = k\mathbf{t}_2$ into the equation of the first surface

$$(\mathbf{R}\cdot\mathbf{t}_2)(\delta\omega\cdot\mathbf{R}) - (\mathbf{t}_2\cdot\delta\omega)(\mathbf{R}\cdot\mathbf{R}) + (\mathbf{t}_2\times\mathbf{t}_1)\cdot\mathbf{R} = 0$$

we find that the linear term drops out and that the first two terms become equal to one another. We conclude that a line parallel to $\mathbf{t}_2$ passing through the origin lies entirely in the surface. This means that the surface cannot be an ellipsoid or a hyperboloid of two sheet (or one of their degenerate forms). It must be a hyperboloid of one sheet (or one of its degenerate forms).

The line $\mathbf{R} = k\mathbf{t}_2$ is one of the rulings of the first surface. A hyperboloid of one sheet has two sets of intersecting rulings (see [13, 14]). We expect therefore that a second line passing through the origin is embedded in the surface. It can be verified that $\mathbf{R} = k\mathbf{t}_0$ where

$$\mathbf{t}_0 = \big((\mathbf{t}_2\times\mathbf{t}_1)\times\delta\omega\big)\times(\mathbf{t}_2\times\mathbf{t}_1)$$

is the equation of this second line.

Note that the hyperboloid passes through the origin—there is no constant term in the equation of the surface (this is not an artifact introduced by the substitution $Z_1\mathbf{r} = \mathbf{R}$, as one can show by taking the limit as $Z \rightarrow 0$). We conclude that the viewer must be on the surface being viewed.

The perspective projection equation, $\mathbf{r} = (1/\mathbf{R}\cdot\hat{\mathbf{z}})\mathbf{R}$, does not enforce the condition that $Z > 0$, where $Z = \mathbf{R}\cdot\hat{\mathbf{z}}$. We find that the motion field corresponds in some image regions to points on the surface lying in front of the viewer and in other image regions to points on the surface behind the viewer. A real ambiguity can only arise if attention is restricted to image regions where both $Z_1 > 0$ and $Z_2 > 0$. If either one changes sign in the region there is no ambiguity. We analyze these image regions in detail later.

A second-order polynomial in X, Y, and Z has nine coefficients. The surface defined by setting this polynomial equal to zero is not altered, however, if we multiply all coefficients by some nonzero quantity. Surfaces defined by such an equation thus belong to an eight-parameter family of surfaces. Can an arbitrary hyperboloid of one sheet be a member of a critical surface pair? That is, can we generate an arbitrary second-order polynomial (lacking a constant term) by suitable choice of $\mathbf{t}_2$, $\mathbf{t}_1$, and $\delta\omega$? The answer to this question is most likely "no," since

[3]For classification of quadrics, see section 3.5-3 in Korn and Korn [13].

[4]The coefficient matrix of the quadratic form appearing in the equation of a proper central quadric has three nonzero eigenvalues.

multiplying $\delta\omega$ by k while dividing $\mathbf{t}_2$ by k does not change the polynomial and because $\mathbf{t}_1$ appears only in the cross-product $\mathbf{t}_2 \times \mathbf{t}_1$. We show later that only *certain* hyperboloids of one sheet can occur as members of critical surface pairs.

2.6 Rareness of Ambiguity

Let us summarize what we have learned so far: only a hyperboloid of one sheet passing through the origin can lead to a potentially ambiguous motion field. We have mentioned, but not proven, additional restrictions. In any case, it is clear that ambiguity will be extremely rare. The reader wishing only to be reassured on this point may wish to stop right here.

It is interesting, however, to study the geometry of critical surfaces further. It is hard to do this using the vector form of the equations of their surfaces; so in the next section we use matrix notation. This makes it easier to discover the center of the quadric, as well as its axes, tangent planes, asymptotes, cross sections, and rulings. We also still need to discuss various degenerate cases that may occur.

3 Geometric Properties of Critical Surfaces

Note that

$$(\mathbf{R}\cdot\mathbf{t}_2)(\delta\omega\cdot\mathbf{R}) - (\mathbf{t}_2\cdot\delta\omega)(\mathbf{R}\cdot\mathbf{R}) = \mathbf{R}^T(\mathbf{t}_2\delta\omega^T)\mathbf{R} - (\mathbf{t}_2\cdot\delta\omega)\mathbf{R}^T\mathbf{R}$$

since we can write $\mathbf{a}\cdot\mathbf{b} = \mathbf{a}^T\mathbf{b}$ if we consider the column vectors $\mathbf{a}$ and $\mathbf{b}$ as 3×1 matrixes. We can, as a result, rewrite the equation of the first surface in the form

$$\mathbf{R}^T M_1 \mathbf{R} + 2\mathbf{L}^T\mathbf{R} = 0$$

where the matrix M_1 is given by

$$M_1 = \mathbf{t}_2\delta\omega^T + \delta\omega\mathbf{t}_2^T - 2(\mathbf{t}_2\cdot\delta\omega)I$$

while $\mathbf{L} = \mathbf{t}_2 \times \mathbf{t}_1$. Similarly, the equation of the second surface can be written in the form

$$\mathbf{R}^T M_2 \mathbf{R} + 2\mathbf{L}^T\mathbf{R} = 0$$

where

$$M_2 = \mathbf{t}_1\delta\omega^T + \delta\omega\mathbf{t}_1^T - 2(\mathbf{t}_1\cdot\delta\omega)I$$

The matrices M_1 and M_2 have been constructed in such a way that they are symmetric, in order that the products $\mathbf{R}^T M_1\mathbf{R}$ and $\mathbf{R}^T M_2\mathbf{R}$ are quadratic forms. Each of the matrixes M_1 and M_2 thus has three real eigenvalues with three corresponding orthogonal eigenvectors.

The matrix M_1 determines the shape of the first surface in the critical surface pair—it depends on $\mathbf{t}_2$ and $\delta\omega$. (The vector $\mathbf{L}$ only controls the size and position of this surface.) Similarly, M_2 determines the shape of the second surface—it depends on $\mathbf{t}_1$ and $\delta\omega$. (The vector $\mathbf{L}$ controls its size and position also.)

When there is no likelihood of confusion, we will discuss surfaces of the form

$$\mathbf{R}^T M\mathbf{R} + 2\mathbf{L}^T\mathbf{R} = 0$$

without specifying whether the matrix in the quadratic form happens to be M_1 or M_2.

3.1 Center of the Quadric

The surfaces we are interested in have been defined in implicit form by an equation of the form $f(\mathbf{R}) = 0$. They turn out to be centrally symmetric about a point $\mathbf{C}$ where the gradient of f is zero, that is, where

$$\left.\frac{\partial f}{\partial \mathbf{R}}\right|_{\mathbf{R}=\mathbf{C}} = 0$$

In our case, this leads to

$$M\mathbf{C} + \mathbf{L} = 0$$

The significance of the so-called *center* $\mathbf{C}$ is that the linear term disappears from the implicit equation of the surface if we shift the origin to $\mathbf{C}$. To see this, let

$$\mathbf{R} = \mathbf{C} + \mathbf{R}'$$

then

$$(\mathbf{C} + \mathbf{R}')^T M(\mathbf{C} + \mathbf{R}') + 2\mathbf{L}^T(\mathbf{C} + \mathbf{R}') = 0$$

which, upon expanding and substituting $M\mathbf{C} + \mathbf{L} = 0$, becomes just

$$(\mathbf{R}')^T M\mathbf{R}' = c$$

where the constant term is given by

$$c = \mathbf{C}^T M \mathbf{C}$$

The directions of the axes of the quadric surface are the directions of the eigenvectors of the matrix M, while the lengths of the axes depend on the eigenvalues of M and the constant c, as we see later.

3.2 Eigenvalue—Eigenvector Decomposition

We are dealing with matrices of the form

$$M = \mathbf{t}\delta\boldsymbol{\omega}^T + \delta\boldsymbol{\omega}\mathbf{t}^T - 2(\mathbf{t}\cdot\delta\boldsymbol{\omega})I$$

where $\mathbf{t} = \mathbf{t}_2$ for M_1 and $\mathbf{t} = \mathbf{t}_1$ for M_2. What are the eigenvalues and eigenvectors of such a matrix?

Let $\hat{\mathbf{t}}$ and $\delta\hat{\boldsymbol{\omega}}$ be unit vectors in the directions $\mathbf{t}$ and $\delta\boldsymbol{\omega}$ respectively. Also, let

$$\sigma = \|\mathbf{t}\|\,\|\delta\boldsymbol{\omega}\| \qquad \text{and} \qquad \tau = \hat{\mathbf{t}}\cdot\delta\hat{\boldsymbol{\omega}}$$

It is clear that $\sigma > 0$ and $0 < |\tau| < 1$. It is easy to verify that

$$M(\mathbf{t}\times\delta\boldsymbol{\omega}) = -2(\mathbf{t}\cdot\delta\boldsymbol{\omega})(\mathbf{t}\times\delta\boldsymbol{\omega})$$

so $\mathbf{t}\times\delta\boldsymbol{\omega}$ is an eigenvector with eigenvalue $-2(\mathbf{t}\cdot\delta\boldsymbol{\omega})$. The other two eigenvectors must lie in a plane perpendicular to $\mathbf{t}\times\delta\boldsymbol{\omega}$, that is, they must be linear combinations of $\mathbf{t}$ and $\delta\boldsymbol{\omega}$. In fact, $(\hat{\mathbf{t}} + \delta\hat{\boldsymbol{\omega}})$ is an eigenvector with eigenvalue $(\|\mathbf{t}\|\,\|\delta\boldsymbol{\omega}\| - \mathbf{t}\cdot\delta\boldsymbol{\omega})$, while $(\hat{\mathbf{t}} - \delta\hat{\boldsymbol{\omega}})$ is an eigenvector with eigenvalue $(-\|\mathbf{t}\|\,\|\delta\boldsymbol{\omega}\| - \mathbf{t}\cdot\delta\boldsymbol{\omega})$.

We see then that the *unit* eigenvectors and corresponding eigenvalues of M are

$$\hat{\mathbf{e}}_+ = \frac{1}{\sqrt{2(1+\tau)}}(\hat{\mathbf{t}} + \delta\hat{\boldsymbol{\omega}}) \qquad \lambda_+ = \sigma(1-\tau)$$

$$\hat{\mathbf{e}}_0 = \frac{1}{\sqrt{1-\tau^2}}(\hat{\mathbf{t}} \times \delta\hat{\boldsymbol{\omega}}) \qquad \lambda_0 = -2\sigma\tau$$

$$\hat{\mathbf{e}}_- = \frac{1}{\sqrt{2(1-\tau)}}(\hat{\mathbf{t}} - \delta\hat{\boldsymbol{\omega}}) \qquad \lambda_- = -\sigma(1+\tau)$$

Note that $\lambda_- < 0$, $\lambda_+ > 0$, $\lambda_0 \neq 0$ and $\lambda_0 = \lambda_+ + \lambda_-$. This last constraint tells us that M cannot be an arbitrary symmetric matrix.

One axis of symmetry of the first surface is perpendicular to both $\mathbf{t}_2$ and $\delta\boldsymbol{\omega}$. Another one points in a direction bisecting the angle between $\mathbf{t}_2$ and $\delta\boldsymbol{\omega}$, while the third lies in a direction that bisects the angle between $\mathbf{t}_2$ and $-\delta\boldsymbol{\omega}$.

3.3 Constant Term in Transformed Equation

To determine the length of the axes and to verify that the surface can only be a hyperboloid of one sheet we need to find the constant $c = \mathbf{C}^T M \mathbf{C}$ in the equation

$$(\mathbf{R}')^T M \mathbf{R}' = c$$

Now

$$\mathbf{C}^T M \mathbf{C} = -\mathbf{C}^T\mathbf{L} = \mathbf{L}^T M^{-1}\mathbf{L}$$

since $M\mathbf{C} + \mathbf{L} = 0$. Using the eigenvector-eigenvalue decomposition of M we have

$$\mathbf{C}^T M \mathbf{C} = \lambda_+(\hat{\mathbf{e}}_+\cdot\mathbf{C})^2 + \lambda_0(\hat{\mathbf{e}}_0\cdot\mathbf{C})^2 + \lambda_-(\hat{\mathbf{e}}_-\cdot\mathbf{C})^2$$

and

$$\mathbf{L}^T M^{-1}\mathbf{L} = \frac{1}{\lambda_+}(\hat{\mathbf{e}}_+\cdot\mathbf{L})^2 + \frac{1}{\lambda_0}(\hat{\mathbf{e}}_0\cdot\mathbf{L})^2 + \frac{1}{\lambda_-}(\hat{\mathbf{e}}_-\cdot\mathbf{L})^2$$

At this point we need to be specific about which of the surfaces in the critical pair we are interested in. Consider the first surface, for which

$$M_1 = \mathbf{t}_2\delta\boldsymbol{\omega}^T + \delta\boldsymbol{\omega}\mathbf{t}_2^T - 2(\mathbf{t}_2\cdot\delta\boldsymbol{\omega})I$$

Noting that $\mathbf{L} = \mathbf{t}_2 \times \mathbf{t}_1$, we find

$$\hat{\mathbf{e}}_+\cdot\mathbf{L} = \frac{[\mathbf{t}_2\,\mathbf{t}_1\,\delta\boldsymbol{\omega}]}{\sqrt{2(1+\tau_2)}}$$

$$\hat{\mathbf{e}}_0\cdot\mathbf{L} = \frac{(\mathbf{t}_2\times\delta\boldsymbol{\omega})\cdot(\mathbf{t}_2\times\mathbf{t}_1)}{\sigma_2\sqrt{1-\tau_2^2}}$$

$$\hat{\mathbf{e}}_-\cdot\mathbf{L} = -\frac{[\mathbf{t}_2\,\mathbf{t}_1\,\delta\boldsymbol{\omega}]}{\sqrt{2(1-\tau_2)}}$$

where $\sigma_2 = \|\mathbf{t}_2\|\,\|\delta\boldsymbol{\omega}\|$ and $\tau_2 = \mathbf{t}_2\cdot\delta\boldsymbol{\omega}$. We see that

$$\mathbf{L}^T M_1^{-1}\mathbf{L} = \frac{1}{\lambda_0}(\hat{\mathbf{e}}_0\cdot\mathbf{L})^2$$

since the first and third terms cancel. So c has the same sign as λ_0. Using $\lambda_0 = -2\omega_2\tau_2$ we finally obtain

$$c_1 = \mathbf{L}^T M_1^{-1}\mathbf{L} = -\frac{k_2^2}{2\sigma_2^3\tau_2(1-\tau_2^2)}$$

where $k_2 = (\mathbf{t}_2\times\delta\boldsymbol{\omega})\cdot(\mathbf{t}_2\times\mathbf{t}_1)$. Similarly,

$$c_2 = \mathbf{L}^T M_2^{-1}\mathbf{L} = -\frac{k_1^2}{2\sigma_1^3\tau_1(1-\tau_1^2)}$$

where $k_1 = (\mathbf{t}_1 \times \delta\omega)\cdot(\mathbf{t}_2 \times \mathbf{t}_1)$.

3.4 Signs of the Eigenvalues

If c is non-zero, we can transform the equation

$$(\mathbf{R}')^T M \mathbf{R}' = c$$

into the form

$$(\mathbf{R}')^T M' \mathbf{R}' = 1$$

using $M' = (1/c)M$. The lengths of the axes are given by the square roots of the inverse of the absolute values of the eigenvalues of M'. The type of the surface is determined by the signs of the eigenvalues of M'.

The eigenvalues of M' are equal to the eigenvalues of M divided by c. Let

$$\lambda'_- = \frac{\lambda_-}{c},\ \lambda'_0 = \frac{\lambda_0}{c},\ \lambda'_+ = \frac{\lambda_+}{c} \text{ when } c > 0$$

and

$$\lambda'_- = \frac{\lambda_+}{c},\ \lambda'_0 = \frac{\lambda_0}{c},\ \lambda'_+ = \frac{\lambda_-}{c} \text{ when } c < 0$$

We see that λ'_- is always negative, while λ'_+ is always positive. Finally, and most importantly, λ_0 is always positive since c has the same sign as λ_0.

So M' has two positive eigenvalues and one negative one. This confirms our earlier conclusion that the surface is a hyperboloid of one sheet. Now $\lambda'_+ + \lambda'_- = \lambda'_0$ since $\lambda_+ + \lambda_- = \lambda_0$, so we conclude that

$$\frac{1}{a_+^2} - \frac{1}{a_-^2} = \frac{1}{a_0^2}$$

where $a_+^2 = 1/\lambda'_+$, $a_0^2 = 1/\lambda'_0$, and $a_-^2 = -\lambda'_-$ are the squares of the lengths of the half-axes of the quadric surface. Hyperboloids of one sheet with a given center and given axes form a three-parameter family of surfaces. The above constraint on the semi-major axes implies that critical surfaces form a two-parameter subset of this family. Only certain hyperboloids of one sheet can be critical surfaces.

3.5 The Center Revisited

The center of the first surface is given by

$$\mathbf{C}_1 = -M_1^{-1}\mathbf{L}$$

The eigenvectors of M_1^{-1} are the same as the eigenvectors of M_1, while the eigenvalues of M_1^{-1} are the algebraic inverses of the eigenvalues of M_1. So we have

$$\mathbf{C}_1 = -\frac{1}{\lambda_+}(\hat{\mathbf{e}}_+\cdot\mathbf{L})\hat{\mathbf{e}}_+ - \frac{1}{\lambda_0}(\hat{\mathbf{e}}_0\cdot\mathbf{L})\hat{\mathbf{e}}_0 - \frac{1}{\lambda_-}(\hat{\mathbf{e}}_-\cdot\mathbf{L})\hat{\mathbf{e}}_-$$

Using the values of $(\hat{\mathbf{e}}_+\cdot\mathbf{L})$, $(\hat{\mathbf{e}}_0\cdot\mathbf{L})$, and $(\hat{\mathbf{e}}_-\cdot\mathbf{L})$ found earlier, we obtain

$$\sigma_2^2(1-\tau_2^2)\mathbf{C}_1 = -\upsilon\sigma_2\hat{\mathbf{t}}_2 + \frac{k_2}{2\tau_2}\hat{\mathbf{t}}_2 \times \delta\hat{\omega}$$

where $\upsilon = [\mathbf{t}_2\,\mathbf{t}_1\,\delta\omega]$. It is possible to verify that the linear term in

$$\mathbf{R}^T M_1 \mathbf{R} + 2\mathbf{L}\cdot\mathbf{R} = 0$$

disappears when we substitute $\mathbf{R} = \mathbf{C}_1 + \mathbf{R}'$. In this endeavor it may be helpful to note that, by definition, $M_1\mathbf{C}_1 = -\mathbf{L}$ and that

$$\|\mathbf{t}_2 \times \delta\omega\|^2 = \sigma_2^2(1-\tau_2^2)$$

For $\|\mathbf{C}_1\| = 0$ we must have

$$\upsilon = [\mathbf{t}_2\,\mathbf{t}_1\,\delta\omega] = 0$$

and

$$k_2 = (\mathbf{t}_2 \times \mathbf{t}_1)\cdot(\mathbf{t}_2 \times \delta\omega) = 0$$

This in turn implies that $\|\mathbf{t}_2 \times \mathbf{t}_1\| = 0$ or $\|\mathbf{t}_2 \times \delta\omega\| = 0$. These are both special cases, to be dealt with later, which do not involve a proper central quadric. We conclude that the center cannot be at the origin. (The same conclusion can be reached by noting that a nonsingular quadric does not pass through its own center).

3.6 Tangent Planes

A normal to the surface defined by the implicit equation $f(\mathbf{R}) = 0$ at the point $\mathbf{R}_0$ is given by the gradient of f there:

$$\mathbf{N} = \left.\frac{\partial f}{\partial \mathbf{R}}\right|_{\mathbf{R}=\mathbf{R}_0}.$$

In our case then

$$\mathbf{N} = M\mathbf{R}_0 + \mathbf{L}$$

We see that at the origin $\mathbf{N} = \mathbf{L}$. Thus the normal to the surface where it passes through the projection center is just $\mathbf{t}_2 \times \mathbf{t}_1$ (Longuet-Higgins [15] already showed that $\mathbf{t}_1$ is tangent to the quadric surface.)

The tangent plane to the surface at the point $\mathbf{R}_0$ is given by the linear equation

$$\mathbf{N} \cdot \mathbf{R} = \mathbf{N} \cdot \mathbf{R}_0$$

or

$$\mathbf{R}_0^T M \mathbf{R} + \mathbf{L}^T\mathbf{R} = \mathbf{R}_0^T M \mathbf{R}_0 + \mathbf{L}^T\mathbf{R}_0$$

At the origin this simplifies to $\mathbf{L}^T\mathbf{R} = 0$ or

$$[\mathbf{t}_2\,\mathbf{t}_1\,\mathbf{R}] = 0$$

3.6.1 Critical Image Line. Before we transformed to scene coordinates, we derived the equation

$$\frac{1}{Z_1}[\mathbf{t}_2\,\mathbf{t}_1\,\mathbf{r}] + (\mathbf{r} \times \delta\omega)\cdot(\mathbf{t}_2 \times \mathbf{r}) = 0$$

for the first critical surface. We can solve this for Z_1 to obtain

$$Z_1 = -\frac{[\mathbf{t}_2\,\mathbf{t}_1\,\mathbf{r}]}{(\mathbf{r}\cdot\mathbf{t}_2)(\delta\omega\cdot\mathbf{r}) - (\mathbf{t}_2\cdot\delta\omega)(\mathbf{r}\cdot\mathbf{r})}$$

Similarly, for the second critical surface

$$Z_2 = -\frac{[\mathbf{t}_2\,\mathbf{t}_1\,\mathbf{r}]}{(\mathbf{r}\cdot\mathbf{t}_1)(\delta\omega\cdot\mathbf{r}) - (\mathbf{t}_1\cdot\delta\omega)(\mathbf{r}\cdot\mathbf{r})}$$

These expressions show that the depth of a critical surface is the ratio of a linear polynomial in image coordinates and a quadratic polynomial in image coordinates.

We also note that $Z_1 = Z_2 = 0$ when $[\mathbf{t}_2\,\mathbf{t}_1\,\mathbf{r}] = 0$. The triple product $[\mathbf{t}_2\,\mathbf{t}_1\,\mathbf{r}]$ is linear in image coordinates and so defines a straight line. This line connects the focus of expansion for the first motion

$$\bar{\mathbf{t}}_1 = \frac{1}{\mathbf{t}_1\cdot\hat{\mathbf{z}}}\,\mathbf{t}_1$$

to the focus of expansion of the second motion

$$\bar{\mathbf{t}}_2 = \frac{1}{\mathbf{t}_2\cdot\hat{\mathbf{z}}}\,\mathbf{t}_2$$

This line will here be called the *critical image line.*

Note that Z_1 and Z_2 change sign as one crosses this line. The critical image line is the intersection of the image plane with the tangent plane to the critical surface at the origin. A ray from the center of the projection on one side of this tangent plane will meet the critical surface in front of the viewer, while a ray on the other side will meet the critical surface behind the viewer.

A line intersects a quadric surface in at most two points (unless it is entirely embedded in the surface) [14]. Points in the image correspond to rays through the center of projection. In our case, the center of projection lies on the quadric surface. A ray through the center of projection then will intersect the quadric in at most one other point (unless it is a ruling of the surface). In the case of the hyperboloid of one sheet, the ray will always intersect the surface in exactly one other point (unless it is parallel to a ruling of the asymptotic cone, to be discussed in the next section).

This property is reflected in the fact that the expressions for Z_1 and Z_2 are single-valued and defined for all image points (except those where the denominator becomes zero). The ray may intersect the surface behind the viewer or in front, however. That is, "depth" given by the formulae for Z_1 and Z_2 may be positive or negative.

3.7 Asymptotic Cone

Suppose $\mathbf{R} = \mathbf{C} + k\mathbf{R}'$ where $\mathbf{R}'$ is the direction of a ray from the center $\mathbf{C}$. Then

$$k^2(\mathbf{R}')^T M \mathbf{R}' = \mathbf{C}^T M \mathbf{C}$$

or

$$(\mathbf{R}')^T M \mathbf{R}' = c/k^2$$

If we let $k \to \infty$ we obtain

$$(\mathbf{R}')^T M \mathbf{R}' = 0$$

This is the equation of a cone, called the asymptotic cone, with apex at the center. Rays connecting the origin to points on the hyperboloid that are "infinitely far away" are parallel to lines lying in the asymptotic cone.

The asymptotic cone lies inside the hyperboloid of one sheet. The projection in the image of points infinitely far away on the surface are the points where the denominator of the expression for Z becomes zero. For the first surface this occurs where

$$(\mathbf{r} \cdot \mathbf{t}_2)(\delta\omega \cdot \mathbf{r}) - (\mathbf{t}_2 \cdot \delta\omega)(\mathbf{r} \cdot \mathbf{r}) = 0$$

This is a quadratic in image coordinates x and y and so defines a conic section. This conic section is the intersection of the image plane with the asymptotic cone, when the apex of the cone is shifted to the origin.

3.7.1 Critical Image Curve. The curve where $Z_1 = \infty$ will be called the *critical image curve* for the first surface. As with the critical image line, where $Z_1 = 0$, we find that Z changes sign as we cross the critical image curve. The critical image line and the critical image curve thus divide the image into regions of constant sign. Note that the critical image curves of the two surfaces in a critical surface pair do *not* coincide in general.

The image is divided into four types of regions: where $Z_1 < 0$ and $Z_2 < 0$, $Z_1 < 0$ and $Z_2 > 0$, $Z_1 > 0$ and $Z_2 < 0$, and regions where $Z_1 > 0$ and $Z_2 > 0$. Ambiguity can only arise if the image region where the motion field is known lies entirely in a region where both Z_1 and Z_2 are positive. If one of them changes sign in the region, it cannot be a valid solution.

3.8 Common Ruling

We noted earlier that the line $\mathbf{R} = k\mathbf{t}_0$, where

$$\mathbf{t}_0 = \big((\mathbf{t}_2 \times \mathbf{t}_1) \times \delta\omega\big) \times (\mathbf{t}_2 \times \mathbf{t}_1)$$

lies entirely in the first surface. It also lies entirely in the second surface. Thus, the two surfaces in a critical surface pair not only have a common tangent plane at the origin, but they touch all along a line. Now $\mathbf{t}_0$ is perpendicular to $\mathbf{t}_2 \times \mathbf{t}_1$ and so lies in the plane formed by $\mathbf{t}_1$ and $\mathbf{t}_2$. It can be shown, in fact

$$\mathbf{t}_0 = (\mathbf{t}_2 \times \mathbf{t}_1) \cdot (\mathbf{t}_2 \times \delta\omega)\mathbf{t}_1 - (\mathbf{t}_2 \times \mathbf{t}_1) \cdot (\mathbf{t}_1 \times \delta\omega)\mathbf{t}_2$$

or

$$\mathbf{t}_0 = k_2\mathbf{t}_1 - k_1\mathbf{t}_2$$

where

$$k_2 = (\mathbf{t}_2 \times \mathbf{t}_1) \cdot (\mathbf{t}_2 \times \delta\omega)$$

and

$$k_1 = (\mathbf{t}_2 \times \mathbf{t}_1) \cdot (\mathbf{t}_1 \times \delta\omega)$$

These formulae will prove useful later for they show that $\mathbf{t}_0$ becomes parallel to $\mathbf{t}_2$ when $k_2 = 0$ and parallel to $\mathbf{t}_1$ when $k_1 = 0$.

Consider the projection of the line $\mathbf{R} = k\mathbf{t}_0$ in the image

$$\bar{\mathbf{t}}_0 = \frac{1}{\mathbf{t}_0 \cdot \hat{\mathbf{z}}}\,\mathbf{t}_0$$

Here we just note that $\mathbf{t}_0$ lies in the plane containing $\mathbf{t}_2$ and $\mathbf{t}_1$ and so the projection $\bar{\mathbf{t}}_0$ lines on the line connecting the points $\bar{\mathbf{t}}_1$ and $\bar{\mathbf{t}}_2$. That is, this point lies on the critical image line.

The critical image curve of the first surface intersects the critical image line at $\bar{\mathbf{t}}_2$ and $\bar{\mathbf{t}}_0$ while the critical image curve of the second surface intersects the critical image line at $\bar{\mathbf{t}}_1$ and $\bar{\mathbf{t}}_0$. It can be shown that the two critical image curves are tangent where they cross the critical image line at $\bar{\mathbf{t}}_0$.

3.9 Projections of Rulings

Every point on a doubly ruled surface lies at the intersection of a ruling from one set of rulings with a ruling from the other set. Further, each ruling of one set of rulings of a doubly ruled surface intersects all the rulings of the other set. Now the line $\mathbf{R} = k\mathbf{t}_2$ on the first surface projects into the single image point $\bar{\mathbf{t}}_2$. Thus, all of the rulings that intersect this line project into lines in the image that pass through $\bar{\mathbf{t}}_2$. Similarly, the line $\mathbf{R} = k\mathbf{t}_0$ on the first surface projects into the single image point $\bar{\mathbf{t}}_0$. Thus, all of the rulings that intersect this line project into lines in the image that pass through $\bar{\mathbf{t}}_0$. This completes the description of the projection into the image of the rulings of the first surface.

Similarly, one set of rulings of the second surface project into lines passing through the point $\bar{\mathbf{t}}_1$ while the other set of rulings project into lines passing through the point $\bar{\mathbf{t}}_0$. We conclude that the projections of one set of rulings of the first surface coincide with the projections of one set of rulings of the second surface in a critical surface pair.

3.10 Critical Sections

There are two families of parallel planes that cut an ellipsoid in circular sections. Similarly, there are two families of parallel planes that cut a hyperboloid of one sheet in circular sections [14]. Any plane normal to $\mathbf{t}_2$ satisfies a linear equation of the form

$$\mathbf{R} \cdot \mathbf{t}_2 = d$$

for some constant d. We are to determine the intersection of this plane with the first surface

$$(\mathbf{R} \cdot \mathbf{t}_2)(\delta\omega \cdot \mathbf{R}) - (\mathbf{t}_2 \cdot \delta\omega)(\mathbf{R} \cdot \mathbf{R}) + \mathbf{L} \cdot \mathbf{R} = 0.$$

The intersections will be the same as those of the plane and the new surface

$$d(\delta\omega \cdot \mathbf{R}) - (\mathbf{t}_2 \cdot \delta\omega)(\mathbf{R} \cdot \mathbf{R}) + \mathbf{L} \cdot \mathbf{R} = 0$$

since $\mathbf{R} \cdot \mathbf{t}_2 = d$. The second-order terms in this expression equal $\mathbf{R} \cdot \mathbf{R} = X^2 + Y^2 + Z^2$, so that the above is the equation of a sphere [13]. The intersection of a plane and a sphere is a circle. We conclude that sections of the first surface with planes perpendicular to $\mathbf{t}_2$ are circles. Similar reasoning shows that sections with planes perpendicular to $\delta\omega$ are also circular.

We note that sections of the second surface with planes whose normal is $\mathbf{t}_1$ are circular, as are intersections with planes whose normal is $\delta\omega$. Thus, sections with planes perpendicular to $\delta\omega$ have the unusual property that they cut *both* surfaces in circular sections. It can be shown that the circular section of the first surface is tangent to the circular section of the second surface where they touch.

4 Analysis of Degenerate Cases

We started off by assuming that there exist no special relationships between the vectors $\mathbf{t}_1$, $\mathbf{t}_2$, and $\delta\omega$. In this case, the critical surfaces were shown to be hyperboloids of one sheet. We still have to deal with a number of special cases where there is *no* ambiguity in the motion field as well as other special cases where the critical surface is some degenerate form of a hyperboloid of one sheet.

In some of these situations, we will not be able to use the general equation derived for the critical surface in scene coordinates since that derivation assumed that there were no special relationships between the vectors $\mathbf{t}_1$, $\mathbf{t}_2$, and $\delta\omega$. Instead, we return to the basic equality of two motion fields:

$$\frac{1}{Z_1}((\mathbf{t}_1 \cdot \hat{\mathbf{z}})\mathbf{r} - \mathbf{t}_1) - \frac{1}{Z_2}((\mathbf{t}_2 \cdot \hat{\mathbf{z}})\mathbf{r} - \mathbf{t}_2) = [\mathbf{r}\ \delta\omega\ \hat{\mathbf{z}}]\mathbf{r} - \mathbf{r} \times \delta\omega$$

4.1 Pure Translation or Rotation Known

If $\|\delta\omega\| = 0$ we have

$$\frac{1}{Z_1}((\mathbf{t}_1 \cdot \hat{\mathbf{z}})\mathbf{r} - \mathbf{t}_1) = \frac{1}{Z_2}((\mathbf{t}_2 \cdot \hat{\mathbf{z}})\mathbf{r} - \mathbf{t}_2)$$

Taking the dot-product with $\mathbf{t}_2 \times \mathbf{r}$ we obtain

$$\frac{1}{Z_1}[\mathbf{t}_2\ \mathbf{t}_1\ \mathbf{r}] = 0$$

If this is to be true for all $\mathbf{r}$ in some image region, we must have $\mathbf{t}_2 \parallel \mathbf{t}_1$, or $\mathbf{t}_2 = k\mathbf{t}_1$, say. Then, from the first equation above, we see that $Z_2 = kZ_1$. This confirms the well-known result that if the motion is known to be purely translational, the motion field determines the motion and the surface uniquely up to a scale factor.

The same applies when the rotation is known, because then we can predict the rotational component of the motion field and subtract it to reduce the situation to the one just discussed.

Now suppose that $\mathbf{t}_2 \parallel \mathbf{t}_1$. Then, if we take the dot-product of the basic equation with $\mathbf{t}_1 \times \mathbf{r}$ we obtain

$$0 = -(\mathbf{r} \times \delta\omega) \cdot (\mathbf{t}_1 \times \mathbf{r})$$

or,

$$(\mathbf{r} \cdot \mathbf{r})(\mathbf{t}_1 \cdot \delta\omega) = (\mathbf{r} \cdot \mathbf{t}_1)(\delta\omega \cdot \mathbf{r})$$

This can only hold true if $\|\delta\omega\| = 0$ (or $\|\mathbf{t}_1\| = 0$). Thus, $\mathbf{t}_2$ is parallel to $\mathbf{t}_1$ only in the case that the two motions have the same rotational components (as, for example, when they are both known to be purely translational) and so we will assume from now on that

$$\|\mathbf{t}_2 \times \mathbf{t}_1\| \neq 0$$

4.2 Pure Rotation

If $\|\mathbf{t}_2\| = 0$, we have

$$\frac{1}{Z_1}\big((\mathbf{t}_1 \cdot \hat{\mathbf{z}})\mathbf{r} - \mathbf{t}_1\big) = [\mathbf{r}\ \delta\omega\ \hat{\mathbf{z}}]\mathbf{r} - \mathbf{r} \times \delta\omega$$

Taking the dot-product with $\mathbf{t}_1 \times \mathbf{r}$ we obtain again

$$0 = (\mathbf{r} \times \delta\omega) \cdot (\mathbf{t}_1 \times \mathbf{r})$$

or

$$(\mathbf{r} \cdot \mathbf{r})(\mathbf{t}_1 \cdot \delta\omega) = (\mathbf{r} \cdot \mathbf{t}_1)(\delta\omega \cdot \mathbf{r})$$

If this is to be true for all $\mathbf{r}$ in some image region, we must have $\|\mathbf{t}_1\| = 0$ or $\|\delta\omega\| = 0$. We have already dealt with the latter case. In the case that $\|\mathbf{t}_1\| = 0$, we have

$$[\mathbf{r}\ \delta\omega\ \hat{\mathbf{z}}]\mathbf{r} - \mathbf{r} \times \delta\omega = 0$$

which again implies that $\|\delta\omega\| = 0$, that is, $\omega_2 = \omega_1$.

This confirms the well-known result that the motion field uniquely determines the motion in the case of pure rotation. From now on, we may assume that $\|\delta\omega\| \neq 0$, $\|\mathbf{t}_1\| \neq 0$, and $\|\mathbf{t}_2\| \neq 0$.

4.3 Elliptic Cone

When $k_2 = 0$, where

$$k_2 = (\mathbf{t}_2 \times \delta\omega) \cdot (\mathbf{t}_2 \times \mathbf{t}_1)$$

that is, when the plane containing $\mathbf{t}_2$ and $\delta\omega$ is orthogonal to the plane containing $\mathbf{t}_2$ and $\mathbf{t}_1$, we obtain an elliptic cone. This is because the constant term $c_1 = -(k_2^2/2\sigma_2^3\tau_2(1 - \tau_2^2))$ in the equation

$$(\mathbf{R}')^T M \mathbf{R}' = c_1$$

becomes zero. Of course, k_2 is zero when $\|\mathbf{t}_2 \times \mathbf{t}_1\| = 0$ or $\|\mathbf{t}_2 \times \delta\omega\| = 0$, but we have already dealt with these special cases. Another special case is the one where $\delta\omega$ is actually parallel to $\mathbf{t}_2 \times \mathbf{t}_1$. In this case, the elliptic cone degenerates into two intersecting planes, as we show in detail later. Other special cases can be studied by expanding k_2 to yield

$$k_2 = (\mathbf{t}_2 \cdot \mathbf{t}_2)(\delta\omega \cdot \mathbf{t}_1) - (\mathbf{t}_2 \cdot \mathbf{t}_1)(\delta\omega \cdot \mathbf{t}_2)$$

Since $k_2 = 0$, the center may be found at $\mathbf{C}_1$ where

$$\sigma_2(1 - \tau_2)\mathbf{C}_1 = -\upsilon \mathbf{t}_2$$

where $\upsilon = [\mathbf{t}_2\ \mathbf{t}_1\ \delta\omega]$, as always. This is not surprising since all rulings pass through the center and we know that $\mathbf{R} = k\mathbf{t}_2$ is a ruling.

Note that a critical surface cannot be a *circular* cone, because that would require that two of the eigenvalues were equal with the third of opposite sign and nonzero. This cannot happen when $\lambda_- + \lambda_+ = \lambda_0$. We note that a critical surface cannot be an arbitrary elliptic cone since there is a constraint on the length of the axes.

The elliptic cone is a central, but improper, quadric. It can be thought of as a degenerate hyperboloid of one sheet obtained by shrinking all of the semi-axes in such a way that their ratios are maintained. The two sets of rulings collapse into one, since $\mathbf{t}_0$ becomes parallel to $\mathbf{t}_2$ when $k_2 = 0$.

4.4 Hyperbolic Paraboloid

Consider now the equation

$$(\mathbf{R} \cdot \mathbf{t}_2)(\delta\omega \cdot \mathbf{R}) - (\mathbf{t}_2 \cdot \delta\omega)(\mathbf{R} \cdot \mathbf{R}) + (\mathbf{t}_2 \times \mathbf{t}_1) \cdot \mathbf{R} = 0$$

for the first critical surface. Suppose that $\mathbf{t}_2 \cdot \delta\omega = 0$, then the equation becomes just

$$(\mathbf{R} \cdot \mathbf{t}_2)(\delta\omega \cdot \mathbf{R}) + (\mathbf{t}_2 \times \mathbf{t}_1) \cdot \mathbf{R} = 0$$

since the term in $(\mathbf{t}_2 \cdot \delta\omega)(\mathbf{R} \cdot \mathbf{R})$ drops out. We have $\tau_2 = 0$ and hence $\lambda_0 = 0$ with $\lambda_+ = \sigma_2$ and $\lambda_- = -\sigma_2$. The degenerate surface we are dealing with here does not have a center. It is a hyperbolic paraboloid with equal axes. Any section with a plane perpendicular to $\mathbf{t}_2 \times \delta\omega$ will yield a hyperbola.

Similarly, if $\mathbf{t}_1 \cdot \delta\omega = 0$, we find that the second critical surface degenerates into a hyperbolic paraboloid. We may note that in these cases the formula for Z as a function of x and y factors so that it is a ratio of a linear polynomial in x and y and a product of two linear polynomials.

A hyperbolic paraboloid is doubly ruled. The lines in each of the two sets of rulings are parallel to a given plane [14]. We can verify this by intersecting the hyperbolic paraboloid first with planes orthogonal to $\mathbf{t}_2$, given by

$$\mathbf{R} \cdot \mathbf{t}_2 = d$$

The intersection of this plane and the hyperbolic paraboloid is the intersection of the plane and the new surface,

$$d(\delta\omega \cdot \mathbf{R}) + (\mathbf{t}_2 \times \mathbf{t}_1) \cdot \mathbf{R} = 0$$

since $\mathbf{R} \cdot \mathbf{t}_2 = d$. But this is the equation of a plane, and two planes intersect in a line. Thus, all intersections with planes orthogonal to $\mathbf{t}_2$ are rulings. Similarly, we find that all intersections with planes orthogonal to $\delta\omega$ are rulings.

The lines of intersection can be thought of as circles with infinite radius and correspond in this way with the circular sections we obtained in the case of the hyperboloid of one sheet. The hyperbolic paraboloid is a proper quadric, but not central. While there is not a center, there is one distinguished point on its surface, the saddle point, which we shall find next.

4.4.1 Saddle Point. If $\mathbf{t}_2 \cdot \delta\omega = 0$, then $\mathbf{t}_2$, $\delta\omega$, and $\mathbf{t}_2 \times \delta\omega$ are orthogonal to one another. Two of the axes of the quadric surface lie in the plane containing $\mathbf{t}_2$ and $\delta\omega$. Shifting the origin to the saddle point $\mathbf{S}$ will make the linear term proportional to $(\mathbf{t}_2 \times \delta\omega) \cdot \mathbf{R}$. It can be shown (tediously) that the saddle-point for the first surface is at

$$\mathbf{S}_1 = -\frac{\upsilon}{\sigma_2^2}\mathbf{t}_2$$

by substituting $\mathbf{R} = \mathbf{S}_1 + \mathbf{R}'$. The equation of the surface then has the form

$$(\mathbf{R}' \cdot \mathbf{t}_2)(\delta\omega \cdot \mathbf{R}') + \frac{k_2}{\sigma_2^2}(\mathbf{t}_2 \times \delta\omega) \cdot \mathbf{R} = 0$$

In proving this result, it is useful to note that when $\mathbf{t}_2 \cdot \delta\omega = 0$,

$$\sigma_2^2(\mathbf{t}_2 \times \mathbf{t}_1) = \upsilon \, \| \mathbf{t}_2 \|^2 \delta\omega + k_2(\mathbf{t}_2 \times \delta\omega)$$

We note that the saddle point is imaged at the focus of expansion $\bar{\mathbf{t}}_2$.

4.5 Intersecting Planes

Now consider the special case when both $\mathbf{t}_2 \cdot \delta\omega = 0$ and $\mathbf{t}_1 \cdot \delta\omega = 0$. In fact, let $\delta\omega = k\mathbf{t}_2 \times \mathbf{t}_1$. Then, the equation for the first surface becomes

$$k(\mathbf{R} \cdot \mathbf{t}_2)[\mathbf{t}_2\, \mathbf{t}_1\, \mathbf{R}] - [\mathbf{t}_2\, \mathbf{t}_1\, \mathbf{R}] = 0$$

since the term $(\mathbf{t}_2 \cdot \delta\omega)(\mathbf{R} \cdot \mathbf{R})$ drops out. We can rewrite this in the form

$$\big((\mathbf{R} \cdot \mathbf{t}_2) + 1\big)[\mathbf{t}_2\, \mathbf{t}_1\, \mathbf{R}] = 0$$

So either $[\mathbf{t}_2\, \mathbf{t}_1\, \mathbf{R}] = 0$ or $k(\mathbf{R} \cdot \mathbf{t}_2) + 1 = 0$. The first equation defines a plane passing through the origin which projects into the image as a line. The second equation defines a plane with normal $\mathbf{t}_2$ with perpendicular distance $1/(k \, \| \mathbf{t}_2 \|)$ from the origin.

Similarly, the equation for the second critical surface yields the intersecting planes $[\mathbf{t}_2\, \mathbf{t}_1\, \mathbf{R}] = 0$ and $k(\mathbf{R} \cdot \mathbf{t}_1) + 1 = 0$. This is the celebrated case of the dual planar solution [3,4,5]. If we ignore the plane $[\mathbf{t}_1\, \mathbf{t}_2\, \mathbf{R}] = 0$, which projects into a line, we find that the plane with normal $k\mathbf{t}_2$ and motion $\{\mathbf{t}_1, \omega_1\}$ yields the same motion field as the plane with normal $k\mathbf{t}_1$ and motion $\{\mathbf{t}_2, \omega_1 + k\mathbf{t}_2 \times \mathbf{t}_1\}$.

It is interesting to note that the critical image curve here degenerates into two intersecting lines, one of which coincides with the critical image line. As a result, Z does *not* change sign across the critical image line in this special case. The expressions for depth as a function of image coordinates x and y simplify to

$$Z_1 = -\frac{1}{k(\mathbf{r} \cdot \mathbf{t}_2)} \quad \text{and} \quad Z_2 = -\frac{1}{k(\mathbf{r} \cdot \mathbf{t}_1)}$$

since the other linear term, $[\mathbf{t}_2\, \mathbf{t}_1\, \mathbf{r}]$, cancels. Note that if one of the critical surfaces consists of a pair of intersecting planes, so does the other (see also Maybank [11]).

4.6 Circular Cylinder

Consider the case when $\|\mathbf{t}_2 \times \delta\omega\| = 0$, that is, when $\mathbf{t}_2 \| \delta\omega$. We have either $\tau = +1$ and $\lambda_+ = 0$, $\lambda_0 = -2\sigma$, $\lambda_- = -2\sigma$ or $\tau = -1$ and $\lambda_+ = 2\sigma$, $\lambda_0 = 2\sigma$, $\lambda_- = 0$. In this case, we have two equal eigenvalues. The surface is a cylinder as we show next.

If $\mathbf{t}_2 = k\delta\omega$, the equation of the surface becomes

$$k\,\|\mathbf{R} \times \delta\omega\|^2 + k[\delta\omega\, \mathbf{t}_1\, \mathbf{R}] = 0$$

where we can cancel the constant k immediately. By moving the origin to a point $\mathbf{A}$ on the axis of the cylinder, we should be able to remove the linear term. It can be shown that

$$\mathbf{A} = -\frac{1}{2\,\|\delta\omega\|^2}(\delta\omega \times \mathbf{t}_1)$$

is such a point—actually the one closest to the origin. Let $\mathbf{R} = \mathbf{A} + \mathbf{R}'$. Then

$$\|\mathbf{R} \times \delta\omega\|^2 = \|\mathbf{R}' \times \delta\omega\|^2 - [\delta\omega\, \mathbf{t}_1\, \mathbf{R}'] + \|\mathbf{A} \times \delta\omega\|^2$$

and

$$\|\mathbf{A} \times \delta\omega\|^2 = \frac{\|\delta\omega \times \mathbf{t}_1\|^2}{4\,\|\delta\omega\|^2}$$

while

$$[\delta\omega\, \mathbf{t}_1\, \mathbf{R}] = [\delta\omega\, \mathbf{t}_1\, \mathbf{R}'] - \frac{\|\delta\omega \times \mathbf{t}_1\|^2}{2\,\|\delta\omega\|^2}$$

The equation of the surface thus becomes

$$\|\mathbf{R}' \times \delta\omega\|^2 = \frac{\|\delta\omega \times \mathbf{t}_1\|^2}{4\,\|\delta\omega\|^2}$$

the equation of a cylinder with axis in the direction $\delta\omega$ through the point $\mathbf{A}$ and with radius $\|\delta\omega \times \mathbf{t}_1\|/(2\|\delta\omega\|)$. This follows from the fact that the length of the vector

$$\frac{\mathbf{R}' \times \delta\omega}{\|\delta\omega\|}$$

is the same as that of the vector connecting $\mathbf{R}'$ to the nearest point on the axis along the direction $\delta\omega$.

4.7 Remaining Special Cases

Nothing particularly interesting happens when $\mathbf{t}_1 \cdot \mathbf{t}_2 = 0$, so we need not have included the constraint $\mathbf{t}_1 \cdot \mathbf{t}_2 \neq 0$ in our initial list.

When $[\mathbf{t}_2\, \mathbf{t}_1\, \delta\omega] = 0$, that is, when $\delta\omega$ is coplanar with $\mathbf{t}_2$ and $\mathbf{t}_1$, some simplification occurs. For example,

$$\mathbf{t}_0 = \|\mathbf{t}_2 \times \mathbf{t}_1\|^2\, \delta\omega$$

and

$$\sigma_2(1 - \tau_2^2)\mathbf{C}_1 = \frac{k_2}{2\sigma_2\tau_2}\,\hat{\mathbf{t}}_2 \times \delta\hat{\omega}$$

in this case. But nothing else of great note occurs.

We are left with a number of multiple degeneracies leading to "solutions" involving only lines or points and so of no real interest.

5 An Example

Suppose that we are given the motions $\{\mathbf{t}_1, \omega_1\}$ and $\{\mathbf{t}_2, \omega_2\}$, where

$$\mathbf{t}_1 = 9(0, 0, 1)^T$$

$$\mathbf{t}_2 = (0, 4, 5)^T$$

and

$$\omega_1 - \omega_2 = (0, -4, 5)^T$$

and are to construct two surfaces that yield the same motion field. Now

$$\mathbf{R}_1 \times \delta\omega = (5Y + 4Z, -5X, -4X)^T$$

while

$$\mathbf{t}_2 \times \mathbf{R}_1 = (-5Y + 4Z, +5X, -4X)^T$$

so

$$(\mathbf{R}_1 \times \delta\omega) \cdot (\mathbf{t}_2 \times \mathbf{R}_1) = -(3X)^2 - (5Y)^2 + (4Z)^2$$

Also

$$[\mathbf{t}_2\, \mathbf{t}_1\, \mathbf{R}_1] = 36X$$

so the equation of one of the critical surfaces is

$$-(3X)^2 - (5Y)^2 + (4Z)^2 + 36X = 0$$

Thus one of the critical surfaces has center $(2, 0, 0)^T$ and principal axes parallel to those of the coordinate system. This hyperboloid is "open" in the direction of the Z-axis. The lengths of the principal axes are 2, (6/5), and (3/2). Note that

$$\left(\frac{1}{2}\right)^2 + \left(\frac{2}{3}\right)^2 = \left(\frac{5}{6}\right)^2$$

Now

$$\mathbf{R}_2 \times \delta\omega = (5Y + 4Z, -5X, -4X)^T$$

while

$$\mathbf{t}_1 \times \mathbf{R}_2 = 9(-Y, X, 0)^T$$

So

$$(\mathbf{R}_2 \times \delta\omega) \cdot (\mathbf{t}_1 \times \mathbf{R}_2) = -9(5Y^2 + 4YZ + 5X^2)$$

while

$$[\mathbf{t}_2\ \mathbf{t}_1\ \mathbf{R}_2] = 36X$$

So the equation of the other critical surface is

$$(5Y^2 + 4YZ + 5X^2) - 4X = 0$$

or

$$4\sqrt{41}(5X - 2)^2 + 5((\sqrt{41} + 5)Y + 4Z)^2 - 5((\sqrt{41} - 5)Y - 4Z)^2 = 16\sqrt{41}$$

This critical surface has center $(2/5, 0, 0)^T$ with principal axes in the directions $(1, 0, 0)^T$, $(0, \sqrt{41} + 5, 4)^T$, and $(0, \sqrt{41} - 5, -4)^T$.

Substituting $X = xZ$ and $Y = yZ$ into the equation of the first surface we find

$$Z_1 = \frac{36x}{9x^2 + 25y^2 - 16}$$

while the equation of the second surface yields

$$Z_2 = \frac{4x}{5x^2 + 5y^2 + 4y}$$

Now

$$\mathbf{r}_t = \frac{1}{Z}((\mathbf{t} \cdot \hat{\mathbf{z}})\mathbf{r} - \mathbf{t}) + [\mathbf{r}\ \omega\ \hat{\mathbf{z}}]\mathbf{r} - \mathbf{r} \times \omega$$

which, for surface Z_1, yields

$$\frac{9x^2 + 35y^2 - 16}{4x}(\mathbf{r} - \mathbf{t}_1) + [\mathbf{r}\ \omega_1\ \hat{\mathbf{z}}]\mathbf{r} - \mathbf{r} \times \omega_1$$

and, for surface Z_2:

$$\frac{5x^2 + 5y^2 + 4y}{4x}(5\mathbf{r} - \mathbf{t}_2) + [\mathbf{r}\ \omega_1\ \hat{\mathbf{z}}]\mathbf{r} - \mathbf{r} \times \omega_1 + [\mathbf{r}\ \delta\omega\ \hat{\mathbf{z}}]\mathbf{r} - \mathbf{r} \times \delta\omega$$

Thus, ignoring the common term $[\mathbf{r}\ \omega_1\ \hat{\mathbf{z}}]\mathbf{r} - \mathbf{r} \times \omega_1$, we have

$$u_1 = \frac{9x^2 + 25y^2 - 16}{4}$$

$$v_1 = \frac{9x^2 + 25y^2 - 16}{4x} y$$

and

$$u_2 = 5\,\frac{5x^2 + 5y^2 + 4y}{4} + (-4x)x - (5y + 4)$$

$$= u_1$$

$$v_2 = \frac{5x^2 + 5y^2 + 4y}{4x}(5y - 4) + (-4x)y + 5x$$

$$= v_1$$

The critical image line is given by $[\mathbf{t}_2\,\mathbf{t}_1\,\mathbf{r}] = 0$, which here is $x = 0$. The critical image curves are $9x^2 + 25y^2 - 16 = 0$ for Z_1 and $5x^2 + 5y^2 + 4y = 0$ for Z_2. These can also be written

$$\left(\frac{x}{4/3}\right)^2 + \left(\frac{y}{4/5}\right)^2 = 1$$

and

$$\left(\frac{x}{2/5}\right)^2 + \left(\frac{y + 2/5}{2/5}\right)^2 = 1$$

The first of these curves passes through the focus of expansion $\bar{\mathbf{t}}_2 = (0, 4/5, 1)^T$, while the second passes through the focus of expansion $\bar{\mathbf{t}}_1 = (0, 0, 1)^T$. Both intersect the critical image line at $\bar{\mathbf{t}}_0 = (0, -4/5, 1)^T$.

6 Gefährliche Flächen

Before a stereo pair can be used to recover surface topography, one has to determine the relationship between coordinate systems fixed in the two cameras at the time of exposure. The *relative orientation* of the cameras can be found from the coordinates of the images of five (or more) points [16, 17, 18]. There is no closed-form solution of the relative orientation problem (as yet), so iterative numerical methods are used in practice.

The problem of recovering rigid body motion from the (estimated) motion fields is related to that of recovering the transformation from one camera position to the other in binocular stereo. The difference is that in motion vision one usually deals with infinitesimal motions and so can use a vector to represent rotation, while in binocular stereo the translations and rotations are finite and an orthonormal matrix (or unit quaternion) is needed to represent rotation. Iterative schemes for solving the relative orientation problem essentially linearize the problem by restricting adjustments of the relative position and orientation of the cameras to infinitesimal quantities. Thus the incremental adjustment in relative orientation is closely related to the problem of recovering camera motion. We expect then that the problem of relative orientation cannot be solved when the given points happen to lie on a critical surface. (Also, we might expect that the relative orientation cannot be found accurately when the points lie near such a surface.)

Surfaces that lead to difficulties in recovering the relative orientation have been studied in stereo-photogrammetry and are called *Gefährliche Flächen* [19, 20]. This term, although usually translated as *critical surfaces* [17, 18], actually means *dangerous surfaces*. It is because of the relationship between the motion vision and binocular stereo problems that I borrowed the term *critical surface* for the discussion here.

7 Conclusions

I have shown that only certain hyperboloids of one sheet and their degenerate forms can give rise to ambiguous motion fields. These special hyperboloids have to be viewed from a point on their surface. Also, even these surfaces lead to motion fields that are ambiguous only when attention is confined to certain image regions. In general, the motion vision problem is not ambiguous.

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