6.977: Reasoning about sequential programs **CSAIL, MIT**
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Exercise set 3

Galois connections and "Finishing School"

Background

We recall that Galois-connected functions f, g are "nearly" inverses. If one starts with an x, then applies f to get some y and finally applies g to get an x' again, then if $x \neq x'$ they are guaranteed nevertheless to be ordered always in the same way — that is we have $x \sqsubseteq x' = g.(f.x)$ for all x.

Thus "sending an x over and back" is a bit like sending someone to finishing school: they return possibly better than before (or worse — depends on the school). But –whichever way it is– it's always the *same* way, for that school, and sending them a second time has no effect since $(g \circ f) \circ (g \circ f) = g \circ f$. The better/worse-than-before analogy is strengthened by the fact that the new value x' is guaranteed to have some properties that the original x did not (*i.e.* social graces, in our analogy), since it's in the image of g whereas x might not have been.

The ceiling/identity and floor/identity Galois connections have been mentioned before; but *as well* we have ceiling/floor directly forming a Galois connection between two copies of the reals. (Is there an easy way of seeing this immediately from what we know about ceiling/identity and floor/identity already? Do those two connections compose?) "Finishing" in this case is applying $\lceil \lfloor \cdot \rfloor \rceil$ (or the dual), and a real number acquires social graces by moving to the next higher integer (or, dually, the next lower), losing that ugly fractional part.

Exercise 3.1

What social graces does a relation $r : S \to \mathbb{P}S_\perp$ gain by being sent to finishing school with $rp \circ wp$? (Note we've used $\mathbb{P}$ and not $\mathbb{P}_\uparrow$.)

Exercise 3.2

What social graces does monotonic transformer $t : \mathbb{P}S \leftarrow \mathbb{P}S$ acquire from the $wp \circ rp$ experience?

Hint: Define angelic choice, and experiment with the non-conjunctive (but still monotonic) transformer $t := wp.[\![n{:\in}\{0,1\} \sqcup n{:\in}\{0,2\}]\!]$, where in general $x{:\in}E$ chooses x demonically from non-empty (finite) E. Thus t is an angelic choice between two demonic choices. Check carefully that the new transformer $t' := wp.(rp.t)$ really is a proper refinement of it. What other property does t' have that t did not?