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Instant probabilistic/demonic powerdomain:

"It's marvellous — you just add water!"

1. Start with a finite state space S.
2. Extend it to $S_\perp$ by adding an extra $\perp$-state for describing non-termination.
3. Make that into a *cpo* by adding the "flat order" $\perp \sqsubset s$ for all $s \neq \perp$ and no others.
4. Impose the Scott topology on that which, when all chains are finite, amounts to defining open sets to be simply the $\sqsubseteq$-up-closed ones.
5. Define *evaluations* (à la Clare Jones) to be measure-like functions (in the sense of probability measures) but limited to the open sets (*i.e.* in $\mathcal{O} \subseteq \mathbb{P}S$ only) rather than to the whole Borel algebra, which would have included closed sets as well.
6. For evaluations $\mu, \mu' \in \mathcal{O} \to [0, 1]$ define a refinement order $\mu \sqsubseteq \mu'$ just when $\mu.O \leq \mu'.O$ for all open sets $O \in \mathcal{O}$.
7. Observe that a *discrete* evaluation (*i.e.* Jones-style measure) over the finite set $S_\perp$, as above, is determined by its values on the proper (*i.e.* non-$\perp$) singleton sets $\{s\}$, in which case we might as well just write $\mu.s$ instead of $\mu.\{s\}$.
8. Observe further that we then have $\mu \sqsubseteq \mu'$ in this simple case just when $\mu.s \leq \mu.s'$ for all $s \neq \perp$ (and that $\mu.\perp$ is not defined, anyway, since it would stand for $\mu.\{\perp\}$ which is not defined because $\{\perp\}$ is not open.)

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9. And so we can just take our measures/evaluations to be discrete *sub*-distributions over S alone (no $\perp$), that is $\delta \in S \to [0, 1]$ with $\sum \delta \leq 1$, and then refinement is just pointwise $\leq$ of those functions.
10. Then refinement- up-close those sets of sub-distributions (as for the Smyth powerdomain) and add convex- and Cauchy closure:
    instant probabilistic/demonic powerdomain.