Ohm's Law, Kirchoff 's Law and the Drunkard's Walk

2. The Drunkard's Walk

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We change gears in this second part of the article and discuss one of the oldest problems in probability theory, viz. the simple random walk. The recurrence and transience of the random walk in different dimensions is later derived using the material from the first part.

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Imagine a drunkard walking on the street depicted by the line in *Figure 1.* At one end of the street is the bar and at the other end is his home. The bar and his home are on the same street and separated by n blocks. The drunkard starts walking from the crossing k $(0 \leq k \leq n)$ and walks a block left or right randomly with equal probability. When he reaches the next crossing he again randomly chooses the direction of his walk. If he reaches either the bar or his home he stays there. Here we could ask the question: what is the probability $p(k)$ that the man starting from k reaches home before he reaches the bar?

Clearly, if he starts at the bar (i.e. the point 0), then the probability of reaching home before the bar is 0, and if he starts from his home (i.e. the point n) then the probability of reaching home before the bar is 1. So we have

$$p(0) = 0 \text{ and } p(n) = 1. \tag{1}$$

Moreover, starting from k, $(1 \leq k \leq n-1)$, he could first

Figure 1.

walk a block left to $k-1$ with probability $1/2$ and from there the probability that he reaches his home before he reaches the bar is $p(k-1)$; or he could first walk a block right to $k+1$ with probability $1/2$ and from there the probability that he reaches his home before he reaches the bar is $p(k+1)$. (A mathematically rigorous argument may be provided for the previous statement by using conditional probability.) So we have

$$p(k) = \frac{1}{2}p(k-1) + \frac{1}{2}p(k+1) \text{ for } 1 \le k \le n-1. \qquad (2)$$

Here we note the similarity of the above equations with those in (1) and (2) of Part 1, and so we have the solution of (1) and (2) given by

$$p(k) = k/n \text{ for } k = 0, 1, \ldots, n. \qquad (3)$$

Of course it is unreasonable to expect that the drunkard's home and the bar would be on the same street, so instead let us look at a grid as in *Figure 2* of Part 1. Let (k,l) denote the coordinates of a point on the grid, and $p(k,l)$ denote the probability that the drunkard reaches the 'upper' boundary before the 'south-eastern' boundary. Then, as before, we have the boundary conditions

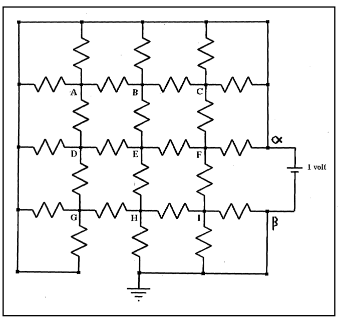

Figure 2 of part I.

$$p(k,l) = \begin{cases} 0 & \text{for } (k,l) \text{ on the 'south-east' boundary} \\ 1 & \text{for } (k,l) \text{ on the 'upper' boundary.} \end{cases} \qquad (4)$$

An argument, similar to that used to obtain (2), based on the block, out of the four possible, which the drunkard takes when he starts his walk from (k,l), yields

$$p(k,l) = \frac{p(k+1,l) + p(k-1,l) + p(k,l+1) + p(k,l-1)}{4}. \qquad (5)$$

These equations too are similar to those we obtained in (5) and (4) of Part 1, and so the solution is exactly the same as that obtained there.

More generally if we have a graph $\mathcal{G}$ which consists of a vertex set $\mathcal{V}$ and an edge set $\mathcal{E}$, a simple random walk is defined to be a random walk such that the probability P_{xy} of going from a vertex $x \in \mathcal{V}$ to a vertex $y \in \mathcal{V}$ in one step is given by

$$P_{xy} = \begin{cases} \dfrac{1}{\text{(number of vertices}} & \text{(if } y \text{ is adjacent to } x\text{)} \\ \text{adjacent to the vertex } x\text{)} & \\ 0 & \text{(if } y \text{ is not adjacent to } x\text{).} \end{cases}$$

Here two vertices are adjacent if they are connected by an edge in $\mathcal{E}$.

Identify two vertices α and β on this graph $\mathcal{G}$ and let $p(x)$ denote the probability of reaching α before β. This quantity $p(x)$ can also be obtained by using an electrical circuit. For this we start with the same graph $\mathcal{G}$ and its vertex set $\mathcal{V}$ and edge set $\mathcal{E}$. We replace each edge by an r ohm resistor and apply a potential of 1 volt between the vertices α and β which we identified earlier. We also ground the vertex β. Let $v(x)$ denote the voltage at the vertex x. Then the following equations in f, for some function f taking values between 0 and 1,

$$f(\alpha) = 1 \text{ and } f(\beta) = 0, \tag{6}$$

and

$$f(x) = \frac{\sum_y f(y)}{\text{number of vertices adjacent to the vertex } x} \tag{7}$$

where the sum is over all vertices y adjacent to the vertex x, must be satisfied for both $f = p$ and $f = v$ and as such the solution $p(x)$ to our probability problem and the solution $v(x)$ to our electricity problem must coincide, i.e.

$$p(x) = v(x). \tag{8}$$

For the mathematically alert reader, the similarity of the equations above with those in section 1 of Part 1 should not be surprising, because they stem from the same *harmonic function* f satisfying (6) and (7) above. Having identified the similarity of the two problems, let us make some probabilistic sense of the electrical notions introduced earlier. Clearly,

$$P_{xy} = \frac{C_{xy}}{C_x}$$

and so, in view of (8), the equation (7) of Part 1 reduces to

$$p(x) = \sum_y p(y) P_{xy}, \tag{9}$$

while equation (6) of Part 1 yields

$$\begin{aligned} i_\alpha &= \sum_y (v(\alpha) - v(y)) C_{\alpha y} = \sum_y p(\alpha) C_{\alpha y} - \sum_y p(y) C_{\alpha y} \\ &= \sum_y C_{\alpha y} - \sum_y p(y) \frac{C_{\alpha y}}{C_\alpha} C_\alpha \\ &= C_\alpha - C_\alpha \sum_y p(y) P_{\alpha y} = C_\alpha [1 - \sum_y p(y) P_{\alpha y}], \qquad (10) \end{aligned}$$

where the sum is over all vertices y adjacent to the vertex α. Equation (10), together with (9) of Part 1 yields

$$\sum_y p(y) P_{\alpha y} = 1 - \frac{i_\alpha}{C_\alpha} \;\; = \;\; 1 - \frac{1}{\rho C_\alpha}. \tag{11}$$

Let us now understand the term $\sum_y p(y) P_{\alpha y}$ in (11). Suppose the drunkard starts walking from α. If he goes to y in the first step, then the probability of returning to α from y before going to β is the term $p(y)$ which we defined earlier. However the drunkard could go to any of the adjacent vertices of α at the first step and he chooses the adjacent vertex y with probability $P_{\alpha y}$. Hence $p(y)P_{\alpha y}$ denotes the probability that, starting at α, the drunkard takes the first step to y and thereafter returns to α before going to β. Thus we see that $\sum_y p(y) P_{\alpha y}$ denotes the probability that the drunkard, starting from α, returns to α before going to β.

In the case of an infinite graph as the d–dimensional lattice $\mathbf{Z}^d$ for $d \geq 1$, β is at 'infinity' and so we have

$$P(\text{the drunkard starting at 0 returns to 0 in } \mathbf{Z}^d) = 1 - \frac{1}{\rho_d C_\alpha}, \tag{12}$$

where ρ_d is the effective resistance of the grid $\mathbf{Z}^d$ each edge of which is an r ohms resistor.

This leads us to the main point of this article. If the graph $\mathcal{G}$ is the d–dimensional integer lattice $\mathbf{Z}^d$ and α is the origin 0, then we have

$$\rho_d \begin{cases} = \infty \text{ for } d = 1 \text{ or } 2 \\ < \infty \text{ for } d \geq 3 \end{cases}$$

and hence,

$$\text{for } d = 1 \text{ or } 2, P(\text{the drunkard starting at 0 returns to 0 in } \mathbf{Z}^d) = 1, \quad (13)$$

$$\text{for } d \geq 3, \ \mathrm{P}(\text{the drunkard starting at 0 returns to 0 in } \mathbf{Z}^d) < 1. \quad (14)$$

After the first return to 0 it is as if the drunkard is starting his walk all over again, and so, for $d = 1$ or 2, with probability 1 he will again return to 0. (In any case, if 0 is the bar, then a few drinks after his first return to the bar the drunkard would have quite certainly forgotten about his previous sojourn!) This argument can be repeated for each of his returns to 0 and so we have, for $d = 1$ or 2,

$$P(\text{the drunkard starting at 0 returns to 0 infinitely often in } \mathbf{Z}^d) = 1. \quad (15)$$

However in 3 or higher dimensions, let p denote the probability that the drunkard, starting at 0 returns to 0, i.e.,

$$p = P(\text{the drunkard starting at 0 returns to 0 in } \mathbf{Z}^d). \quad (16)$$

After the drunkard's first return to 0, it is as if he starts his walk again, so the probability that after his first return to 0 he returns again is p. Hence

$$P(\text{the drunkard starting at 0 returns to 0 at least twice in } \mathbf{Z}^d) = p^2.$$

Repeating this argument we have

$$P(\text{the drunkard starting at 0 returns to 0 at least } n \text{ times in } \mathbf{Z}^d) = p^n.$$

Suggested Reading

- Polya G. Uber eine aufgabe der wahrscheinlichkeitsrechnung bettreffend die irrfahrt in strassennetz. *Mathematische Annalen.* Vol. 84. pp. 149-160, 1921.
- Feller W. *Introduction to probability theory and its applications.* Vols. 1 and 2. Wiley Eastern. New Delhi, 1977.
- Doyle P and Snell J L. Random walk and electrical networks. *Carus mathematical monograph no. 22.* American Mathematical Association. Washington DC, 1984.

For $d \geq 3$, by (14), $p < 1$, so $p^n \to 0$ as $n \to \infty$, hence, for $d \geq 3$

$$P(\text{the drunkard starting at 0 returns to 0 infinitely often in } \mathbf{Z}^d) = 0.$$

The above explains why in probability jargon it is said that a simple random walk is recurrent in 2 or lower dimensions and transient in 3 or higher dimensions.

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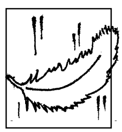
End notes: The original question of recurrence and transience of random walks was asked and solved by George Polya in a celebrated paper in 1921. This method of looking at the question through electrical networks is based on a monograph by Doyle and Snell, where besides the question of recurrence and transience, many other questions of simple random walks are addressed.



"Believe nothing
merely because you have been
told it
Or because it is traditional
Or because you yourself have
imagined it
Do not believe what your
teacher tells you
merely out of respect for the
teacher
But whatever after due
examination and analysis
you find to be conducive to the
good
the benefit,
the welfare of all beings,
that doctrine believe and cling to
and take it as your guide".

The Buddha