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Notes on Galois connections

Two definitions

One definition

We have two partially ordered sets X, Y and functions $f\colon X \to Y$ and $g\colon Y \to X$ satisfying

$$f.x \sqsubseteq y \quad \text{iff} \quad x \sqsubseteq g.y \; . \tag{1}$$

The functions f, g are a *Galois connection* between X and Y. It's an interesting (and useful) construction because there are so many nice things that can be easily proved; for example

1. $x \sqsubseteq g.(f.x)$ because $f.x \sqsubseteq f.x$.
2. $f.(g.y) \sqsubseteq y$ because $g.y \sqsubseteq g.y$.
3. $f.x \sqsubseteq f.x'$ if $x \sqsubseteq x'$ because $x \sqsubseteq x' \sqsubseteq g.(f.x')$, the last from (1.) above.
4. $g.y \sqsubseteq g.y'$ if $y \sqsubseteq y'$ because $f.(g.y) \sqsubseteq y \sqsubseteq y'$, the first from (2.) above.

Another definition

A second definition is just the properties (1.–4.) above; note that the last two are just monotonicity of f and g, a reasonable assumption that –nevertheless– did not have to be stated in the first definition: it was imcluded automatically.

Can you prove that the four conditions imply the original definition (1)?