Probabilistic deterministic relational semantics

The state space is S; the type of program denotations is $S \to \overline{S}$ where $\overline{S}$ is the set of *discrete sub-distributions* over S. Note that we do not use $\perp$ explicitly.

For $f\colon S \to \overline{S}$ define its *probabilistic lifting* to be $f^\dagger\colon \overline{S} \to \overline{S}$ so that we have $f^\dagger.\delta := \mathsf{exp}.\delta.f$ for $\delta \in \overline{S}$. Notice that f and $f^\dagger$ agree on points, since $f^\dagger.\overline{s} = f.s$, where the *point distribution* $\overline{s}$ at s assigns probability 1 to s and 0 to all other points. Note that strictness is handled automatically by our †-definition, since $f^\dagger.\perp = \perp$ where the "bottom sub-distribution" is defined $\perp.s := 0$ for all s.

Here is the relational semantics of our simple imperative, probabilistic deterministic language, given by a function $rp.[\![\cdot]\!]$ taking syntax to semantics.

$$
\begin{array}{rcl}
rp.[\![\,\textbf{abort}\,]\!].s & := & \perp \\
rp.[\![\,\textbf{skip}\,]\!].s & := & \overline{s} \\
rp.[\![\,\textbf{assign}\ E\,]\!].s & := & \overline{[\![\,E\,]\!].s} \\
rp.[\![\,S;T\,]\!] & := & (rp.[\![\,T\,]\!])^\dagger \circ rp.[\![\,S\,]\!] \\
rp.[\![\,S\ {}_p\!\oplus T\,]\!].s & := & rp.[\![\,S\,]\!].s\ {}_{p.s}\!\oplus rp.[\![\,T\,]\!].s \\
 & & \\
rp.[\![\,\textbf{while}\ B\ \textbf{do}\ D\ \textbf{od}\,]\!] & := & \\
 & & (\mu f \cdot (\lambda s \cdot f^\dagger.(rp.[\![\,D\,]\!].s) \ \lhd [\![\,B\,]\!].s \rhd\ \overline{s}\,))
\end{array}
$$

- In general $(\mu x \cdot E)$ means the least fixed point of function $(\lambda x \cdot E)$.