# Geometric Signal Models for Compressive Sensing

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May 31, 2011

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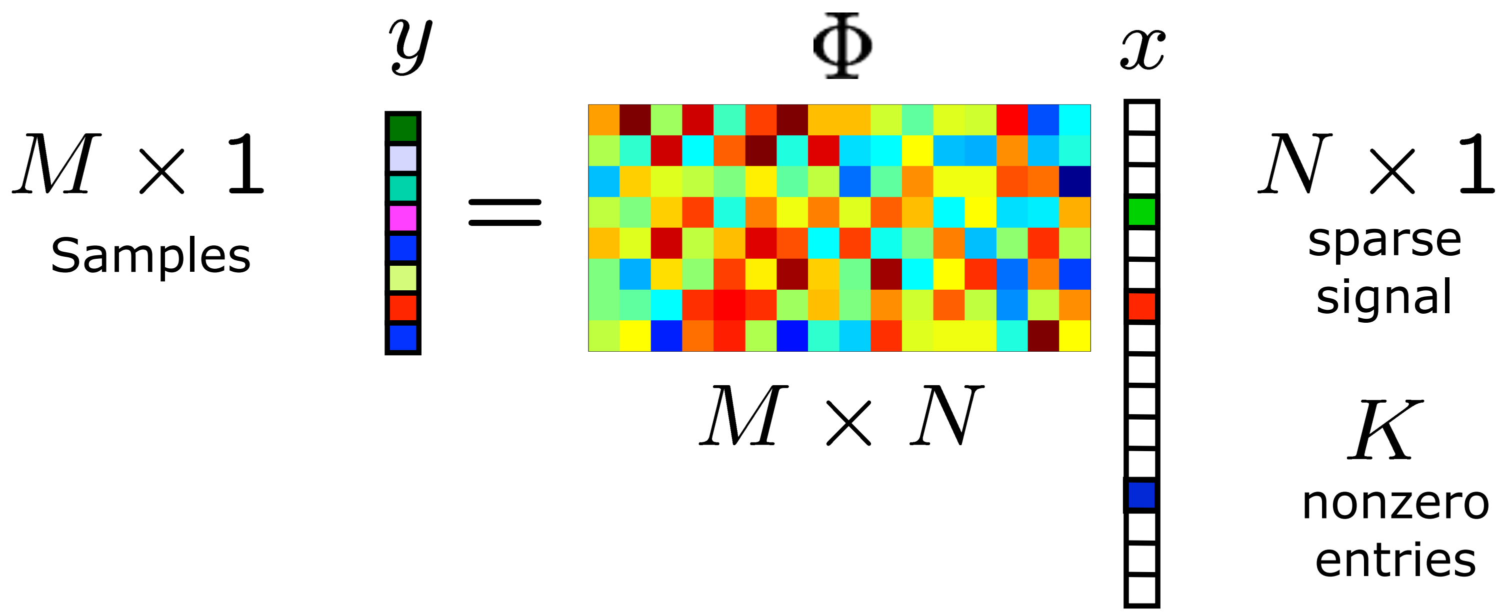
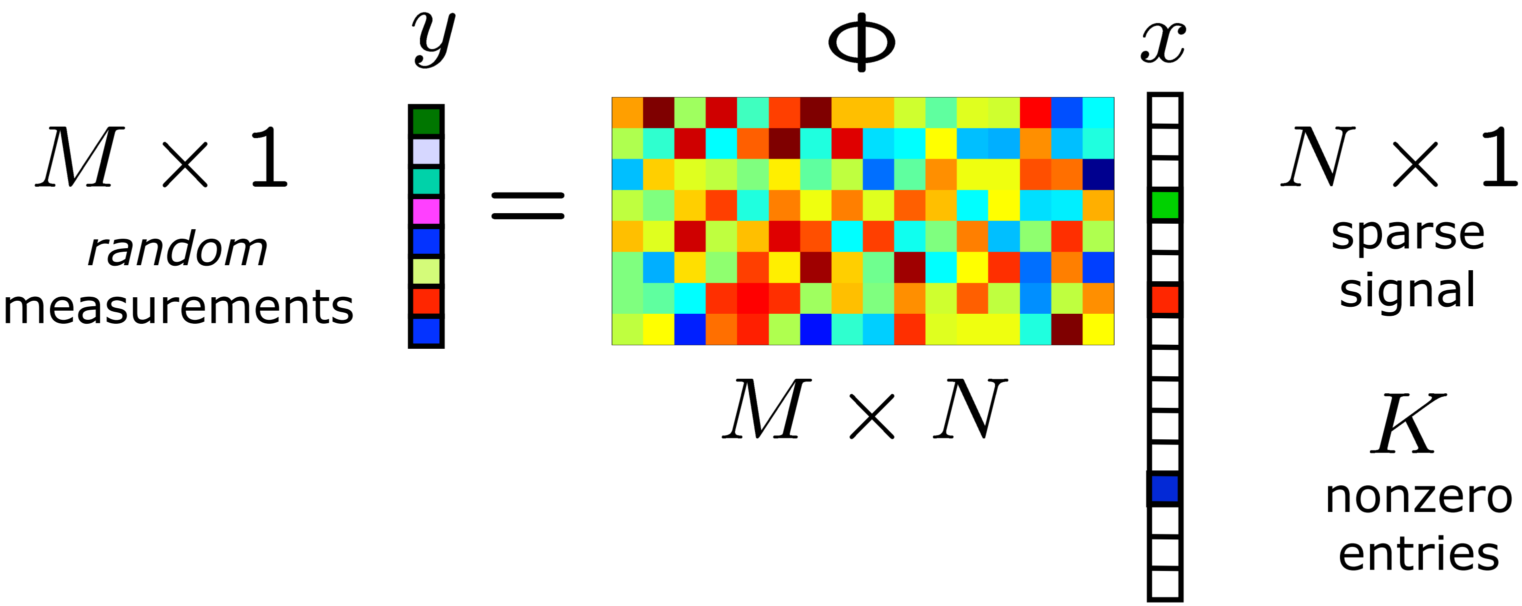
# Compressive Sensing (CS)



# Compressive Sensing (CS)

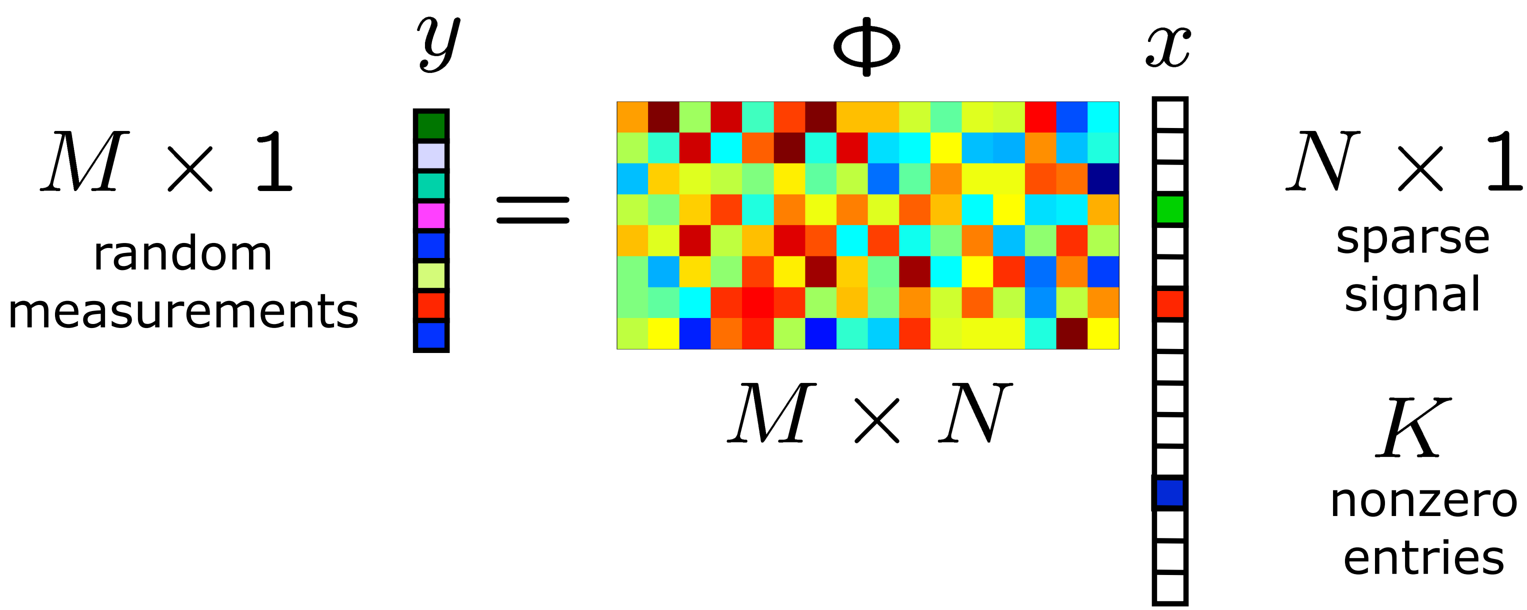
# CS : Sampling




- *Random* subgaussian matrix $\Phi$ has the **RIP** w.h.p. if

$$M = O(K + \log \binom{N}{K}) = O(K \log(N/K))$$

# CS : Recovery

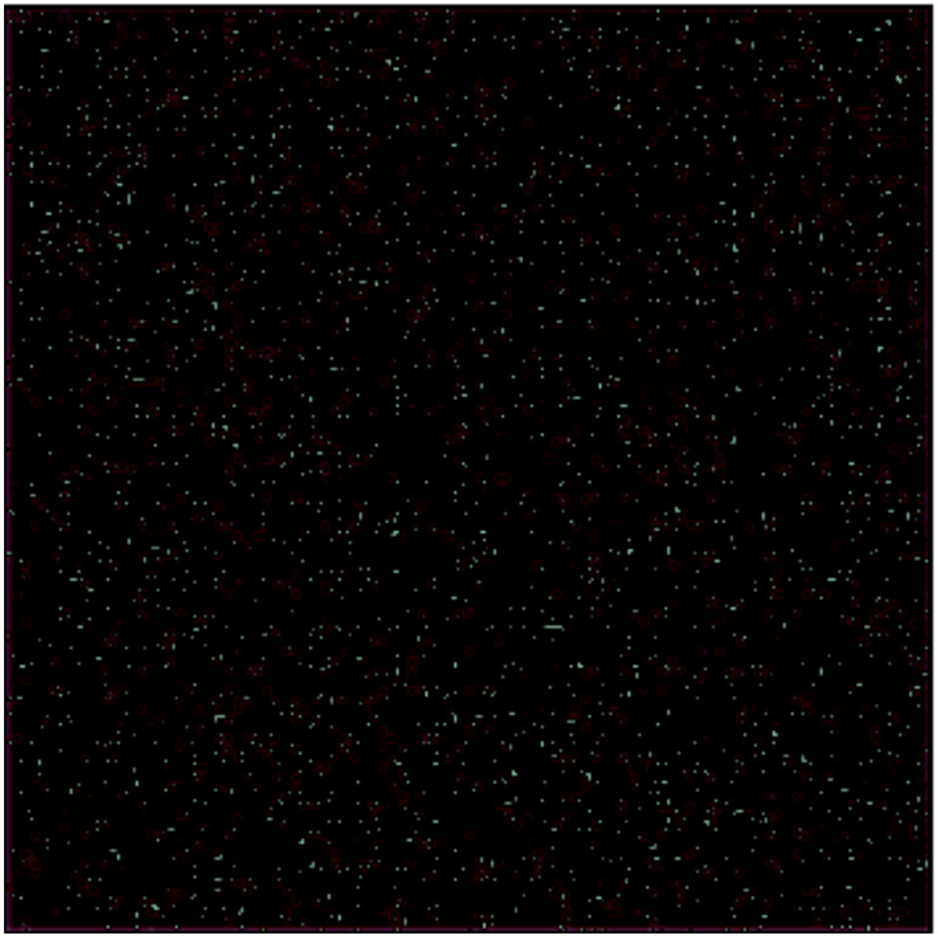



- $\ell_1$-optimization
  [C, R, T]; [D]; [F,W,N]; [H,Y,Z]

- Greedy algorithms
  - OMP [G, T]
  - iterated thresholding [N, F]; [D, D, DeM]; [B, D]
  - CoSaMP [N,T]; Subspace Pursuit [D,M]

# Sparsity

- Sparsity captures **primary structure**



5% sparse “image”

# Structure

- Most real-world signals exhibit **additional structure**

5% sparse image

# How to exploit structure / prior?

## Key idea: **Use Geometry**

- **Linear** models
- **Bilinear** models
- **Manifold** models

# Geometry: model

- **Sparse** signal:

 $x$

  - only *K* out of *N* coefficients nonzero

# Geometry: model

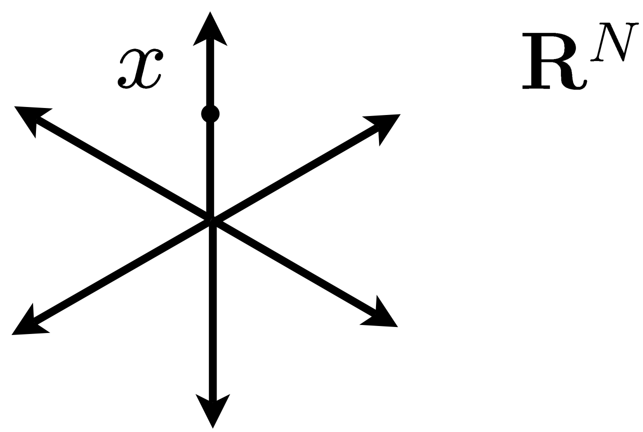
- **Sparse** signal:

$x$

  - only *K* out of *N* coordinates nonzero

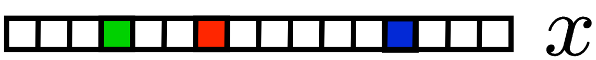
- **Geometry**: *union* of $\binom{N}{K}$ *K*-dimensional subspaces aligned w/ coordinate axes

- *N* = 3, *K* = 1



# Geometry: model

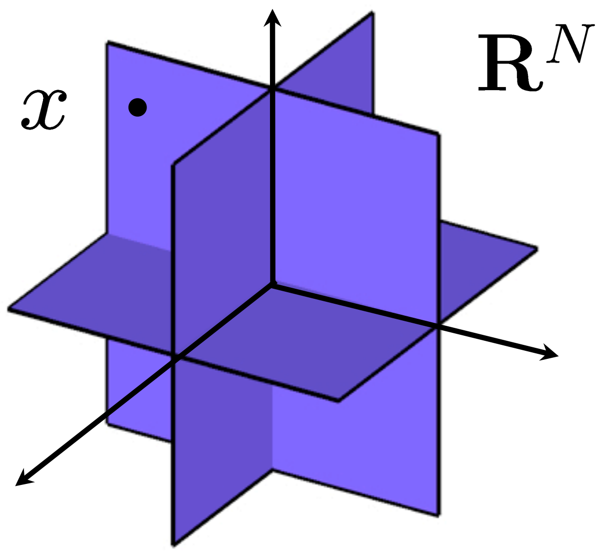
- **Sparse** signal:




  - only $K$ out of $N$ coordinates nonzero

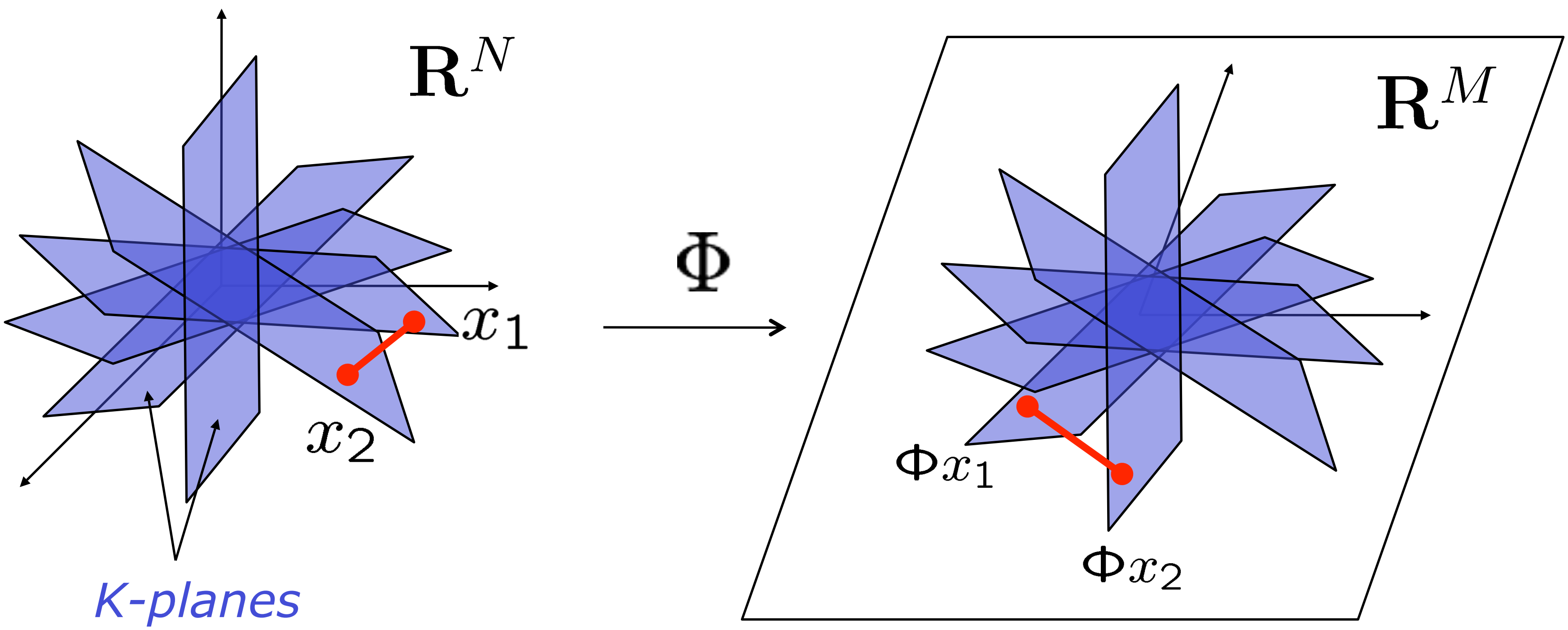
- **Geometry**: *union* of $\binom{N}{K}$ $K$-dimensional subspaces aligned w/ coordinate axes

- $N = 3$, $K = 2$



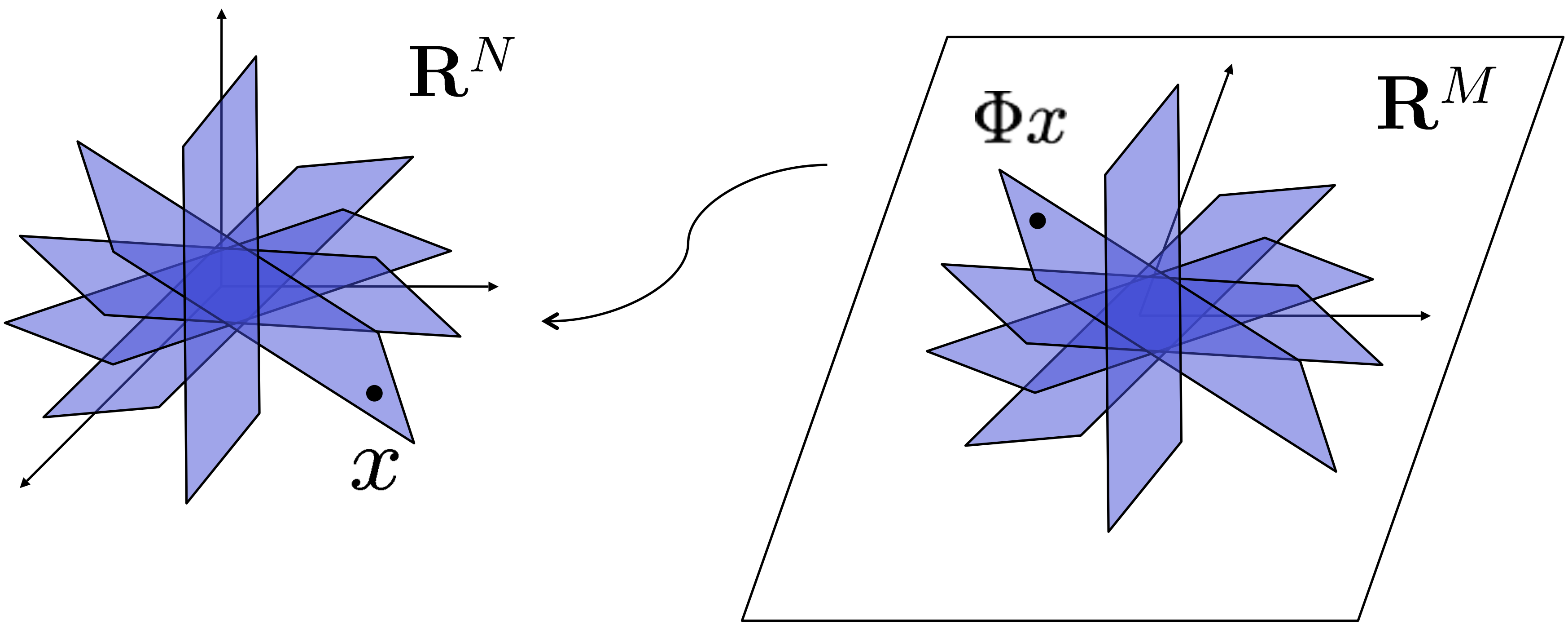
# Geometry : Sampling

- Preserve the structure of sparse signals
- **Restricted Isometry Property (RIP)**



# Geometry : Recovery

- Efficient, stable algorithms that **recover** signal



# Iterated (hard) thresholding

- goal: given $y = \Phi x$ , recover $x \in \Sigma_K$

iterate:

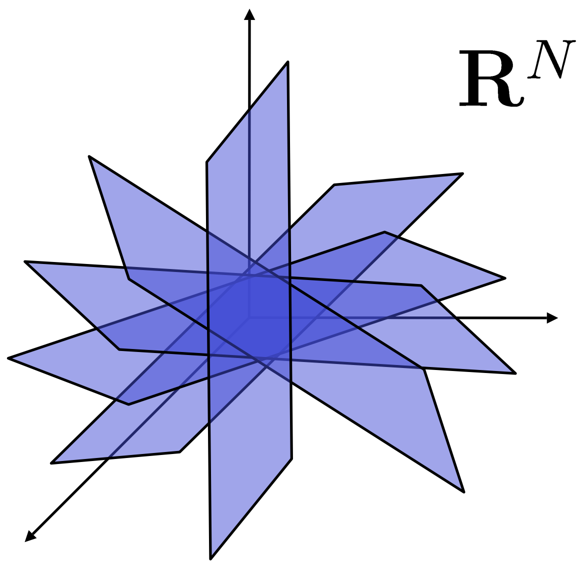
- $\widehat{x}_{i+1} \leftarrow \mathrm{thresh}(\widehat{x}_i + \Phi^T(y - \Phi x_i))$

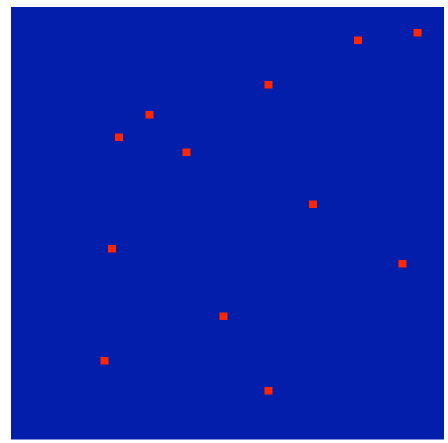
return $\widehat{x} \leftarrow \widehat{x}_i$

# Linear Models

# Sparse signals

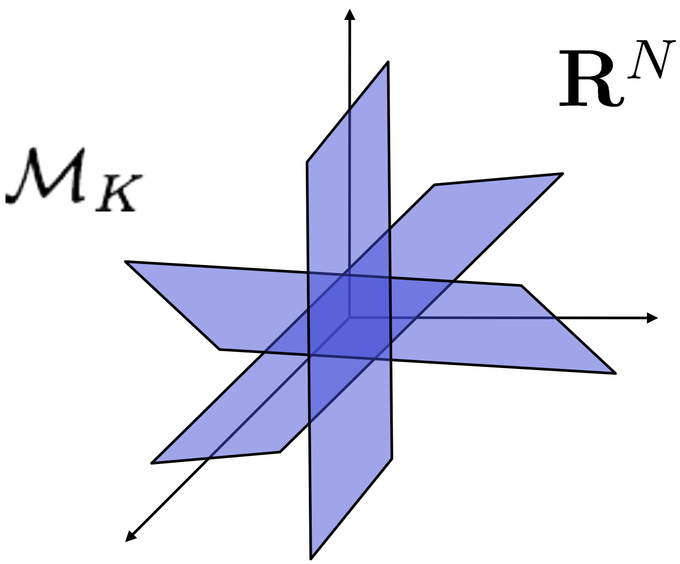
- Defn: ***K*-sparse signals** comprise *all* *K*-dimensional canonical subspaces

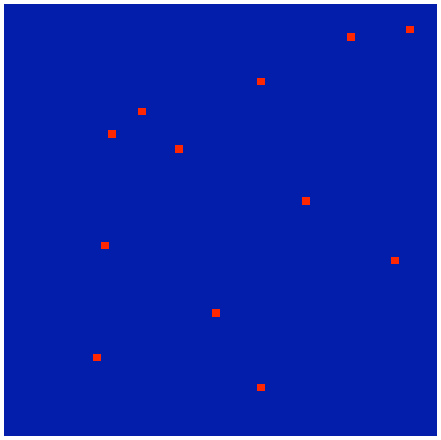


# Model-sparse signals

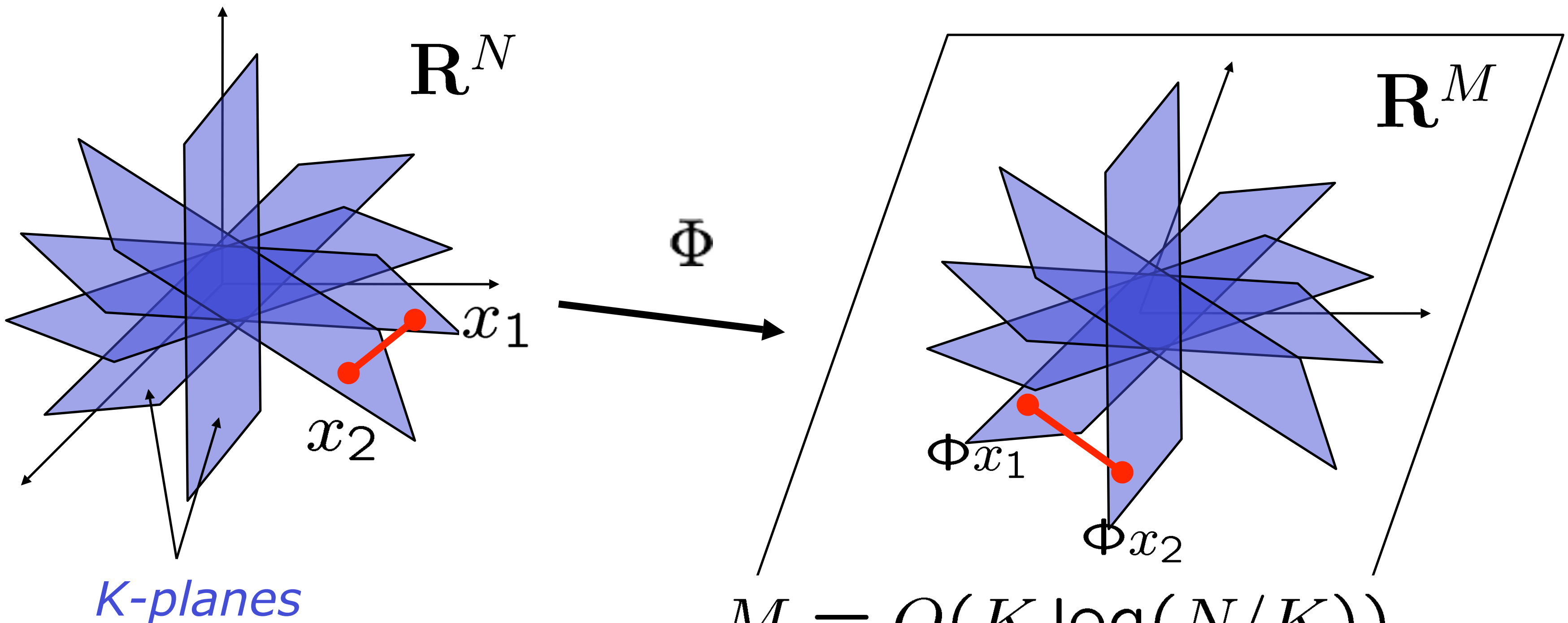
- Def: A ***K*-sparse union-of-subspaces *model*** comprises a particular (*reduced*) set of $L_K$ *K*-dim canonical subspaces



# Sampling bounds

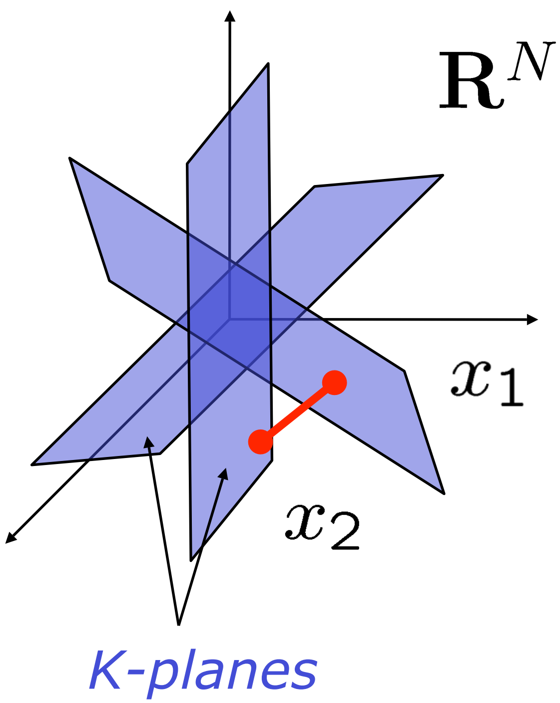
- **RIP:** stable embedding

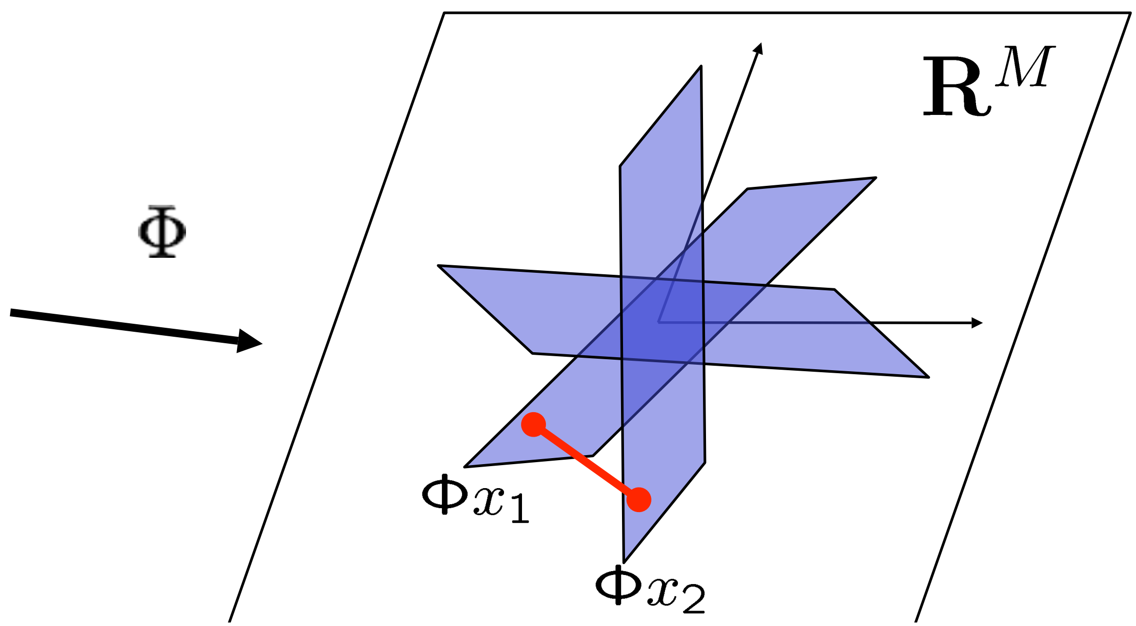



$$M = O(K \log(N/K))$$

# Sampling bounds

- **Model-RIP:** stable embedding
  [B, D]; [B,D,DeV,W]




$$M = O(K + \log(L_K))$$

# Iterated thresholding

- goal: given $y = \Phi x$ , recover $x \in \Sigma_K$

  initialize $i = 0, \quad x_0 = 0$

  iterate:

  - $\widehat{x}_{i+1} \leftarrow \text{thresh}(\widehat{x}_i + \Phi^T(y - \Phi x_i))$

  return $\widehat{x} \leftarrow \widehat{x}_i$

# Iterated model thresholding

- goal: given $y = \Phi x$ , recover $x \in \mathcal{M}_K$

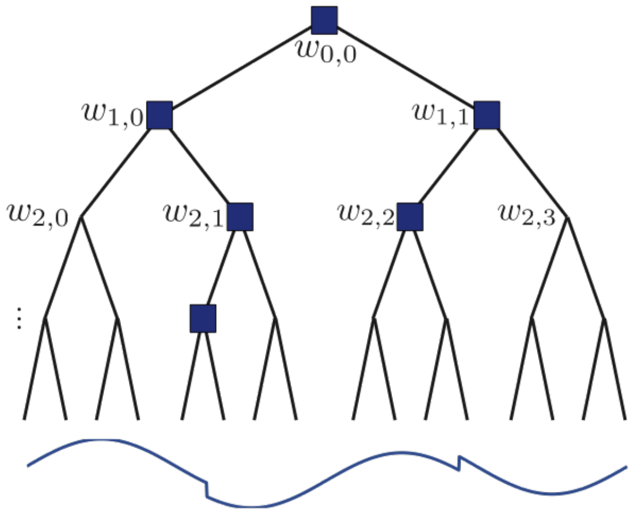
  initialize $i = 0, \quad x_0 = 0$

  iterate:

  - $\widehat{x}_{i+1} \leftarrow \mathcal{M}(\widehat{x}_i + \Phi^T(y - \Phi\widehat{x}_i))$

  return $\widehat{x} \leftarrow \widehat{x}_i$

# E.g. Wavelet trees



# Result - Wavelet trees

Daubechies/CoSaMP - K = 6000 M = 30000

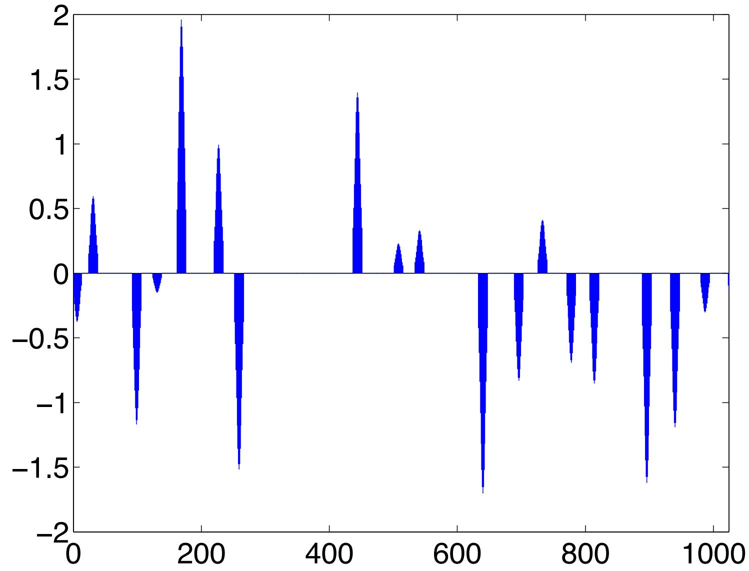
SNR = 13.1361dB

Daubechies/Tree CoSaMP - K = 6000 M = 30000

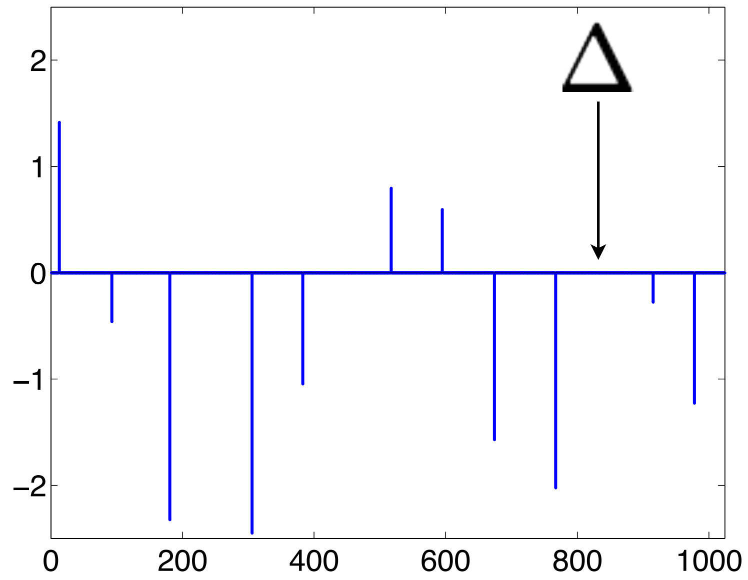
SNR = 17.8263dB

# Union of subspaces models

- Block-sparsity

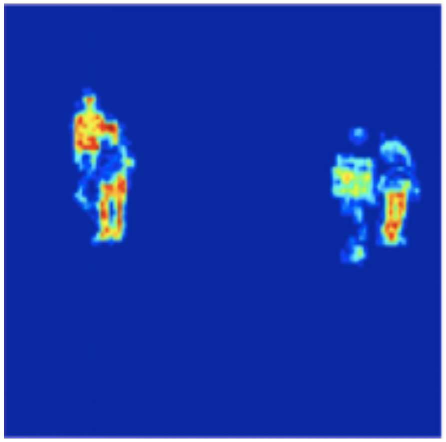



- $\Delta$ - separated spikes



# Union of subspaces models

- Markov Random Fields

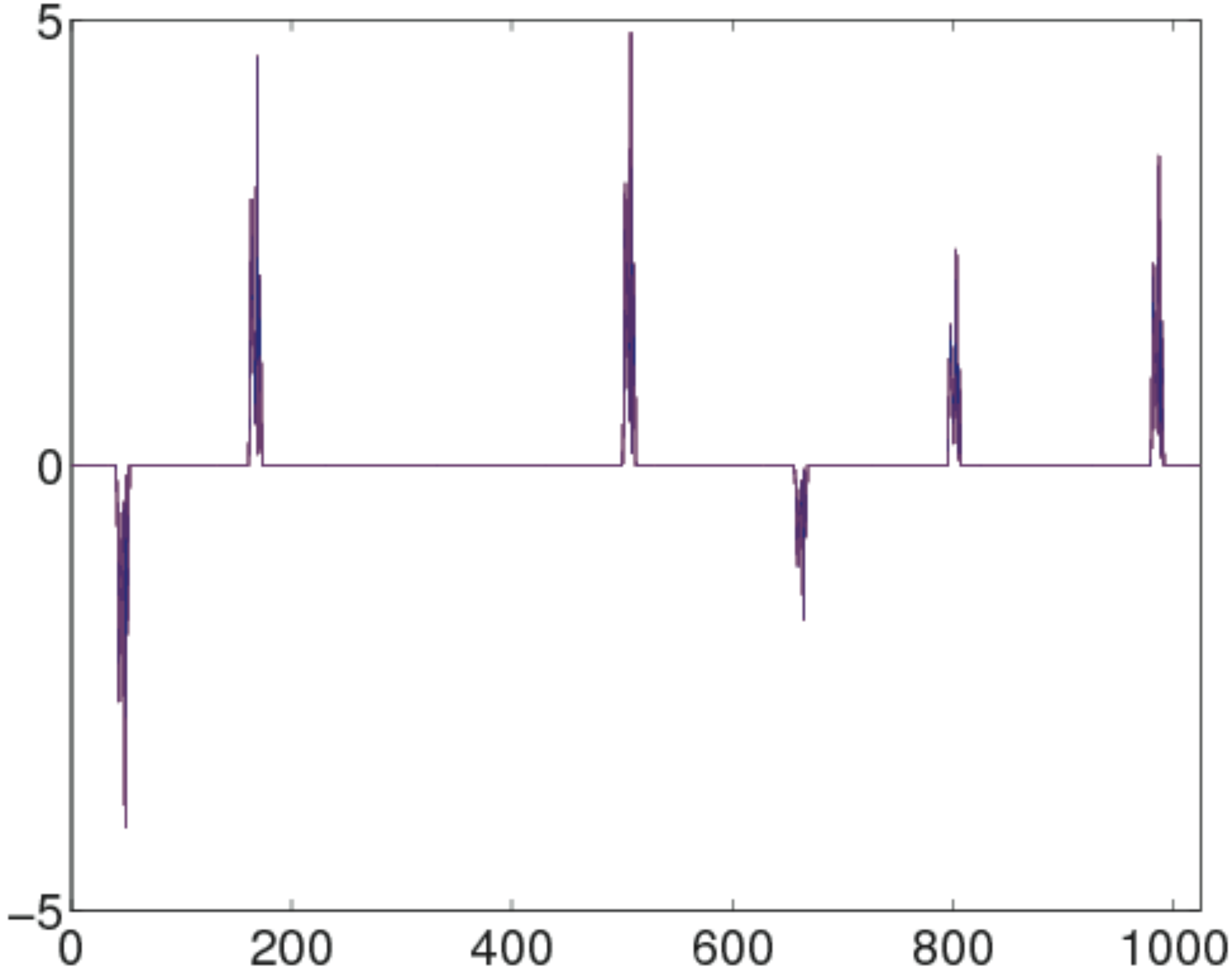


# Recipe for Compressive Sensing

- Geometric model
  - Sampling bound - *M* measurements
  - signal recovery algorithm from M measurements

# Bilinear Models

# When things aren't exactly sparse..




- *6*-sparse signal *convolved* with an *11*-sparse impulse response

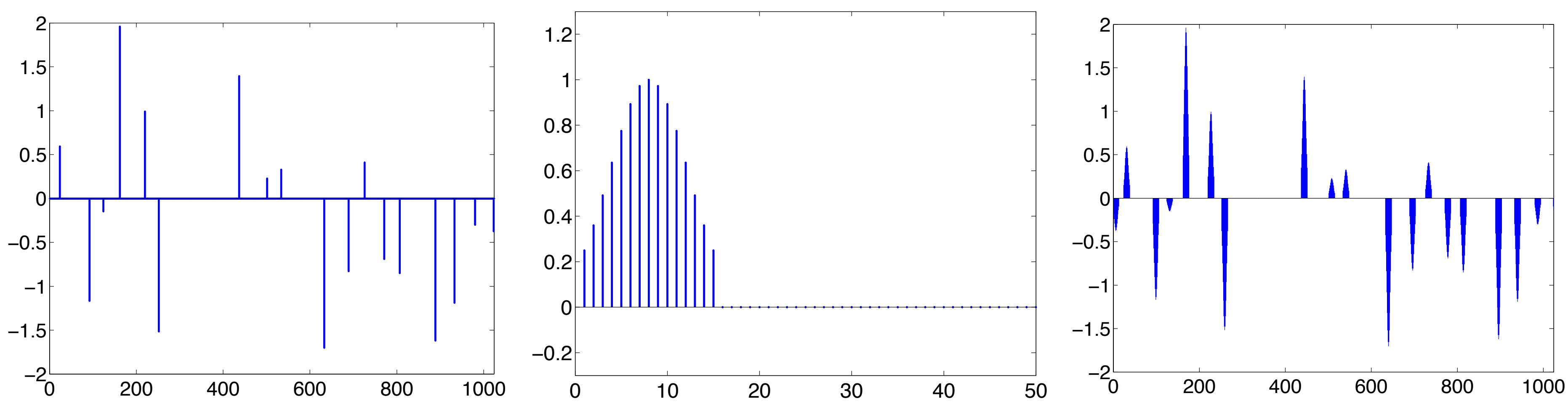
# Bilinear model

- "Pulse stream"

$$z = x * h$$

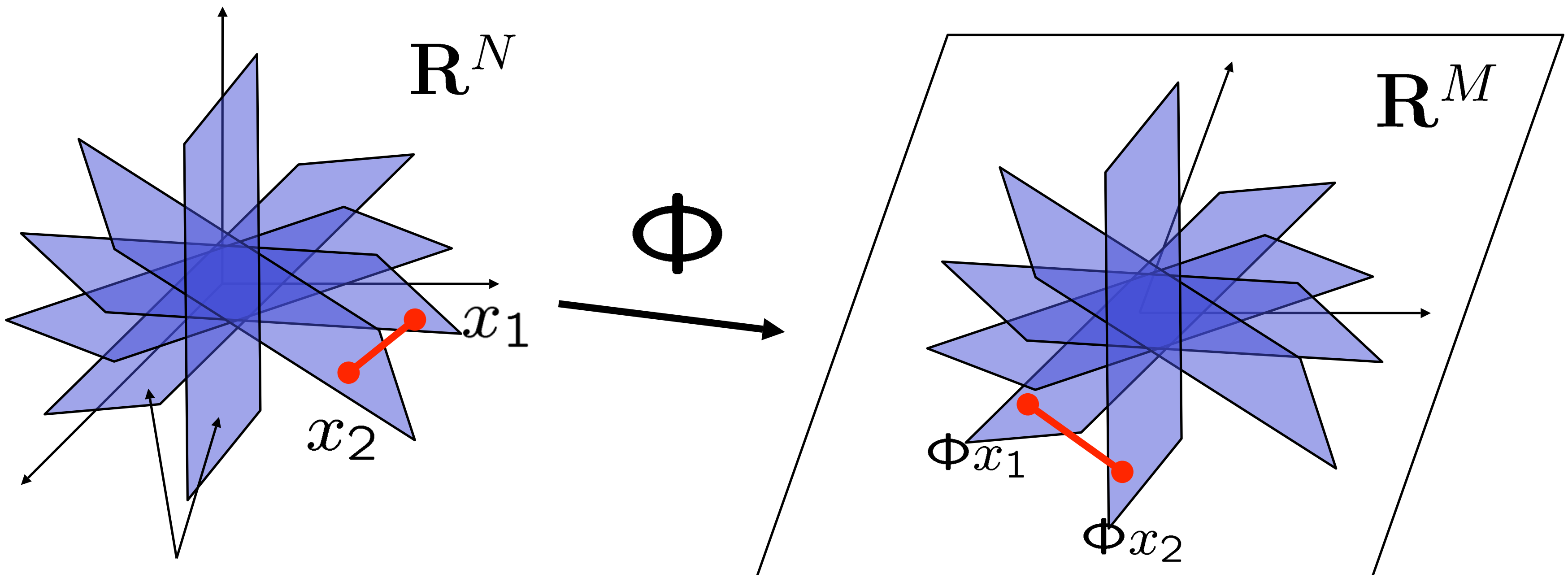
where:

$x \in \mathcal{M}_S$ "spike stream" $\quad$ $h \in \mathcal{M}_F$ "impulse response" (IR)



# Signal model: geometry

- **RIP for pulse streams**




***Infinite*** *union of subspaces*

# Sampling bound

- **Theorem**

$$M = O(S + F + \log(L_S L_F))$$

- In the worst case:

$$\begin{aligned} M &= O(S + F + S\log(N/S) + F\log(N/F)) \\ &\ll O(SF\log N) \ = O(K\log N) \end{aligned}$$

# Recovery

- Problem: recover *z*

$$y = \Phi z = \Phi(x * h) = \Phi H x = \Phi X h$$

# Recovery

- Problem: recover *z*

$$y = \Phi z = \Phi(x * h) = \Phi H x = \Phi X h$$

- Compare to:

$$z = x * h$$

- Blind Deconvolution

# Iterated Model Thresholding

- goal: given $y = \Phi x$ , recover $x \in \mathcal{M}_K$

  initialize $i = 0, \;\; x_0 = 0$

  iterate:

  - $\widehat{x}_{i+1} \leftarrow \mathcal{M}(\widehat{x}_i + \Phi^T(y - \Phi\widehat{x}_i))$

  return $\widehat{x} \leftarrow \widehat{x}_i$

# Iterated Support Estimation

- goal: given $y = \Phi z = \Phi H x = \Phi X h$, recover $z \in \mathcal{M}(S, F, \Delta)$

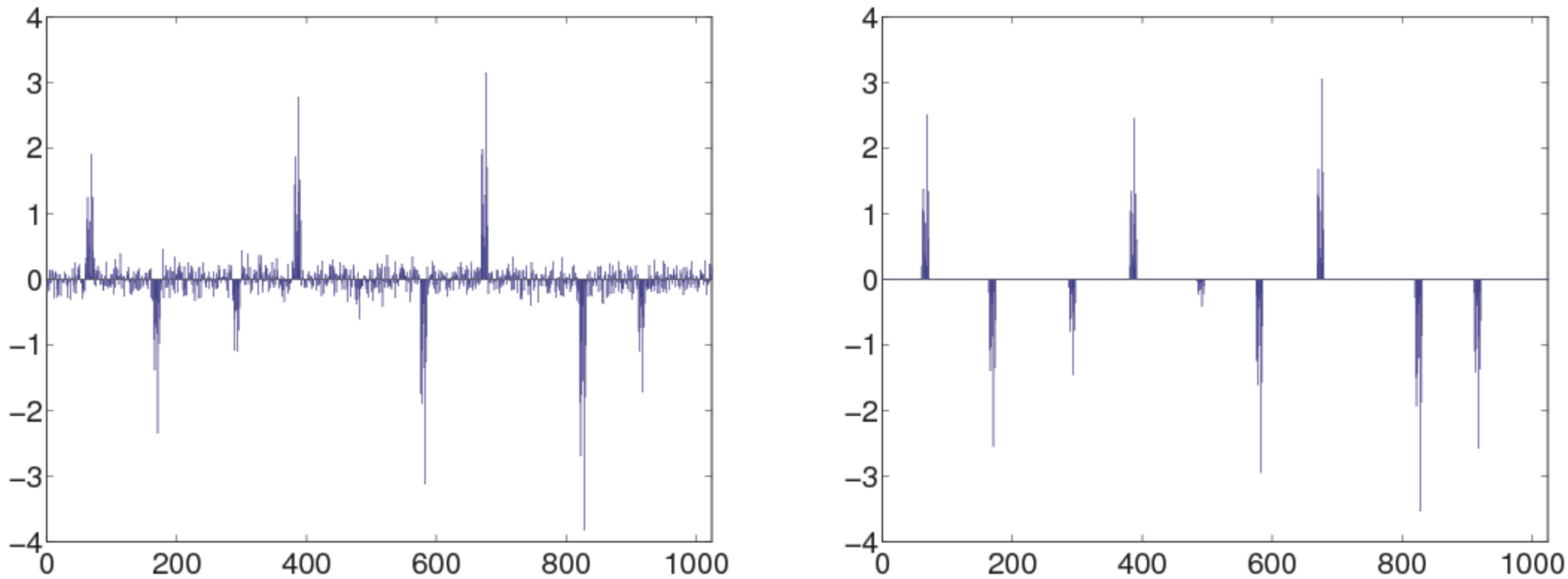
initialize $\widehat{h} = (\mathbf{1}_F^T, 0, \ldots, 0)/\sqrt{F}$
$i = 0, \quad x_0 = 0$

iterate:

- $\widehat{x} \leftarrow \mathcal{M}_S^{\Delta}(\widehat{x} + (\Phi \widehat{H})^T (y - (\Phi \widehat{H} \widehat{x}))$
- $\widehat{h} \leftarrow (\Phi \widehat{X})^{\dagger} y$

return $\widehat{z} \leftarrow \widehat{x} * \widehat{h}$

# Numerical example




$S = 9$, $F = 11$, $K = 99$, $M = 150$

# Astronomical images

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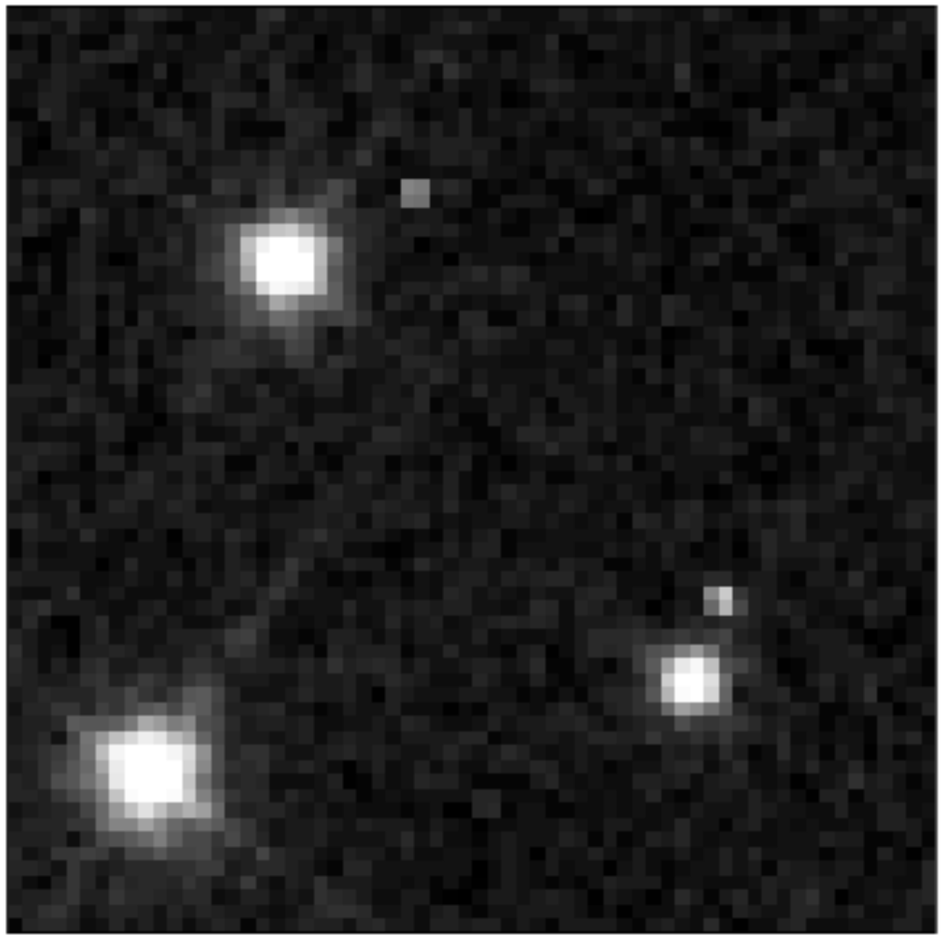
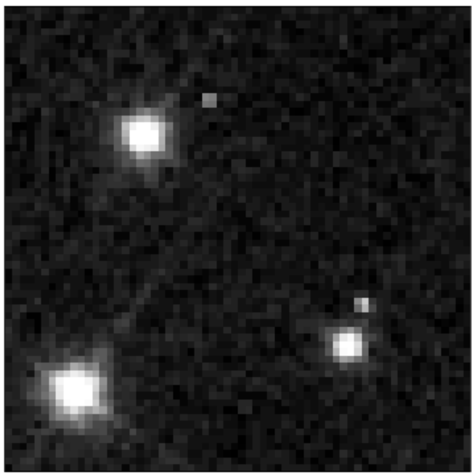
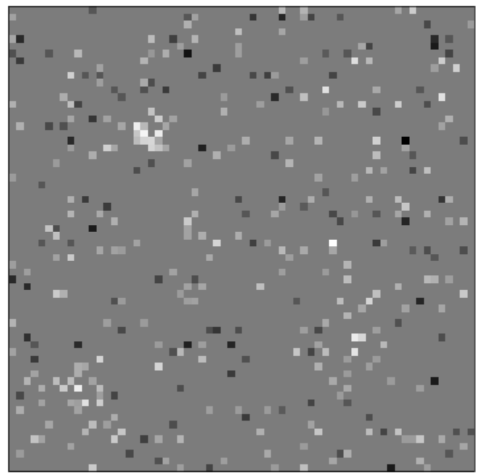


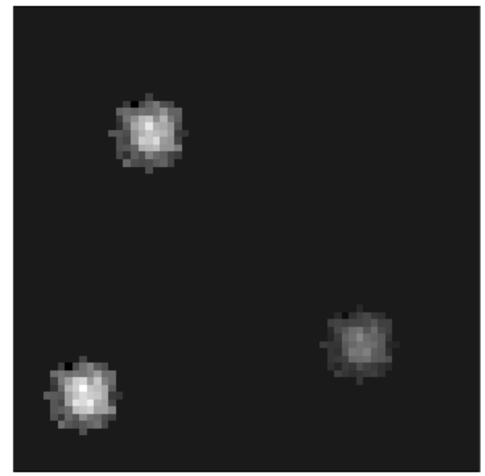
# Astronomical images



Original image

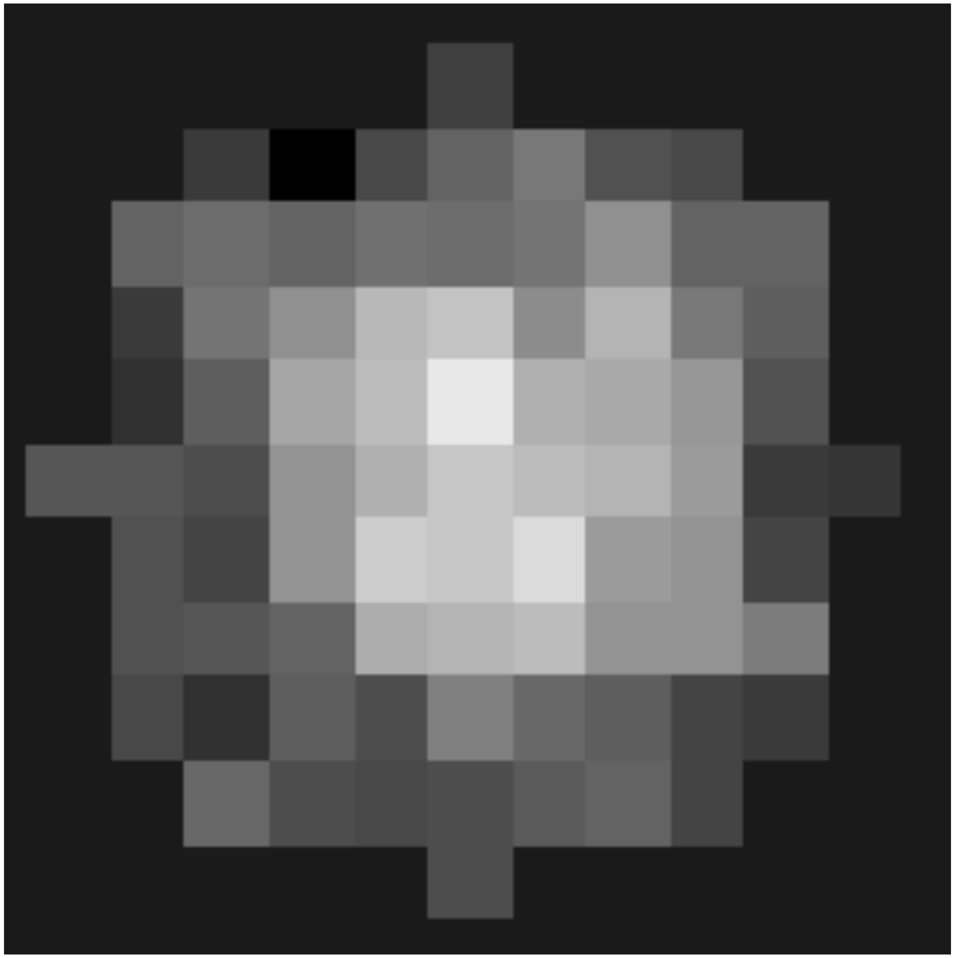


CoSaMP ($M$ = 330)

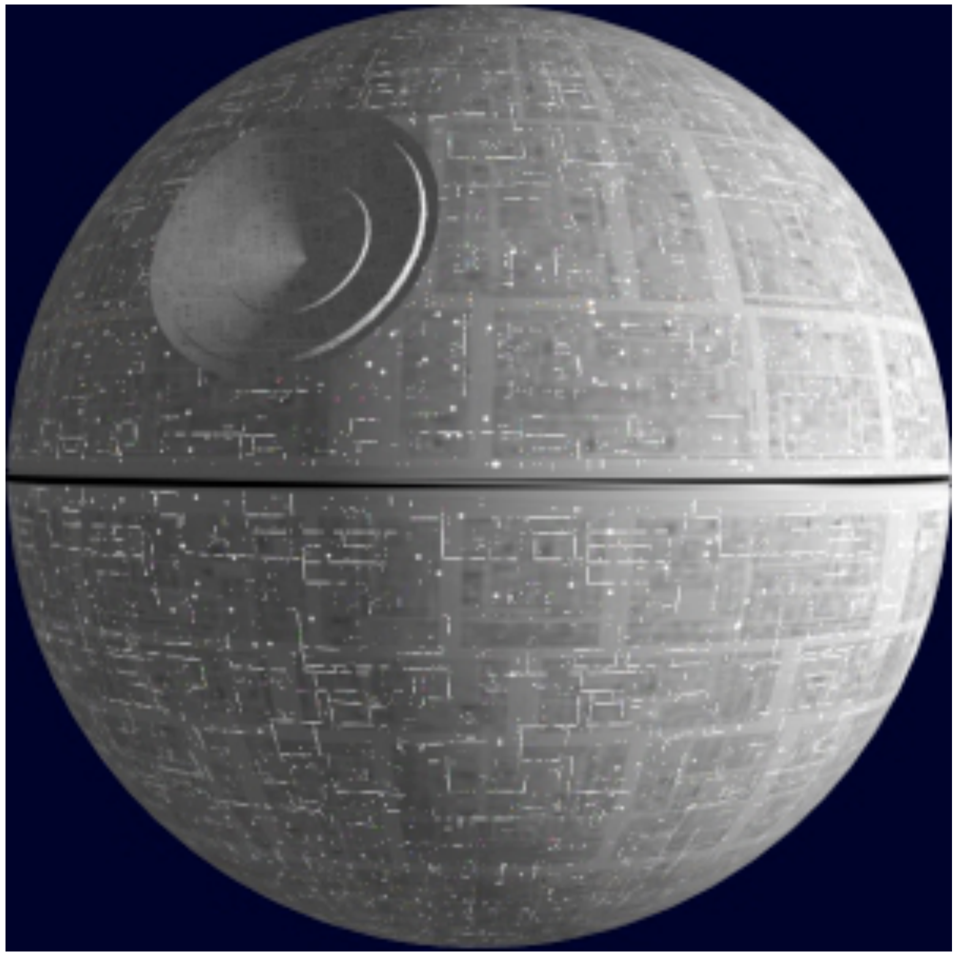


RM#2 ($M$ = 330)

# Reconstruction



# Reconstruction

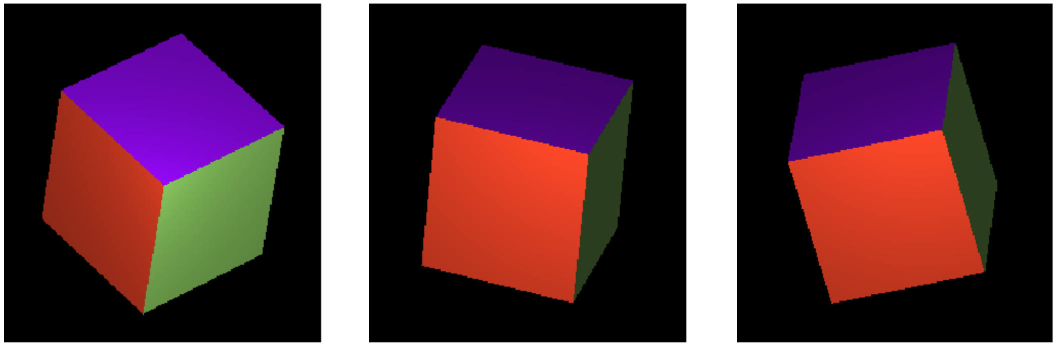


# Nonlinear (manifold) Models

# Manifold Models

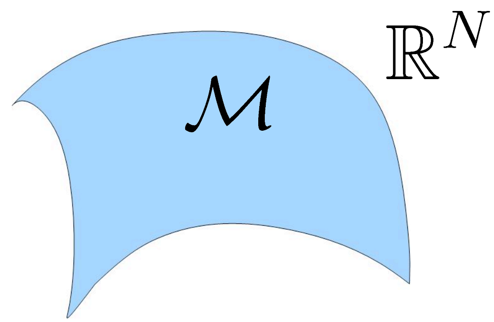
- *K*-dimensional *parameter vector* captures degrees of freedom in signal $x \in \mathbb{R}^N$

$$x = x(\mathbf{z}), z \in \mathbb{R}^K$$

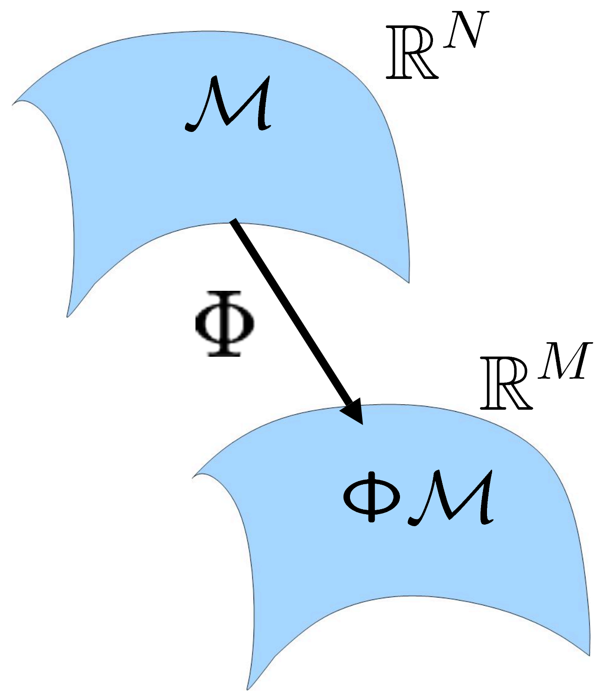


- "Image Articulation Manifold" (IAM)

# Compressive IAM embedding



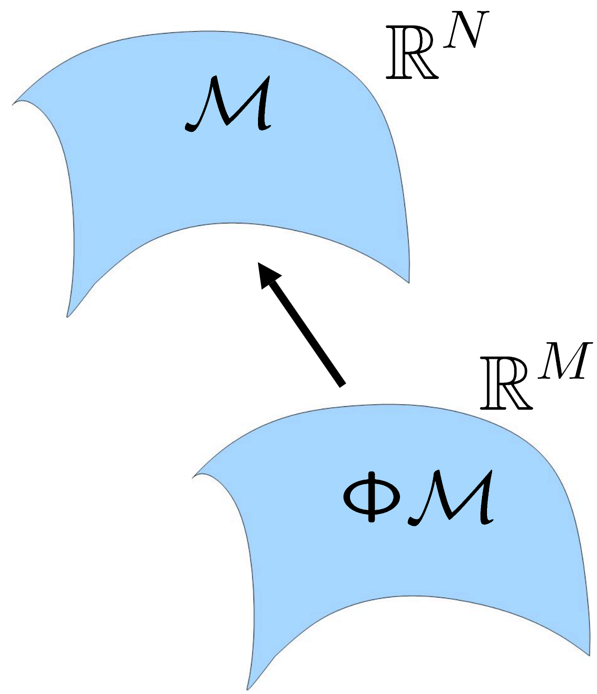
# Compressive IAM embedding



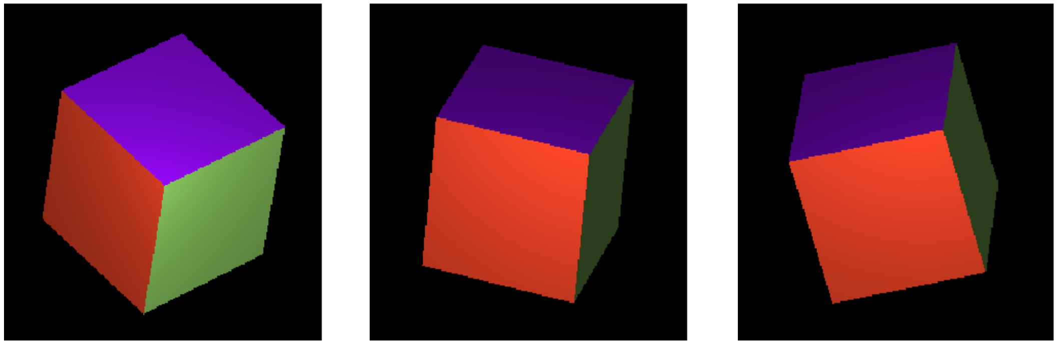

$$M = O\left(\frac{K \log(NV\tau^{-1}\epsilon^{-1}) \log(1/\rho)}{\epsilon^2}\right).$$

[Baraniuk, Wakin 2006]

# Manifold-based CS recovery?

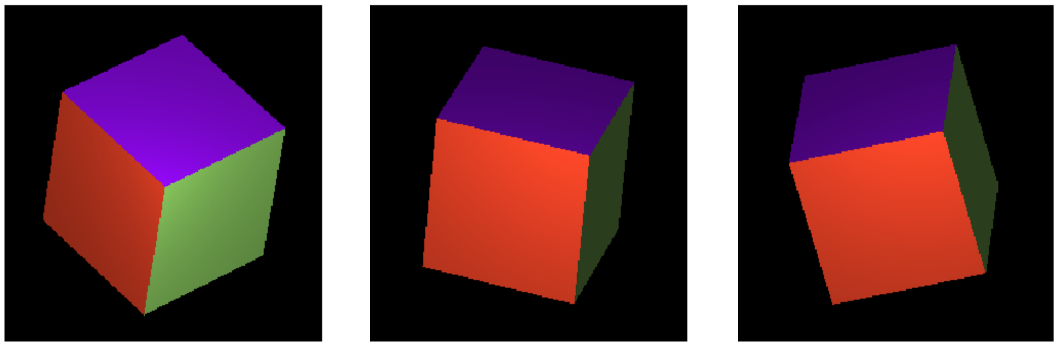


# Lie Operators



$$x = x(\mathbf{z}) = T(\mathbf{z})x_0$$
$$= e^{A\mathbf{z}}x_0$$

# Lie Operators



$$y = \Phi e^{A\mathbf{z}} x_0 = P(\mathbf{z})$$

# Sampling and recovery

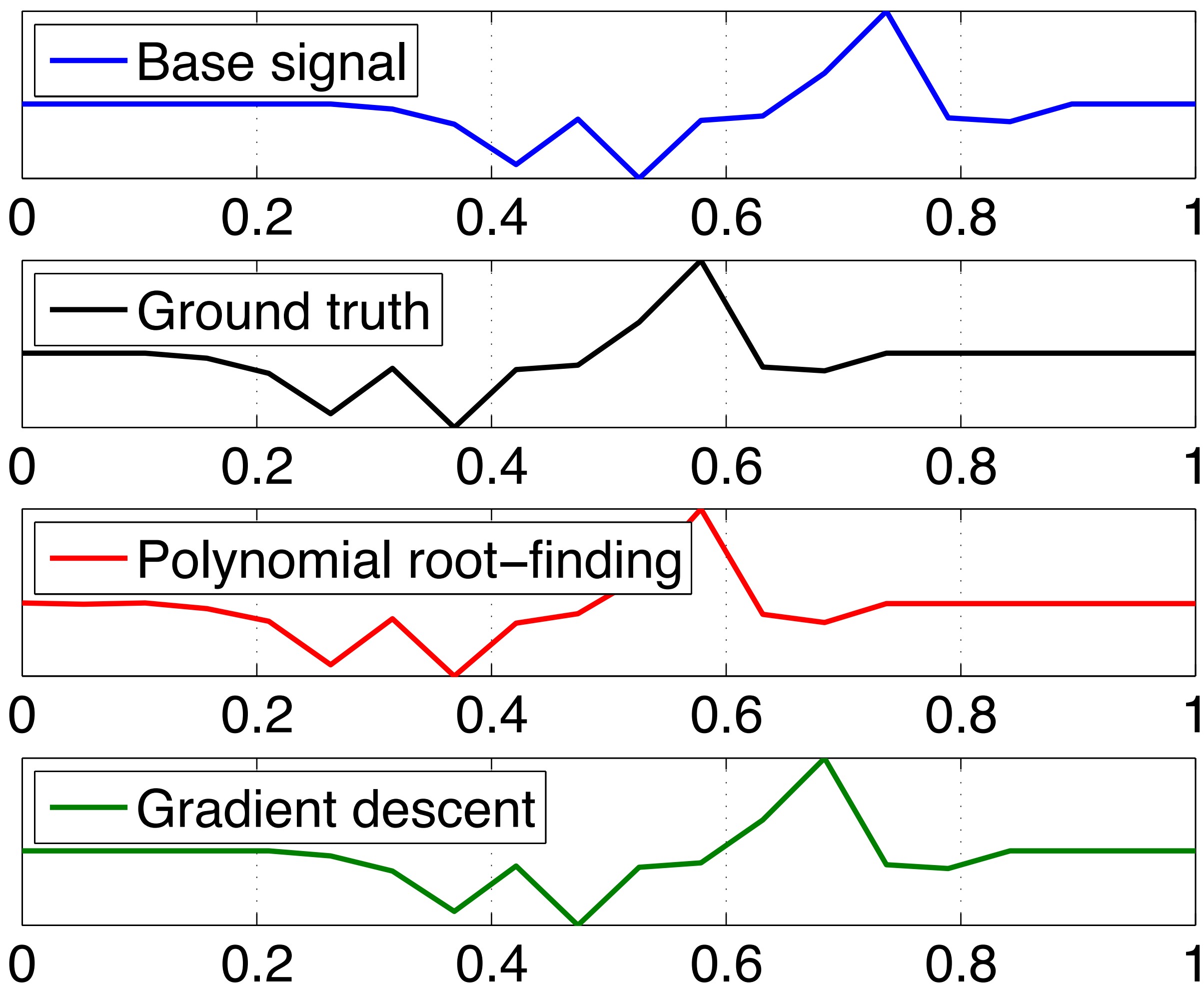
- **Sampling Theorem**
  - **(for these special manifolds)**

$$M = O(K)$$

- Recovery
  - solve a system of multivariate polynomial equations
  - no stable polynomial time algorithm, revert to heuristics

# E.g. Shift estimation

- $M = 25$, $N = 1024$, $K = 1$



# Summary

- Ingredients of CS: a) sampling rate for signal class
  b) algorithm for recovery

- Beyond sparsity

  - UoS/Bilinear/Manifold models

  - If you have prior info, use it! (but how?)

  - **Geometric modeling**