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RICE UNIVERSITY DSP

A Geometric Approach for Compressive Sensing

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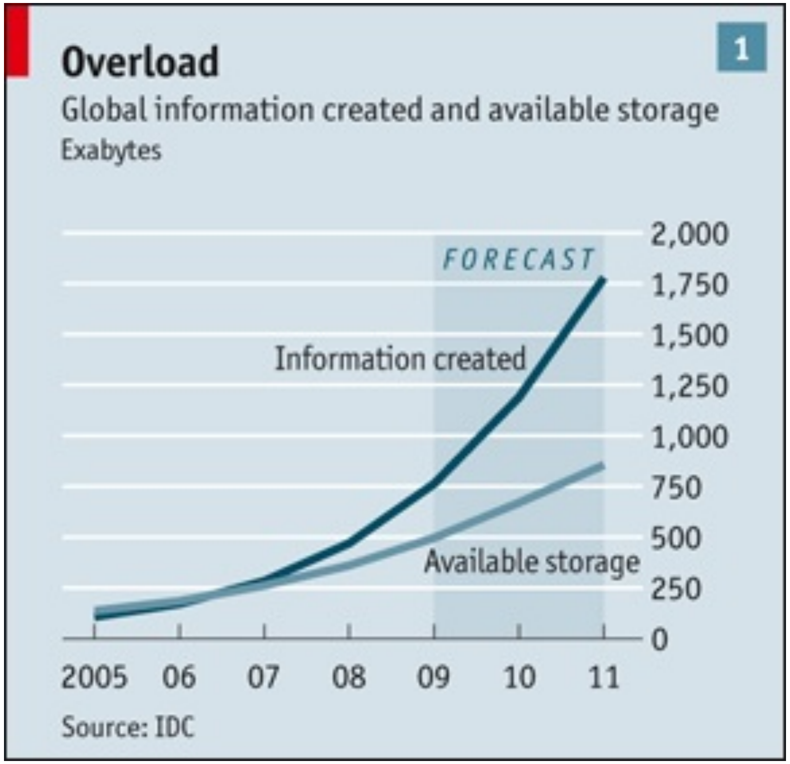
The Data Deluge



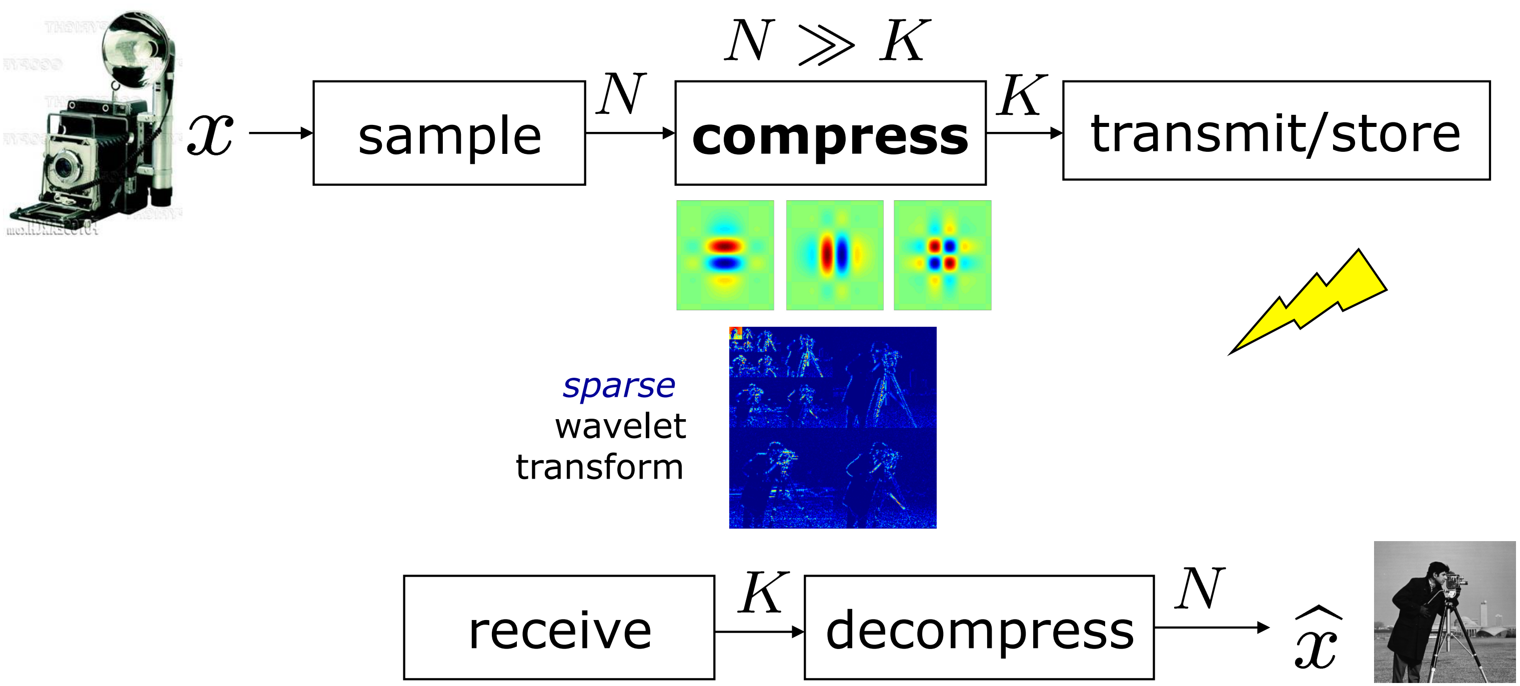
- **>250 billion gigabytes** generated in 2007

  Current: digital bits > stars in the universe

  > Avogadro's number (6.02×10^{23}) in 15 years

Signal Processing Pipeline

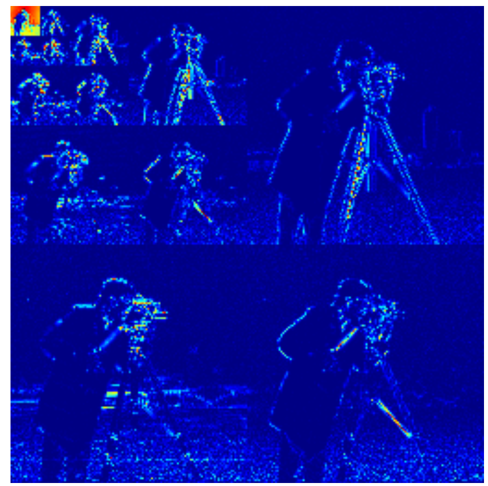
- Established paradigm for digital data acquisition
  - *sample* (sensor)
  - *compress* (processor)
  - *transmit* (network)
  - *reconstruct* (processor)



Sparsity

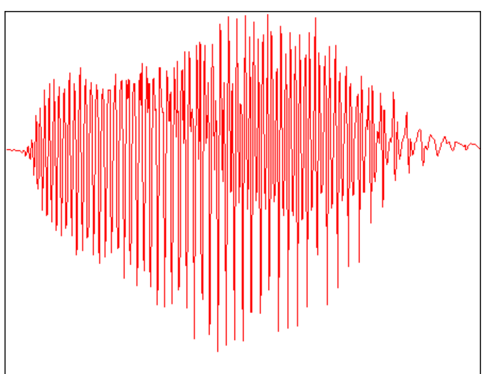
- Many signals can be compressed in some representation/basis (Fourier, wavelets, …)

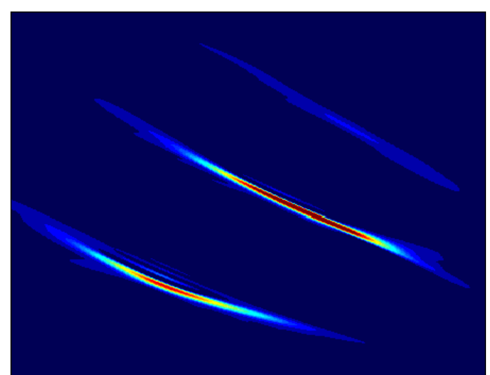
N
pixels



$K \ll N$
large
wavelet
coefficients

N
wideband
signal
samples

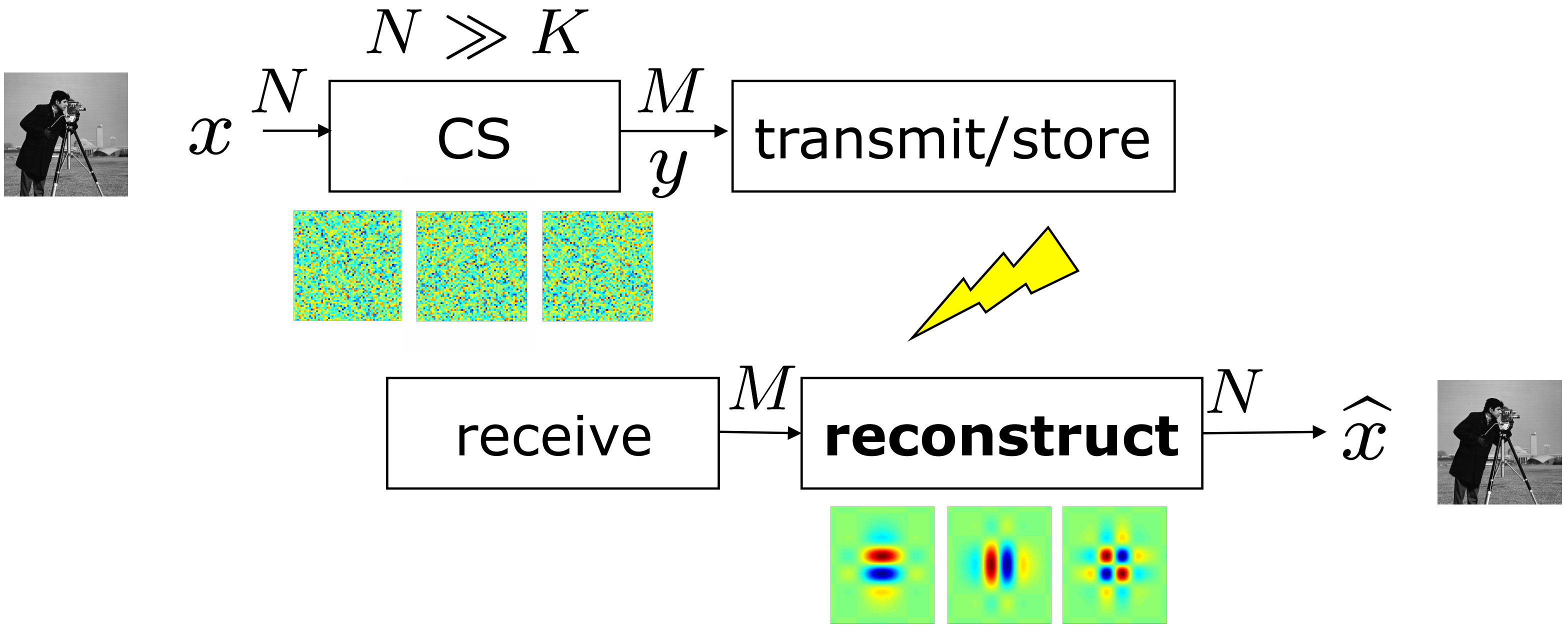




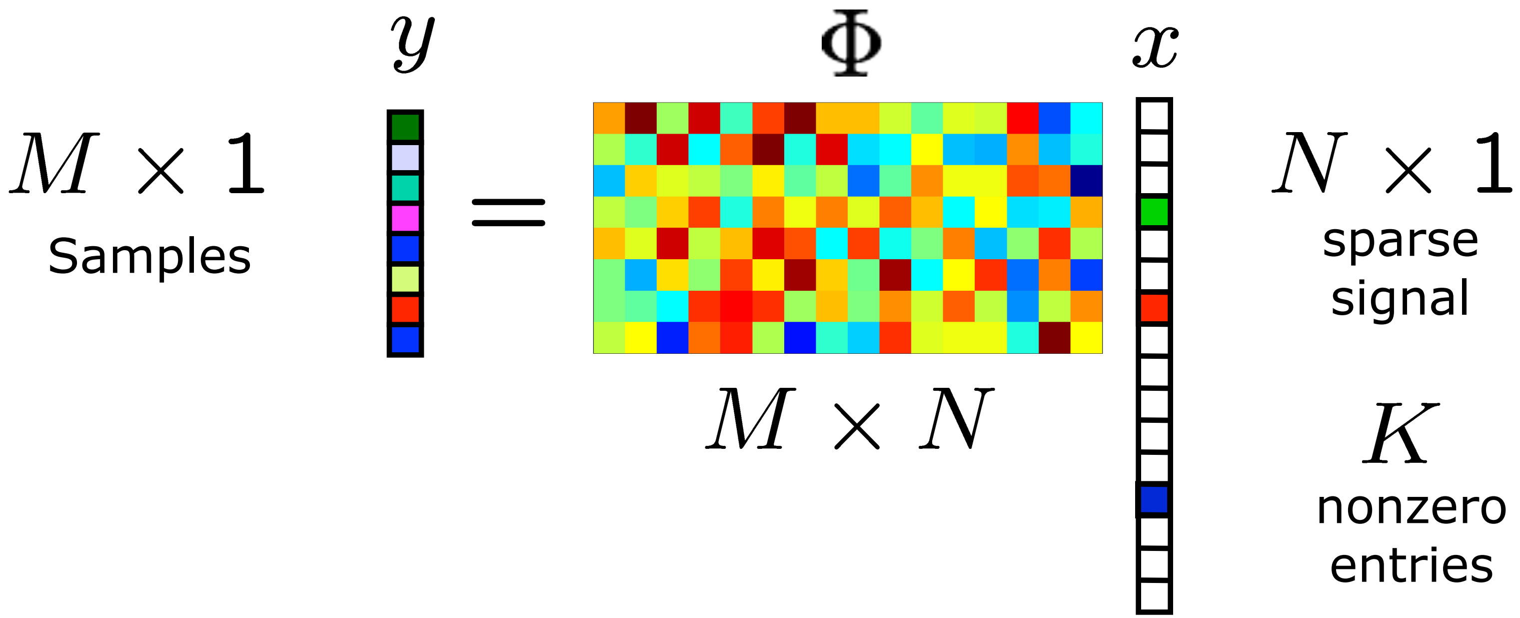
$K \ll N$
large
Gabor
coefficients

Compressive Signal Processing

- Established paradigm for digital data acquisition
  sample and compress
  transmit (network)
  reconstruct (processor)

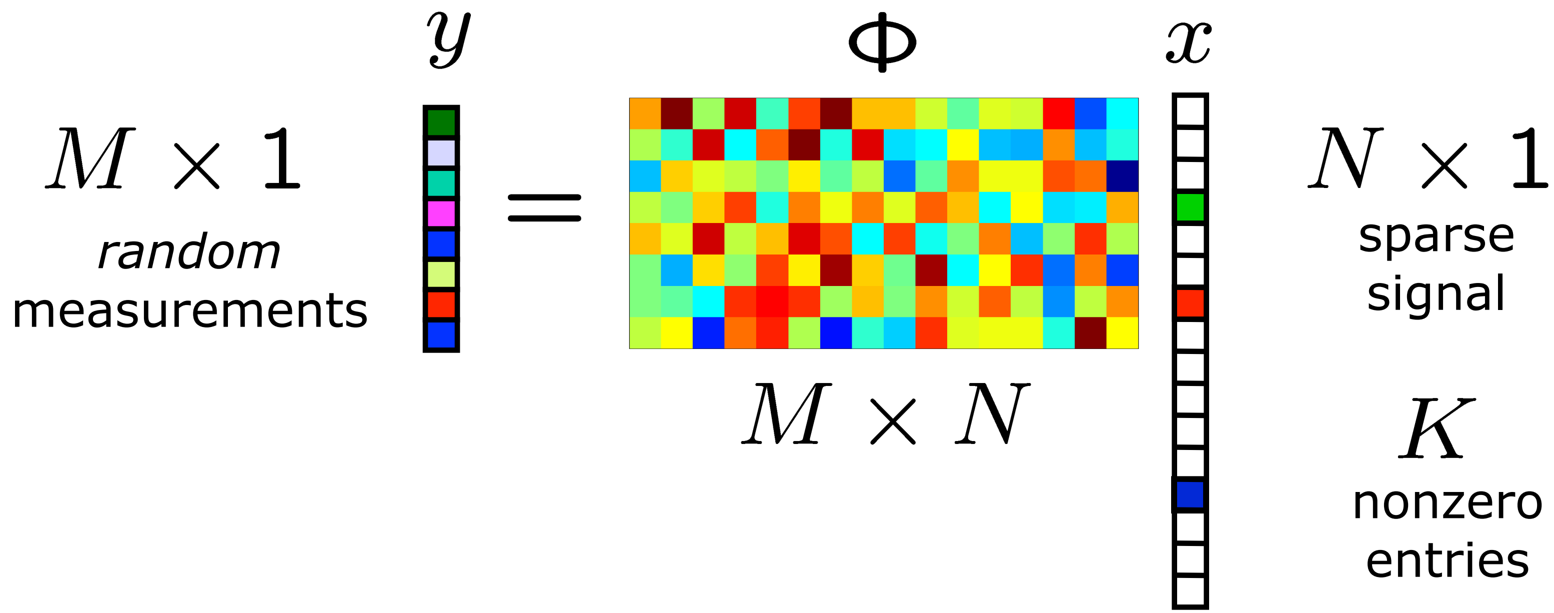


Compressive Sensing (CS)



Compressive Sensing (CS)

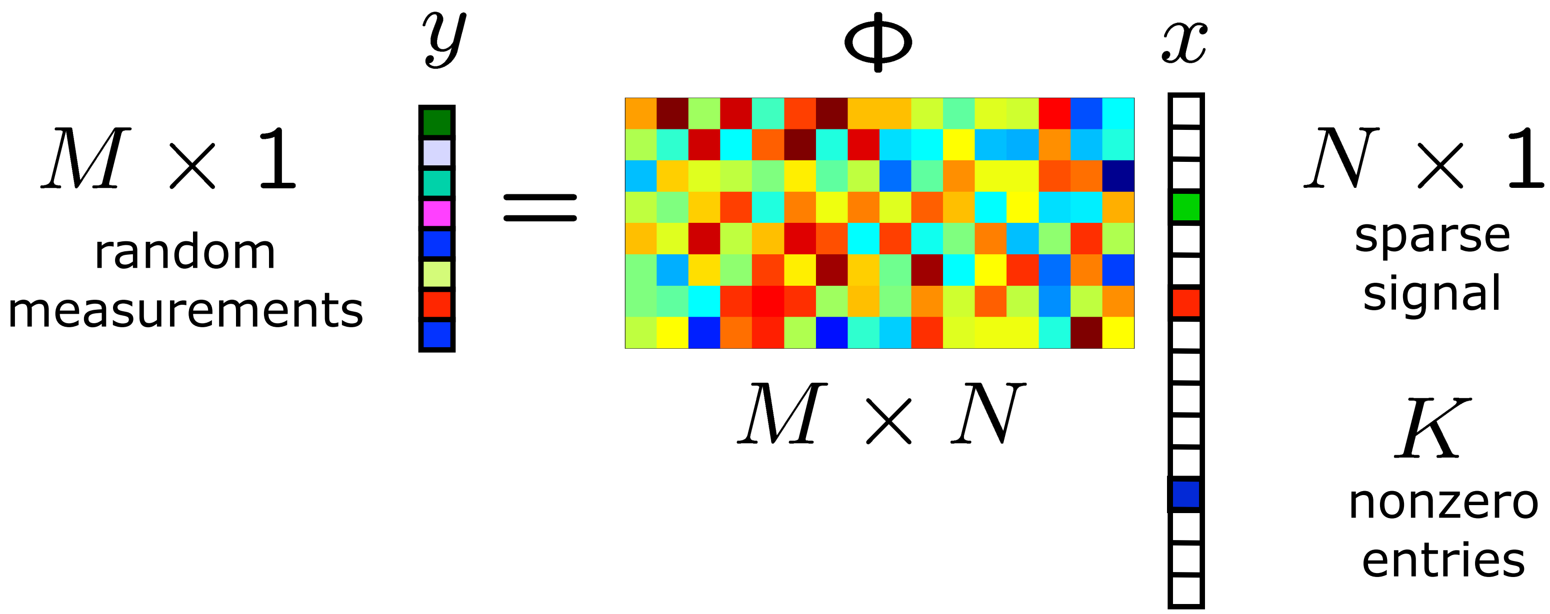
CS : Sampling



- *Random* subgaussian matrix Φ has the **RIP** (*restricted isometry property*) w.h.p. if

$$M = O\left(K + \log \binom{N}{K}\right) = O(K \log(N/K))$$

CS : Recovery



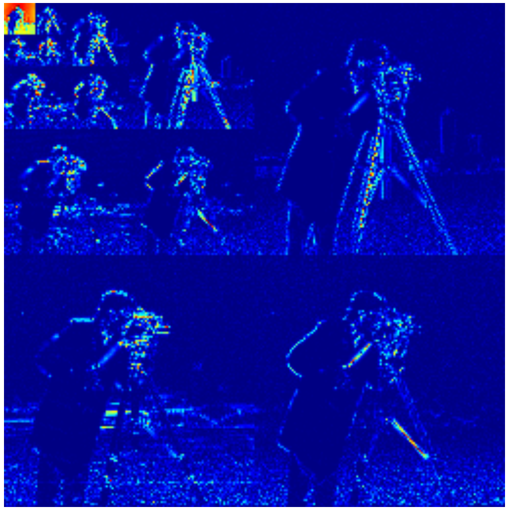
- ℓ_1-optimization
  [C, R, T]; [D]; [F,W,N]; [H,Y,Z]

- Greedy algorithms
  - OMP [G, T]
  - iterated thresholding [N, F]; [D, D, DeM]; [B, D]
  - CoSaMP [N,T]; Subspace Pursuit [D,M]

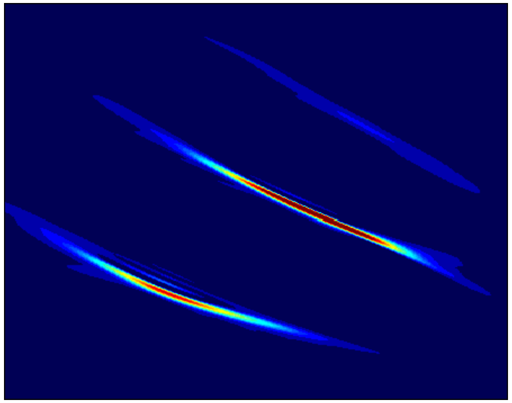
Beyond Sparsity

Signal Structure

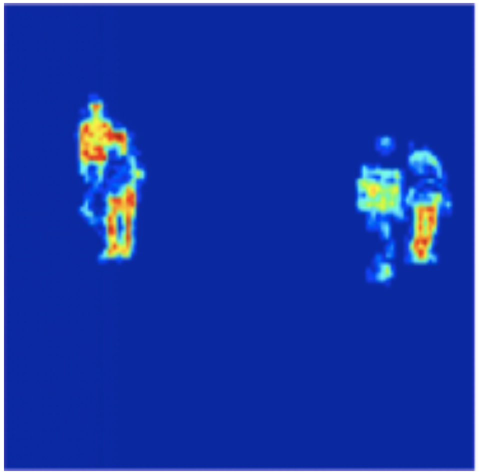
- Sparsity: simplistic, *first-order* assumption
- Many classes of real-world data exhibit **rich, secondary structure**



wavelets:
natural images



Gabor atoms:
chirps/tones



pixels:
background subtracted
images

How to exploit structure / prior?

Key idea: **Use Geometry**

- **Linear** models
- **Bilinear** models
- **Manifold** models

Geometry: Model

- **Sparse** signal:

x

  - only *K* out of *N* coefficients nonzero

Geometry: Model

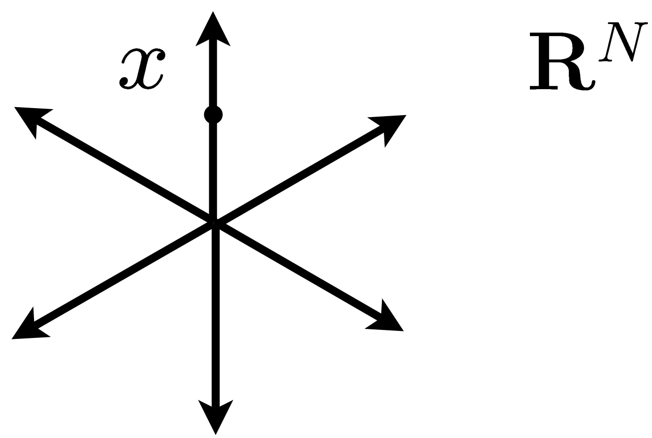
- **Sparse** signal:

x

  - only *K* out of *N* coordinates nonzero

- **Geometry**: *union* of $\binom{N}{K}$ *K*-dimensional subspaces aligned w/ coordinate axes

- *N* = 3, *K* = 1



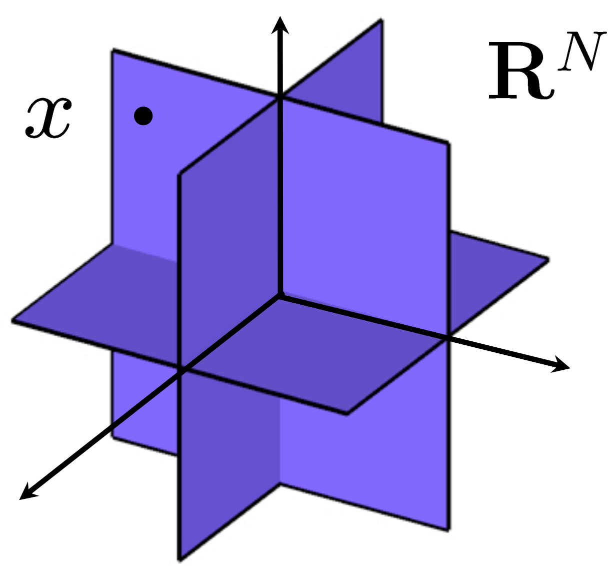
Geometry: Model

- **Sparse** signal:

  - only K out of N coordinates nonzero

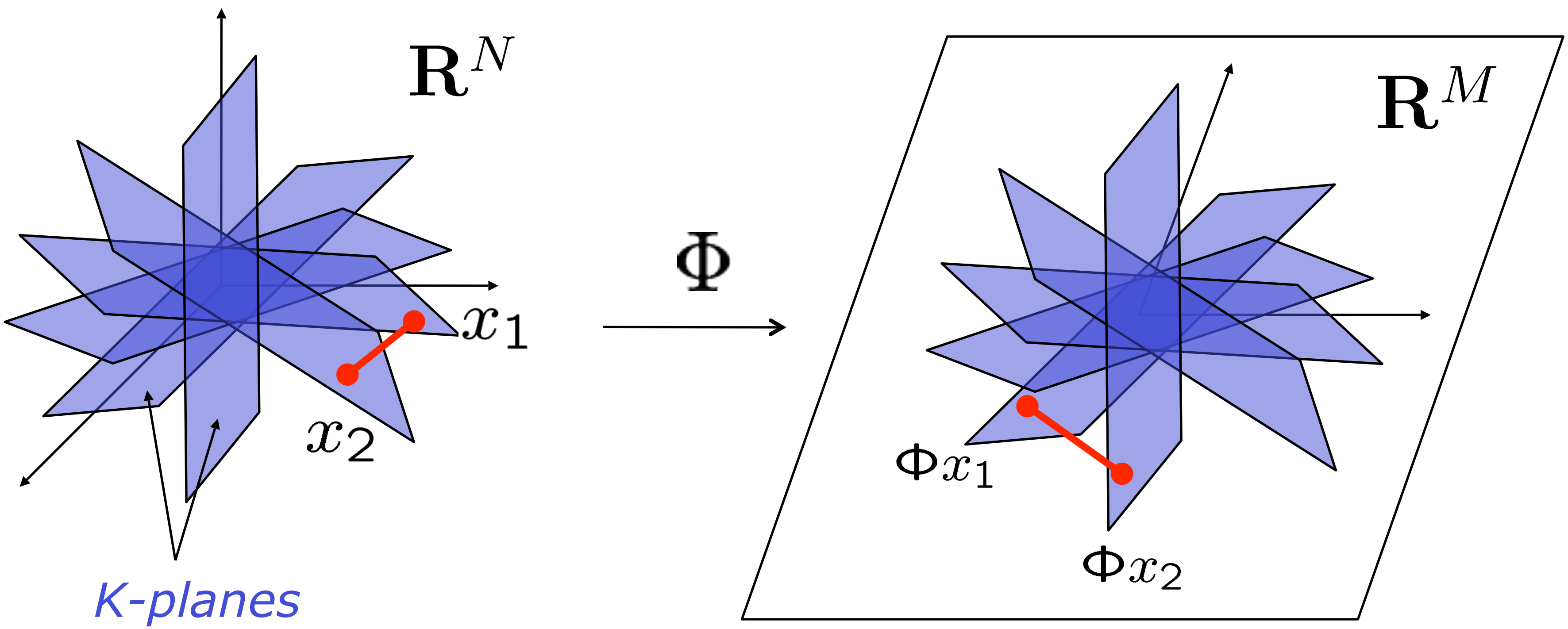
- **Geometry**: *union* of $\binom{N}{K}$ K-dimensional subspaces aligned w/ coordinate axes

- $N = 3$, $K = 2$



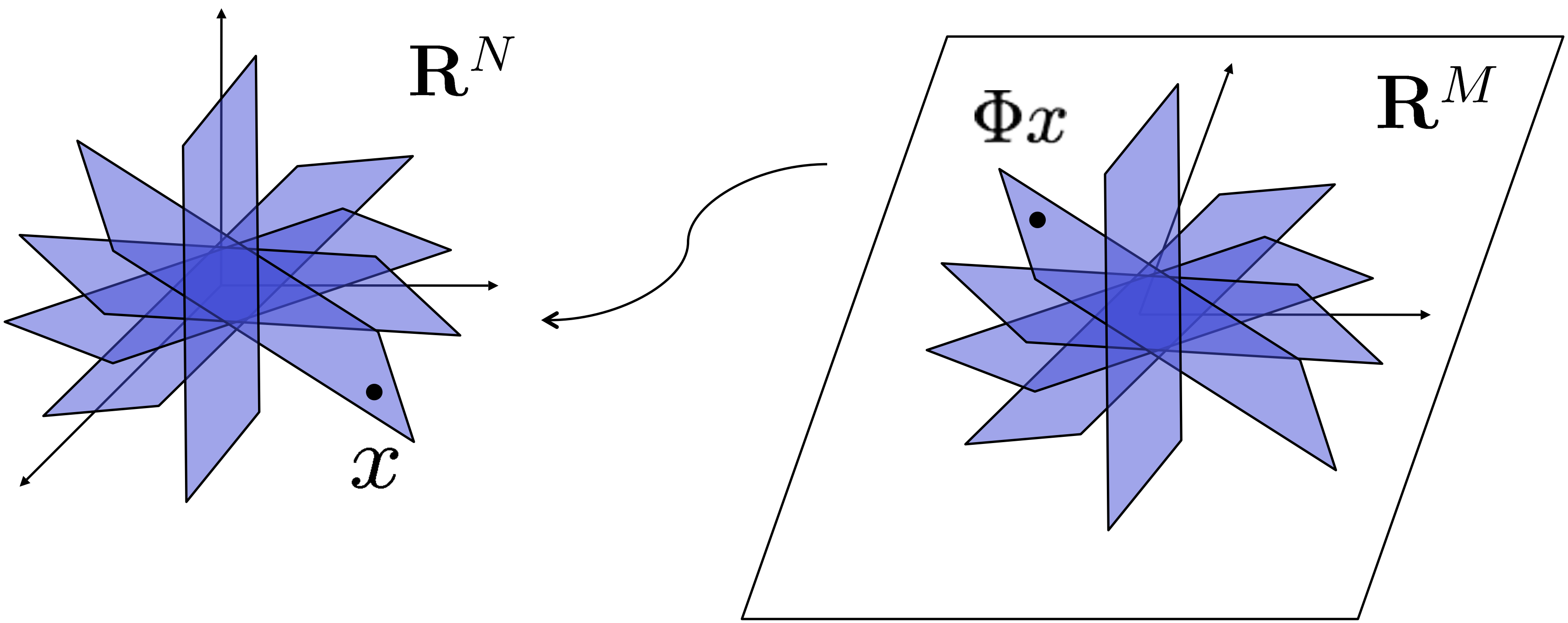
Geometry : Sampling

- Preserve the structure of sparse signals
- **Restricted Isometry Property (RIP)**



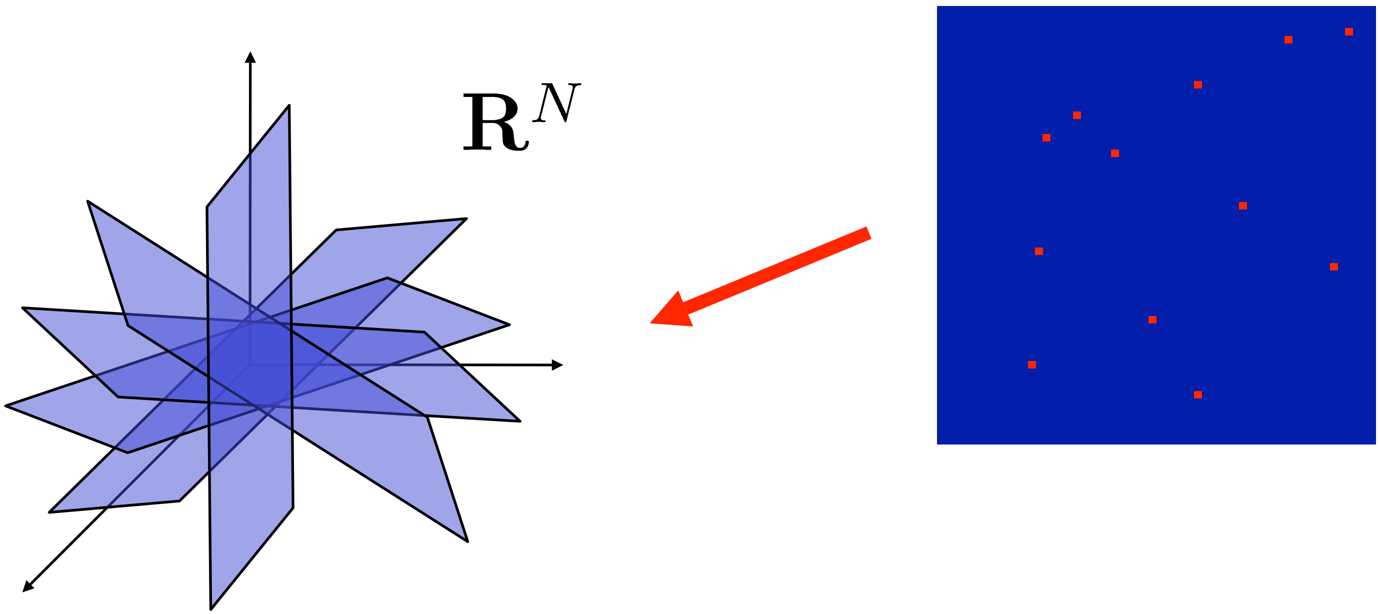
Geometry : Recovery

- Efficient, stable algorithms that **recover** signal



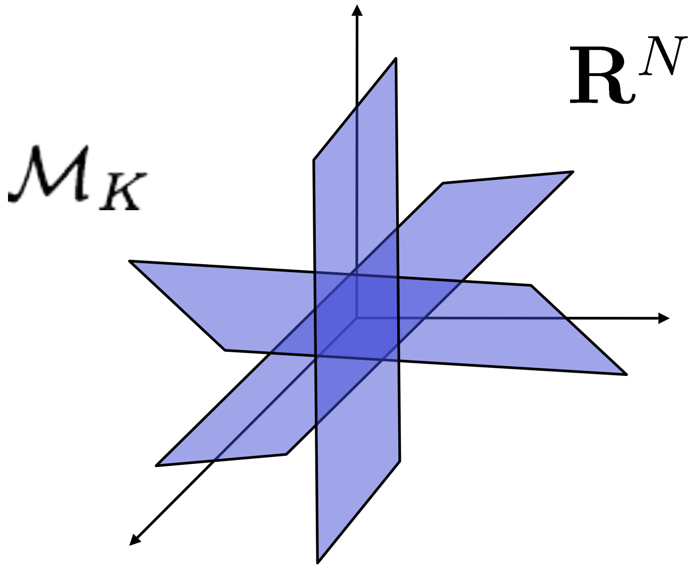
Sparse Signals

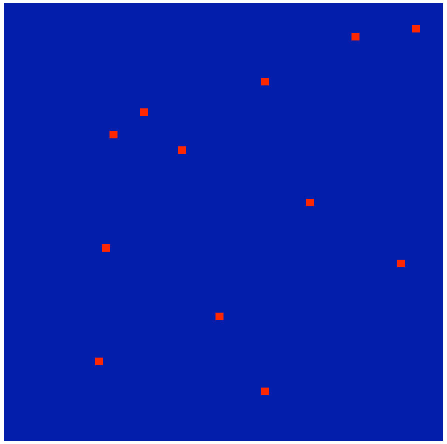
- Defn: ***K*-sparse signals** comprise *all*
  K-dimensional canonical subspaces

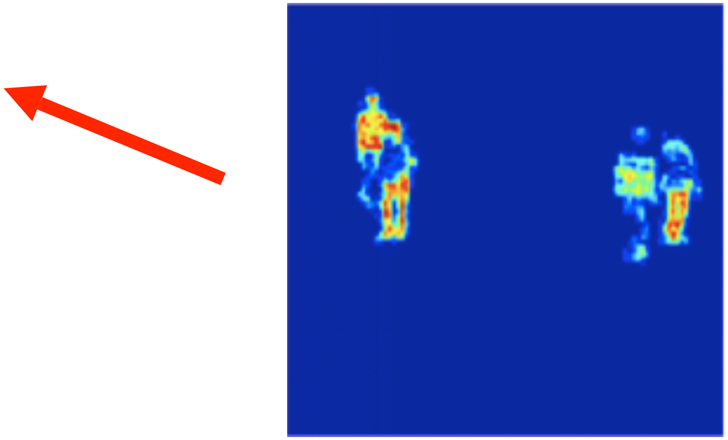


Model-Sparse Signals

- Def: A ***K*-sparse union-of-subspaces *model*** comprises a particular (*reduced*) set of L_K *K*-dim canonical subspaces

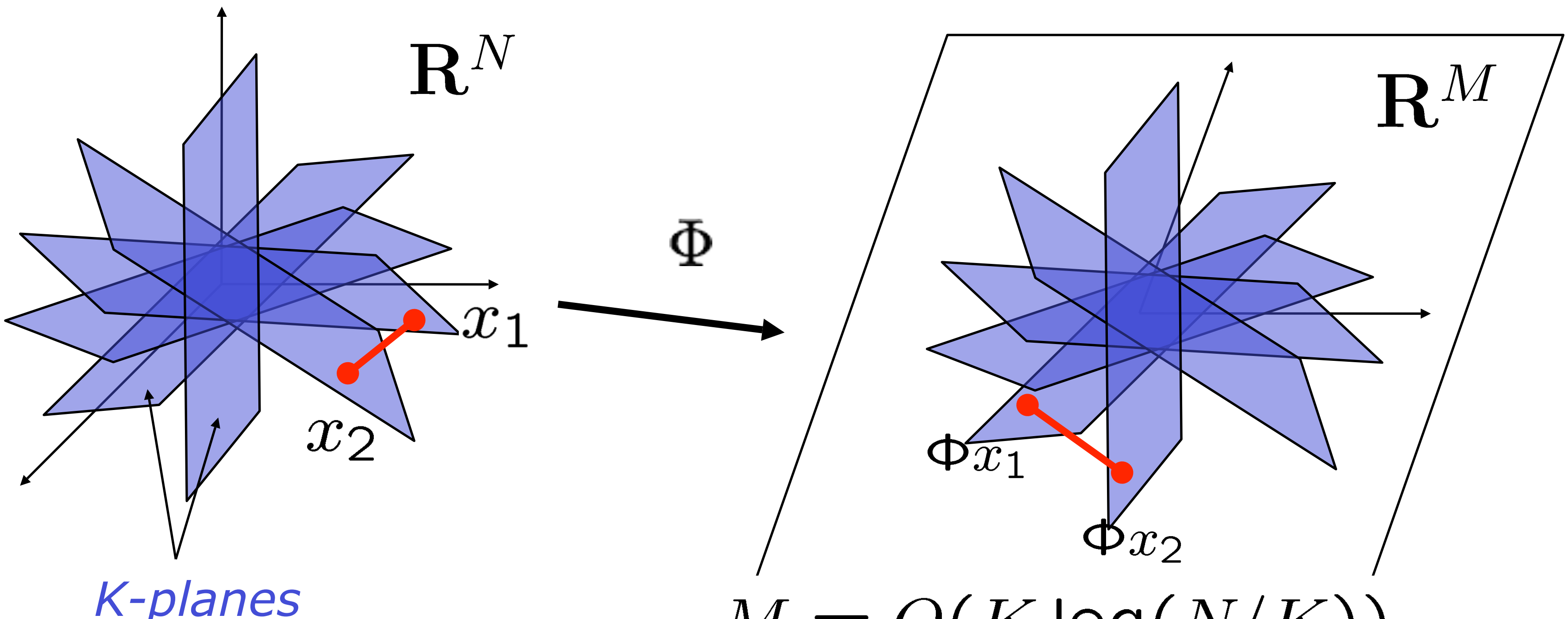






Sampling Bounds

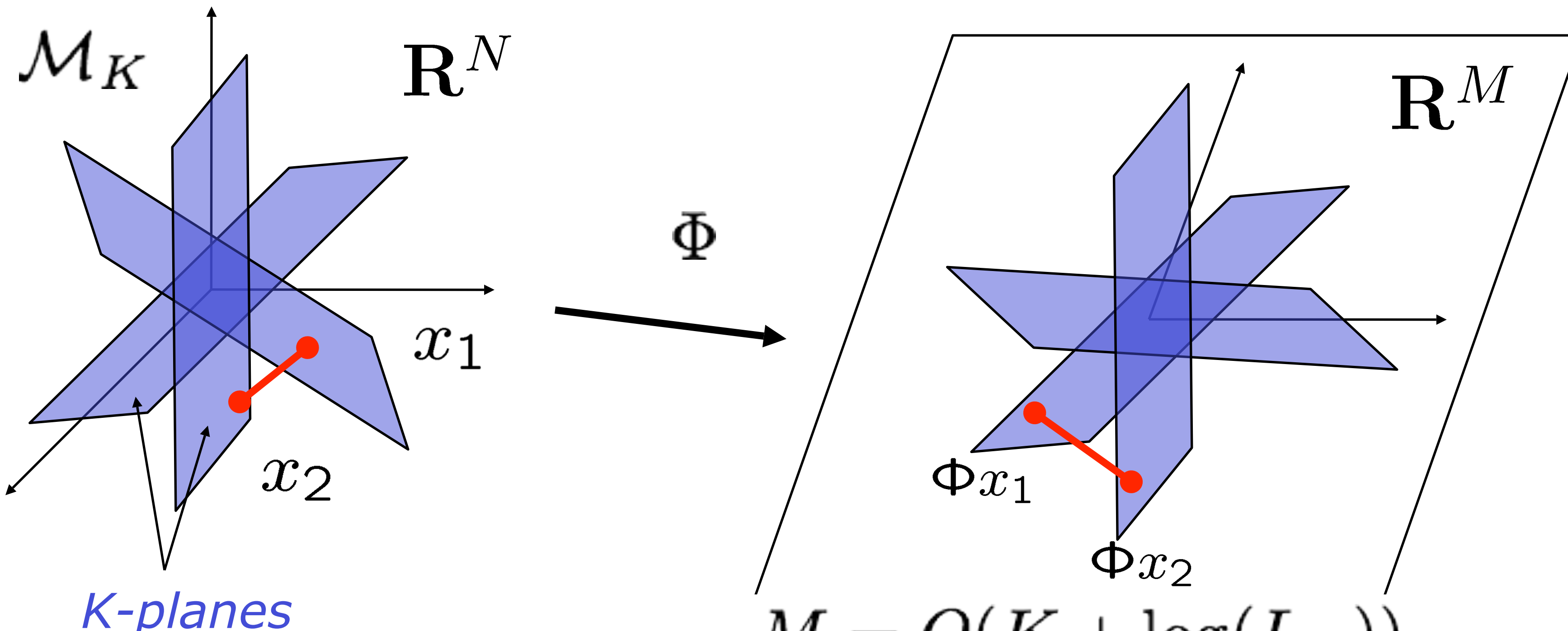
- **RIP:** stable embedding



$$M = O(K \log(N/K))$$

Sampling Bounds

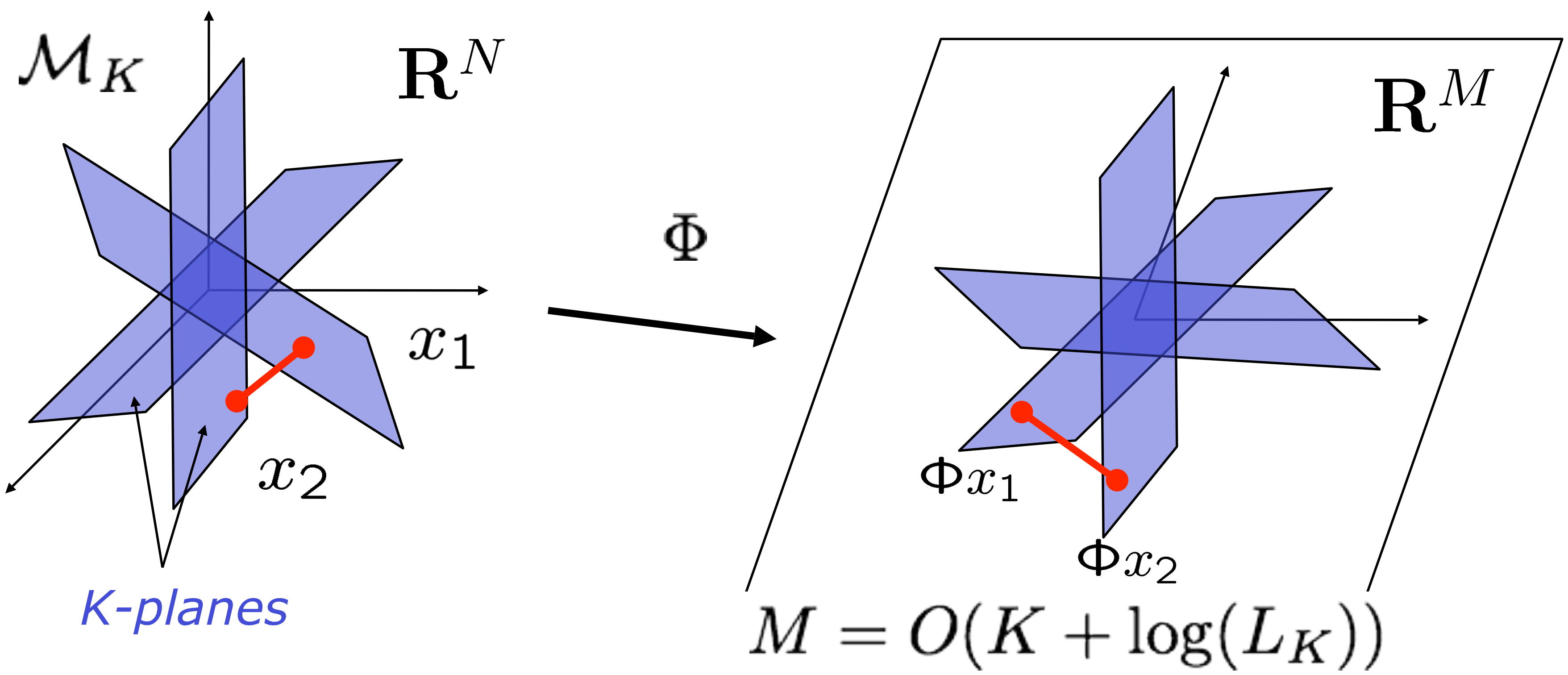
- **Model-RIP:** stable embedding



$$M = O(K + \log(L_K))$$

Sampling Bounds

- **Model-RIP:** stable embedding **(Ingredient 1)**



Iterated Thresholding

- goal: given $y = \Phi x$, recover $x \in \Sigma_K$

  initialize $i = 0, \quad x_0 = 0$

  iterate:

  - $\widehat{x}_{i+1} \leftarrow \text{thresh}(\widehat{x}_i + \Phi^T(y - \Phi x_i))$

  return $\widehat{x} \leftarrow \widehat{x}_i$

Iterated Model Thresholding

- goal: given $y = \Phi x$, recover $x \in \mathcal{M}_K$

  initialize $i = 0, \quad x_0 = 0$

  iterate:

  - $\widehat{x}_{i+1} \leftarrow \mathcal{M}(\widehat{x}_i + \Phi^T(y - \Phi\widehat{x}_i))$

  return $\widehat{x} \leftarrow \widehat{x}_i$

Iterated Model Thresholding

- goal: given $y = \Phi x$, recover $x \in \mathcal{M}_K$

initialize $i = 0, \;\; x_0 = 0$

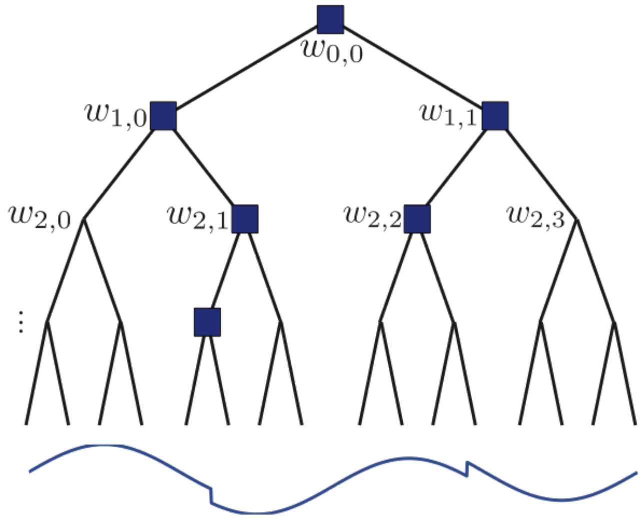
iterate:

- $\widehat{x}_{i+1} \leftarrow \mathcal{M}(\widehat{x}_i + \Phi^T(y - \Phi\widehat{x}_i))$

return $\widehat{x} \leftarrow \widehat{x}_i$

(Ingredient 2)

E.g. Wavelet trees



Daubechies/CoSaMP - K = 6000 M = 30000

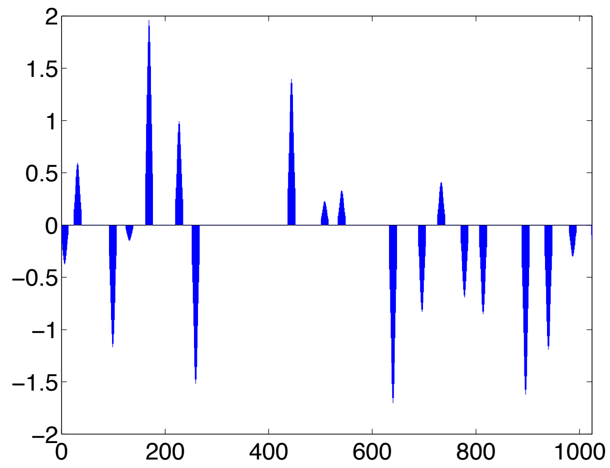
SNR = 13.1361dB

Daubechies/Tree CoSaMP - K = 6000 M = 30000

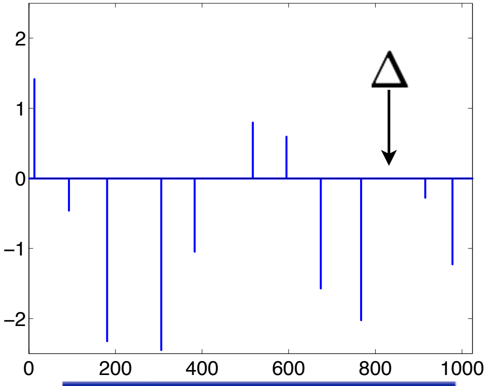
SNR = 17.8263dB

Other Union-of-Subspaces models

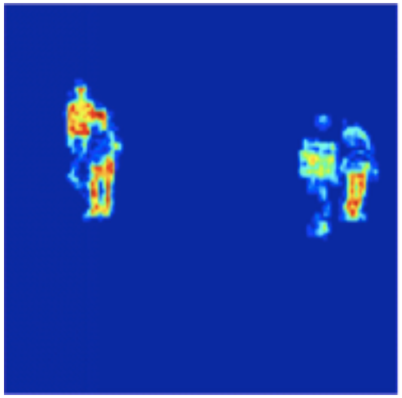
- Block-sparsity



- Δ-separated spikes



- Markov Random Fields



Bilinear Models

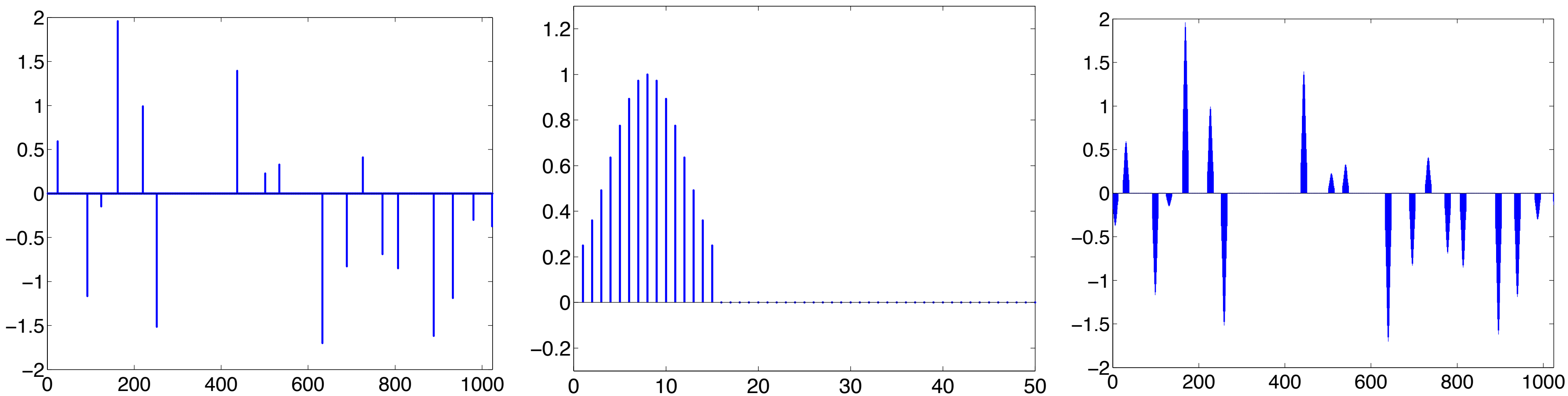
- "Pulse stream"

$$z = x * h$$

where:

$$x \in \mathcal{M}_S \qquad\qquad h \in \mathcal{M}_F$$

"spike stream" "impulse response" (IR)



CS for Bilinear Models

- Problem: recover *z* from compressive measurements

$$y = \Phi z = \Phi(x * h) = \Phi H x = \Phi X h$$

- Compare to:

$$z = x * h$$

"Blind Deconvolution"

Sampling Bound

- **Theorem**

$$M = O(S + F + \log(L_S L_F))$$

- In the worst case:

$$\begin{aligned} M &= O(S + F + S\log(N/S) + F\log(N/F)) \\ &\ll O(SF\log N) \;= O(K\log N) \end{aligned}$$

- Proof Technique: **Uses geometry**
  - Johnson-Lindenstrauss lemma + covering argument

Iterated Support Estimation

- goal: given $y = \Phi z = \Phi H x = \Phi X h$, recover $z \in \mathcal{M}(S, F, \Delta)$

  initialize $\widehat{h} = (\mathbf{1}_F^T, 0, \ldots, 0)/\sqrt{F}$

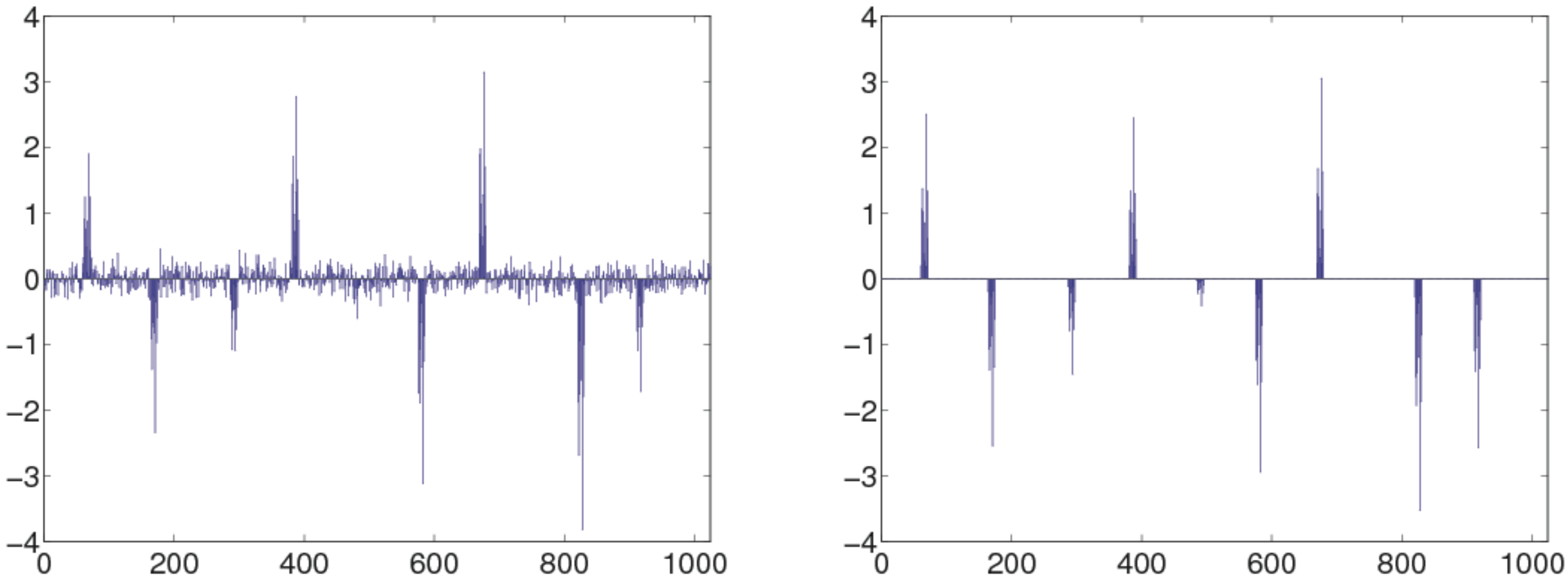
  $i = 0, \quad x_0 = 0$

  iterate:

  - $\widehat{x} \leftarrow \mathcal{M}_S^{\Delta}(\widehat{x} + (\Phi \widehat{H})^T (y - (\Phi \widehat{H} \widehat{x}))$
  - $\widehat{h} \leftarrow (\Phi \widehat{X})^{\dagger} y$

  return $\widehat{z} \leftarrow \widehat{x} * \widehat{h}$

Bilinear Models: Example

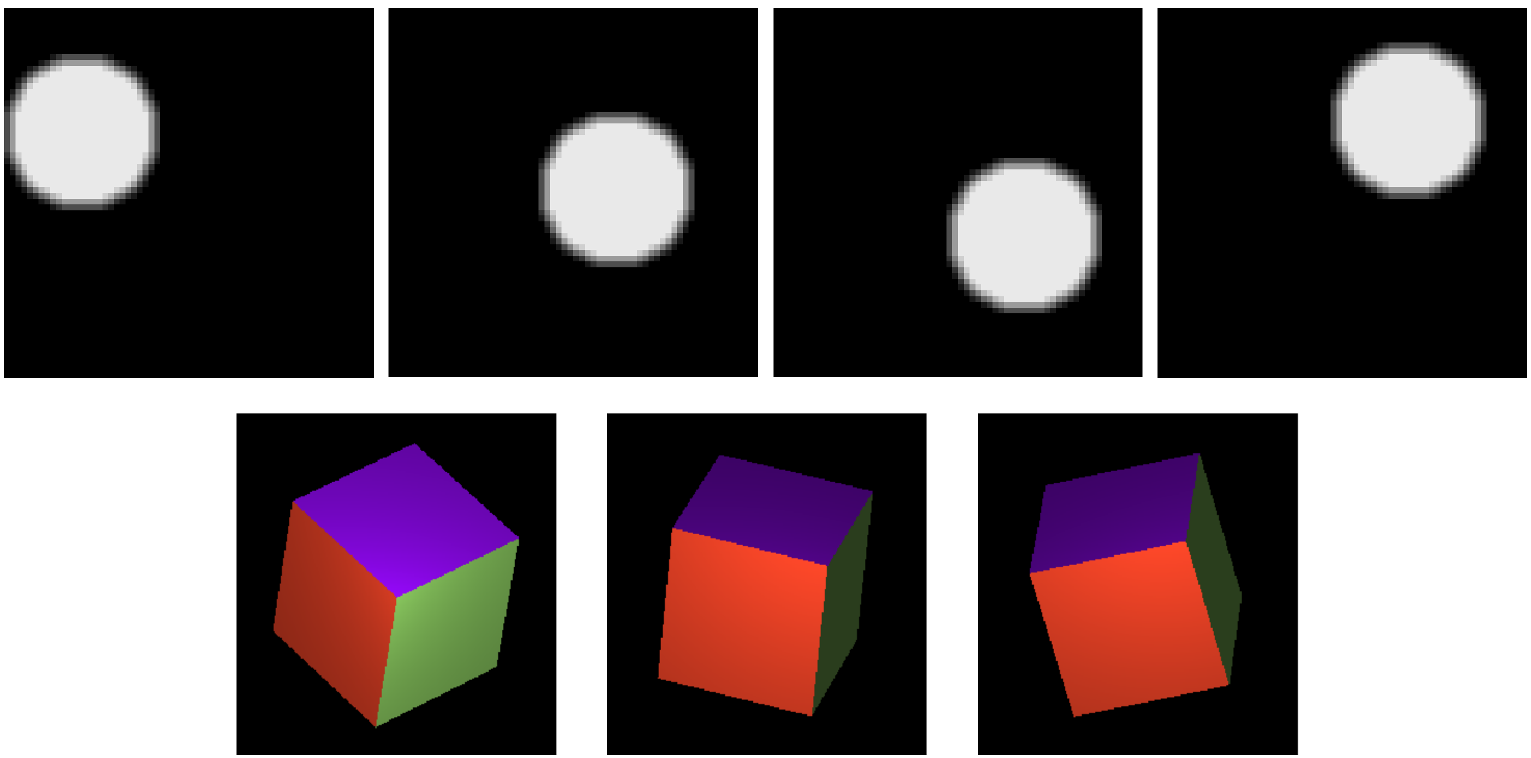


S = 9, F = 11, K = 99, M = 150

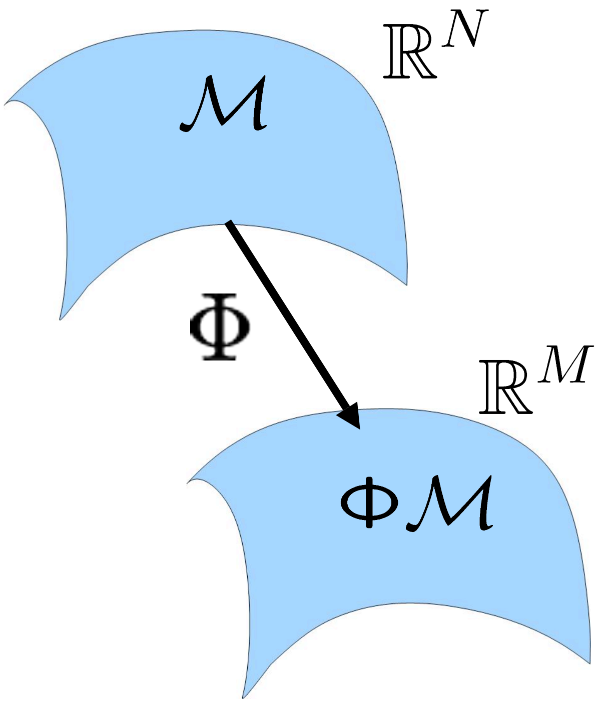
Manifold Models

- *K*-dimensional *parameter vector* captures degrees of freedom in signal $x \in \mathbb{R}^N$

$$x = x(\mathbf{z}), z \in \mathbb{R}^K$$



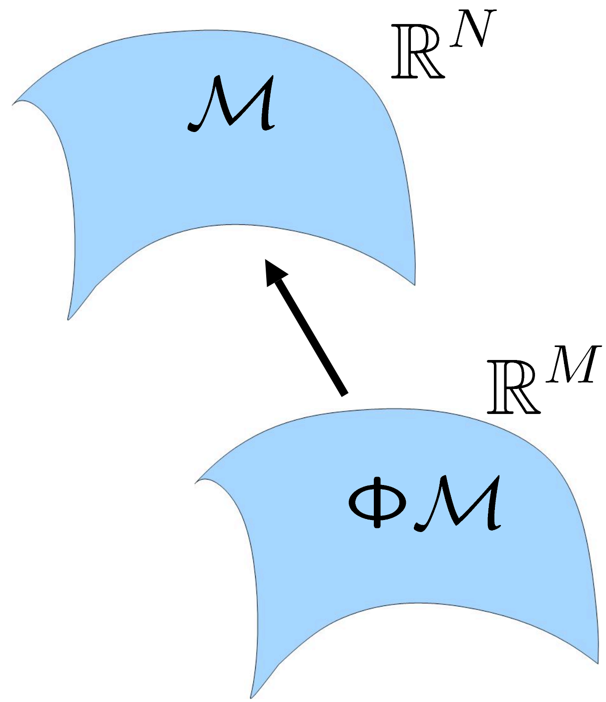
CS for Manifolds: Sampling



$$M = O\left(\frac{K \log(NV\tau^{-1}\epsilon^{-1}) \log(1/\rho)}{\epsilon^2}\right).$$

[Baraniuk, Wakin 2006]

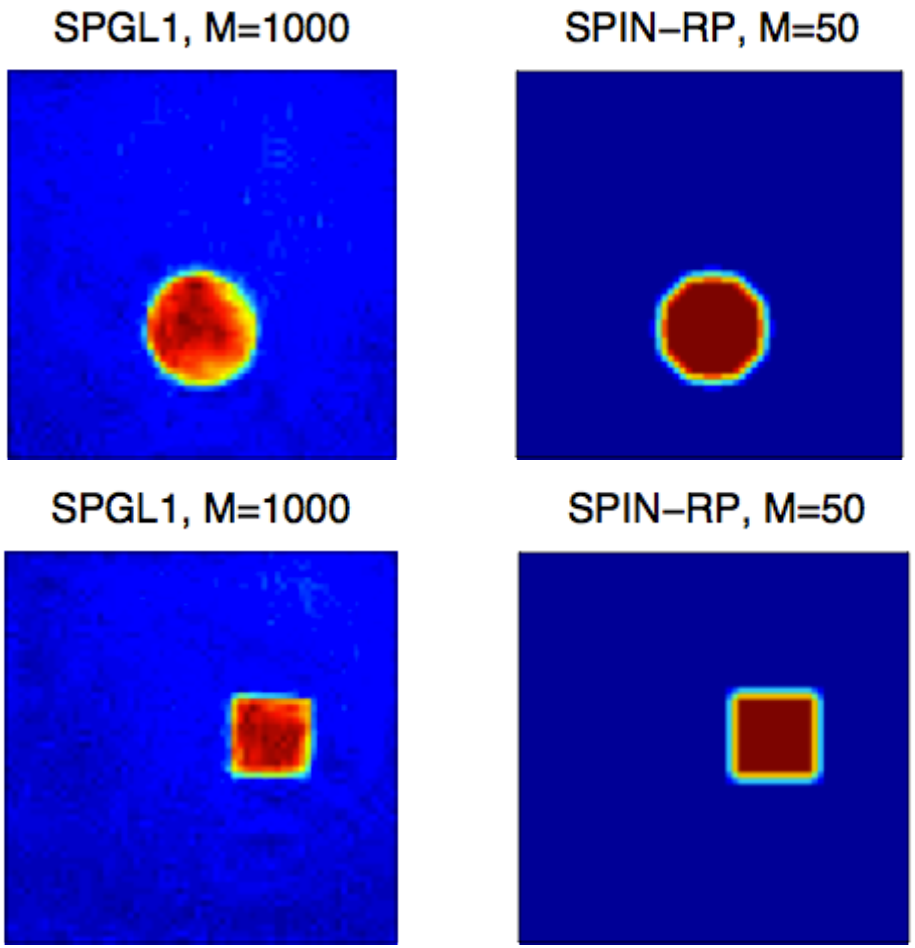
CS for Manifolds: Recovery

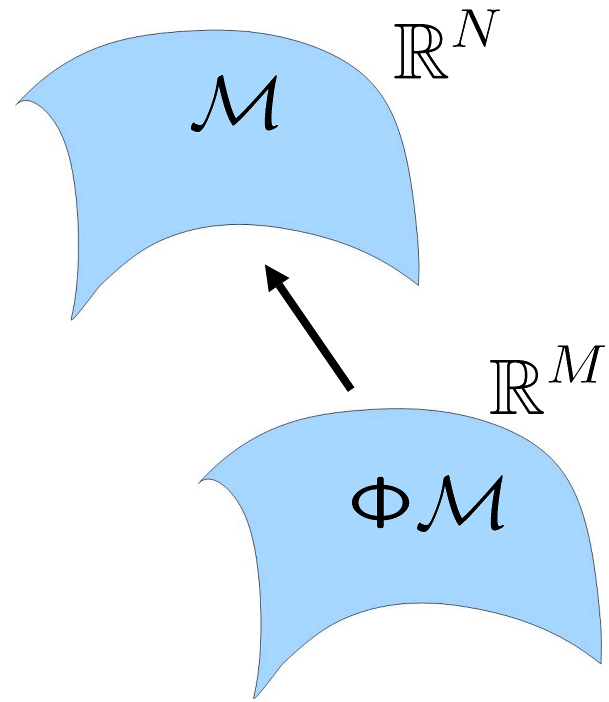


$$\widehat{x}_{i+1} \leftarrow \mathcal{M}(\widehat{x}_i + \Phi^T(y - \Phi\widehat{x}_i))$$

[Shah, Chandrasekaran, 2011]

CS for Manifolds: Recovery





Joint work with Kelly Lab

Summary

- Ingredients of CS: a) sampling bound for signal class
  b) algorithm for recovery

- Beyond sparsity
  - UoS/Bilinear/Manifold models
  - If you have prior info, use it! (but how?)
  - One (nice) method: **Geometric approach**
  - Advantages of the geometric approach:
    - *concise framework for characterization of systems*
    - *ability to generalize to a large class of problems*

- Applications: imaging, video sensing, radar, etc.

Review article: Duarte and Eldar [2011]

What's Next?

- Beyond sparse models: **Matrix models**
  - Affine rank minimization
  - Low-rank + sparse decompositions
  - Bounded degree/coherence matrices

- Beyond randomized sampling: **Adaptivity**
  - Design measurements according to signal/task prior
  - Closed loop sensing + reconstruction

- Beyond signal reconstruction: **Inference**
  - Estimate a *function* of the signal: anomaly detection, etc.
  - Data streaming methods