# Signal Recovery on Incoherent Manifolds

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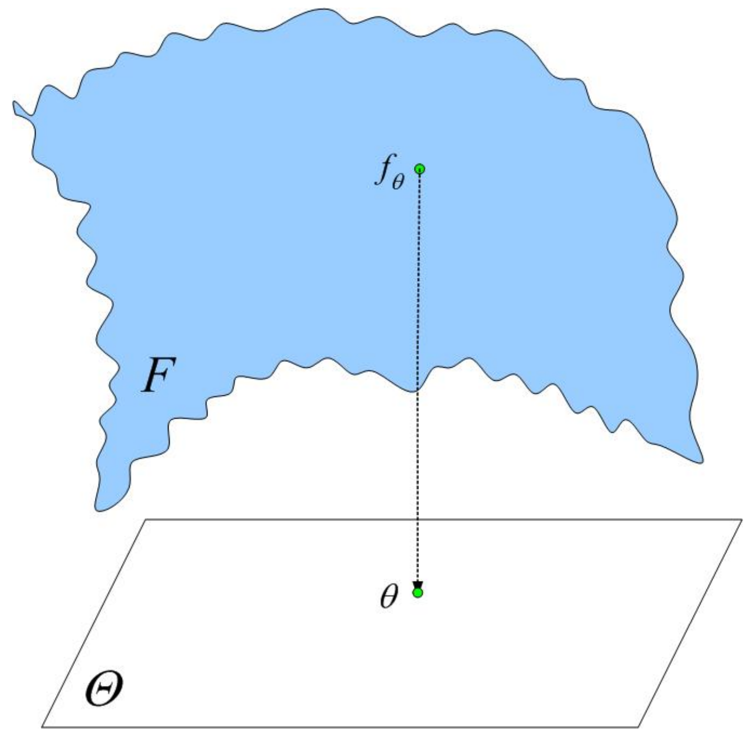
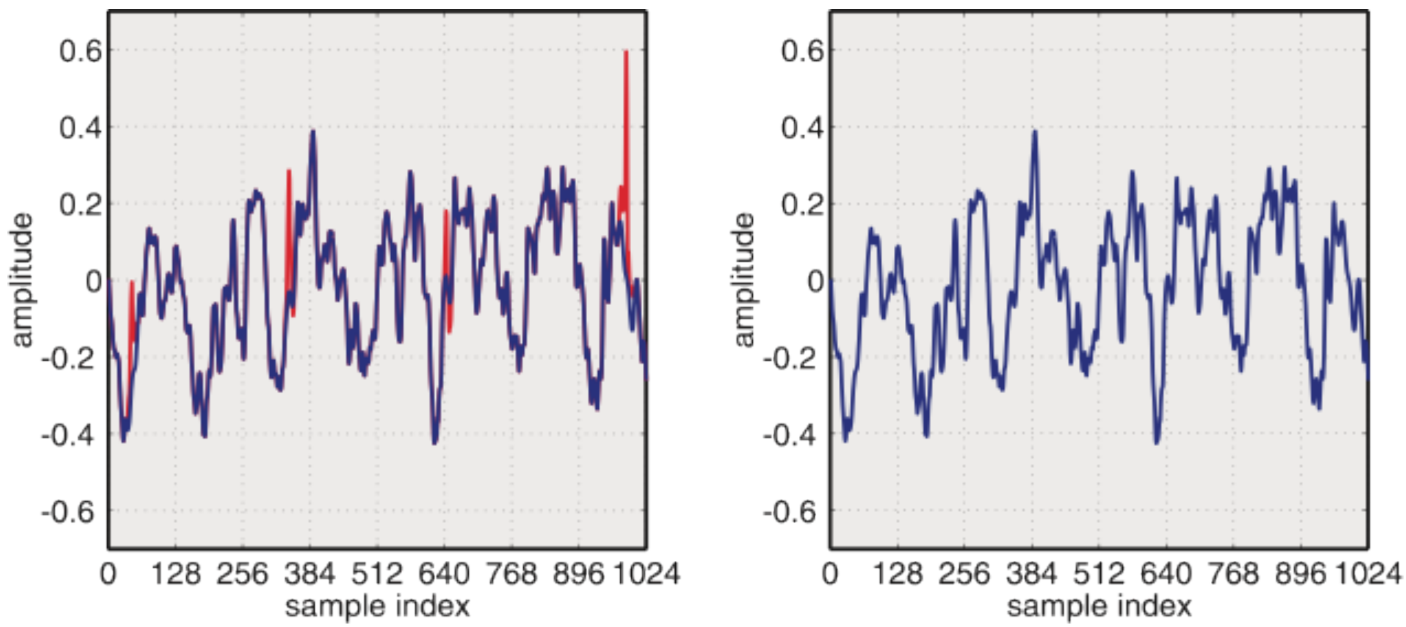
DSP



# Signal Separation

- Audio click removal



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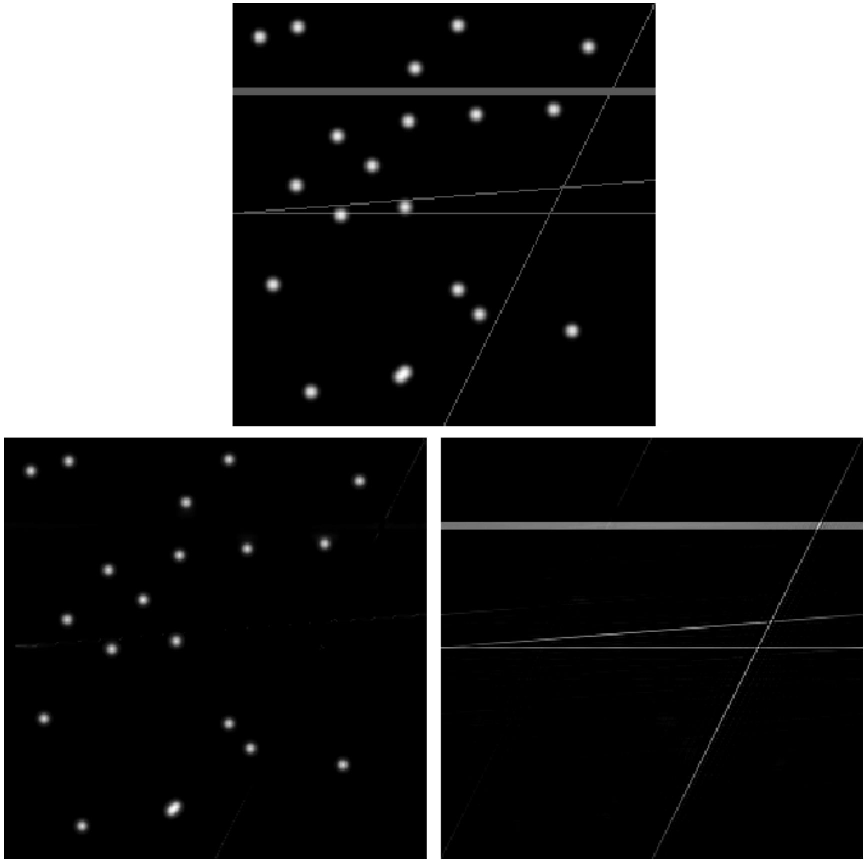
- Audio click removal
- Morphological Components Analysis (MCA)



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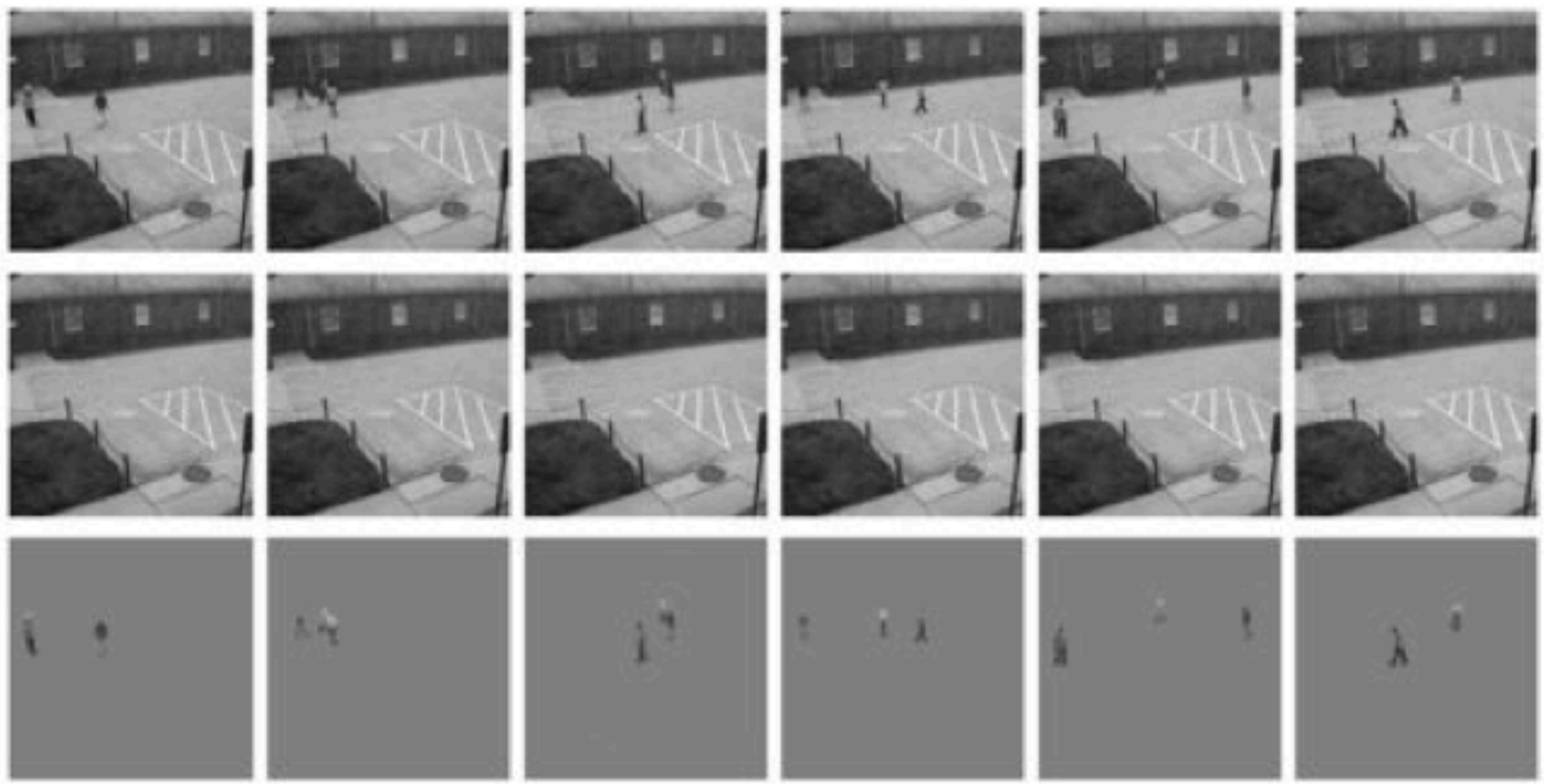
- Audio click removal
- Morphological Components Analysis (MCA)
- Background subtraction in video



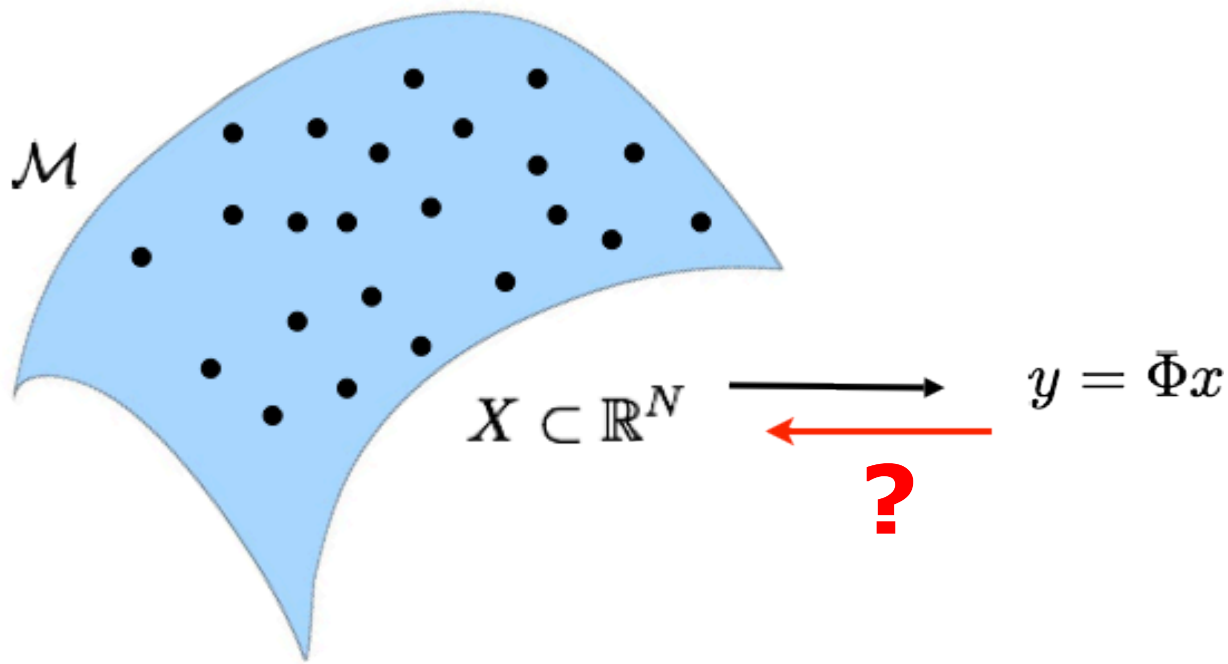
# Signal Separation

- Audio click removal
- Morphological Components Analysis
- Background subtraction in video
- …
- Numerous models have been explored
  - Spikes and sines
  - Points and lines
  - “Robust” compressive recovery
  - **Low-rank + sparse matrices**

**Goal:**

*Solve* ***all*** *such signal recovery problems in a unified, computationally efficient, stable framework*

# Signal Recovery on Manifolds



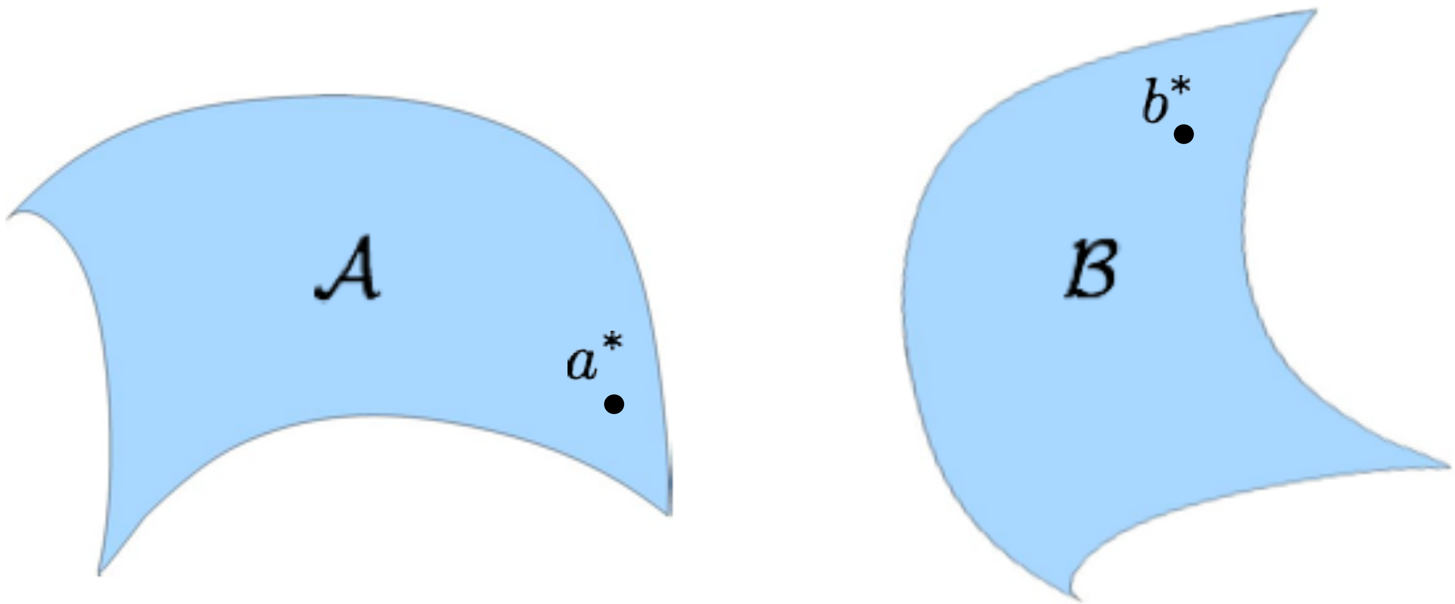

Question asked (and answered) by several people

[CRT06, Don06, FRP07, BD09, BCDH10, … , **SC11**]

# Multi-Manifold Model

- Signal of interest:

$$x = a^* + b^*, \quad a^* \in \mathcal{A}, \ b^* \in \mathcal{B}$$



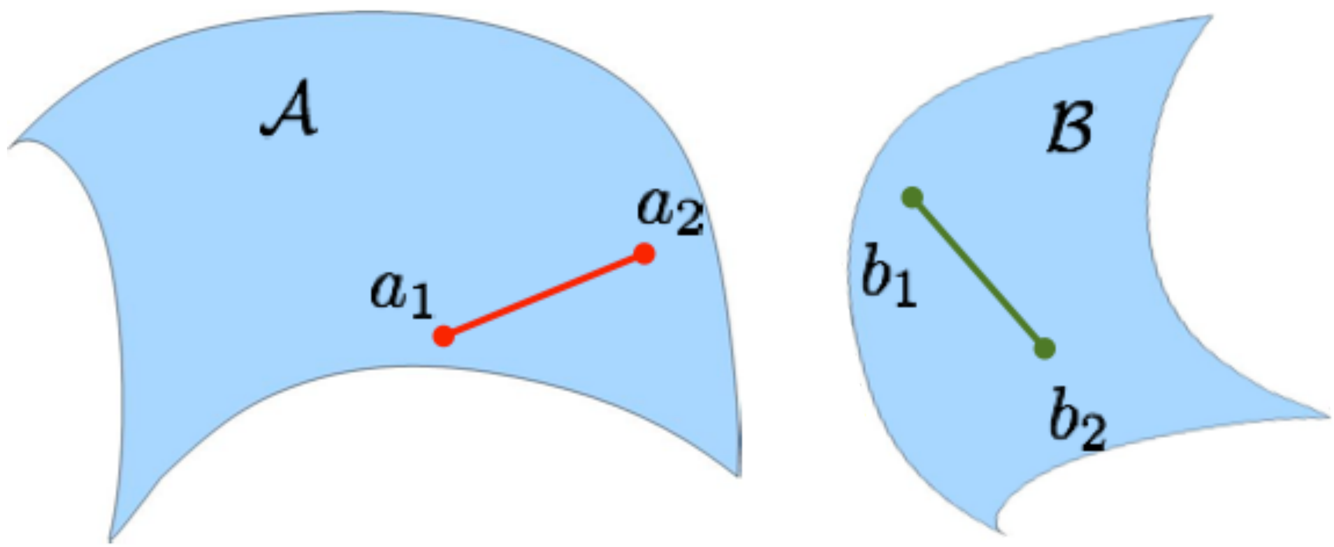

- Noisy linear measurements:

$$y = \Phi x + e = \Phi(a^* + b^*) + e$$

# Signal Separation

# Geometric Ingredient #1

- **Manifold incoherence**

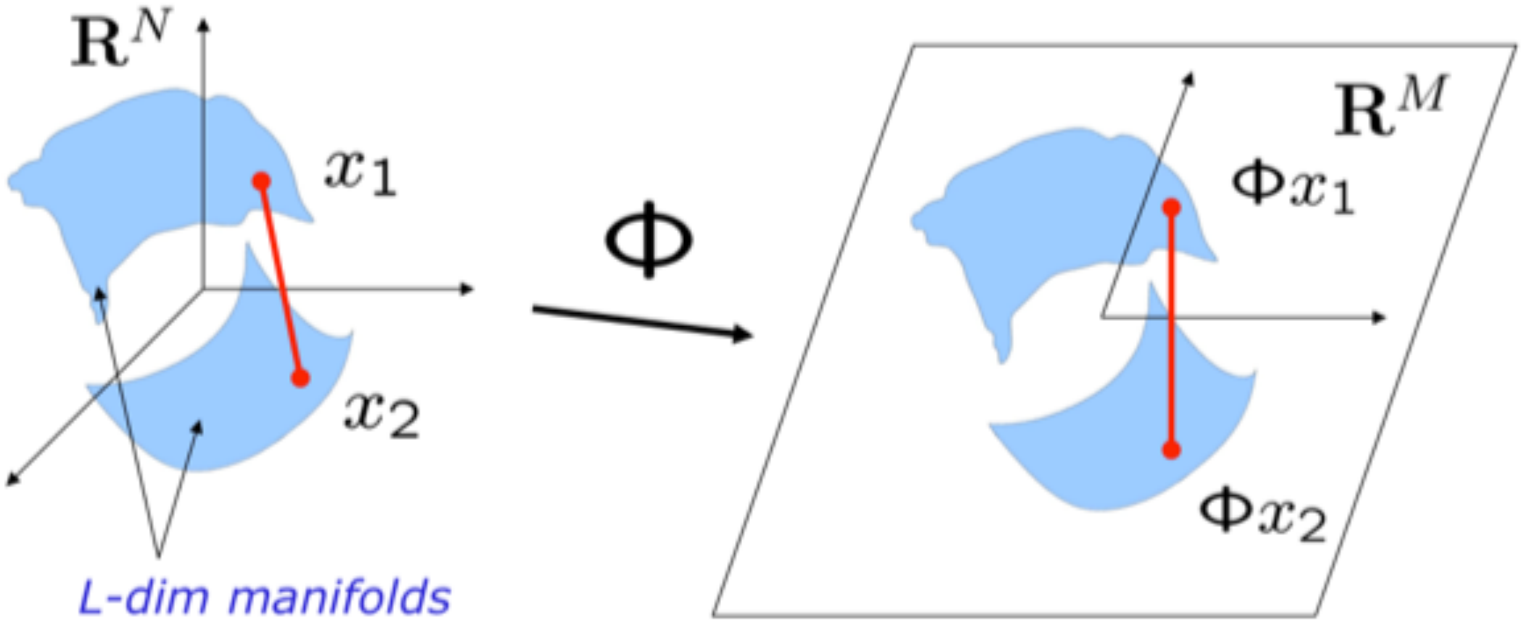



$$\left| \left\langle \frac{a_1 - a_2}{\|a_1 - a_2\|}, \frac{b_1 - b_2}{\|b_1 - b_2\|} \right\rangle \right| \leq \epsilon$$

- *Local* vs. *global* incoherence

# Geometric Ingredient #2

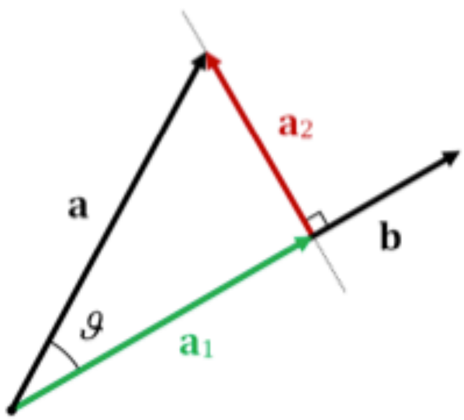
- **Restricted isometry**

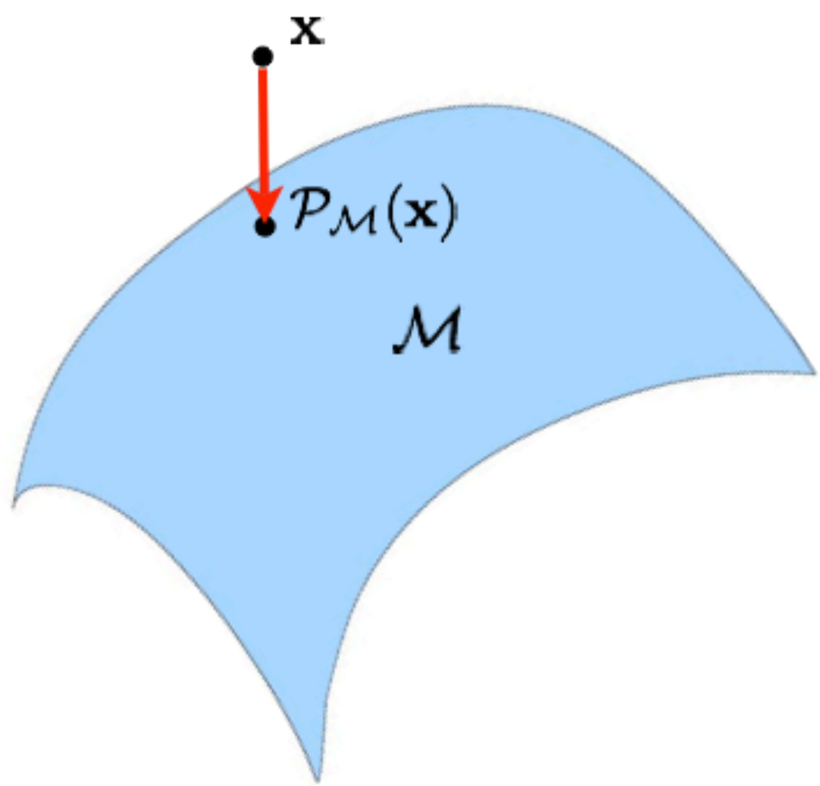



$$1 - \delta \leq \frac{\|\Phi(x_1 - x_2)\|^2}{\|x_1 - x_2\|^2} \leq 1 + \delta$$

# Geometric Ingredient #3

- **Projections onto manifolds**



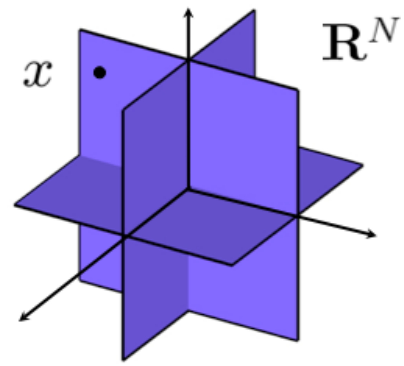




$$\mathcal{P}_{\mathcal{M}}(\mathbf{x}) = \arg \min_{\mathbf{x}' \in \mathcal{M}} \|\mathbf{x}' - \mathbf{x}\|_2^2$$

# Geometric Ingredient #3

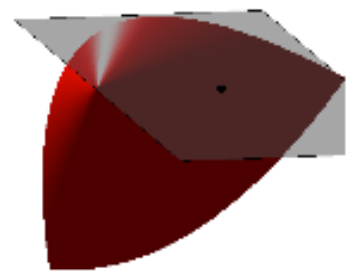
- **Projections onto manifolds**

  – Manifold of *K*-sparse signals



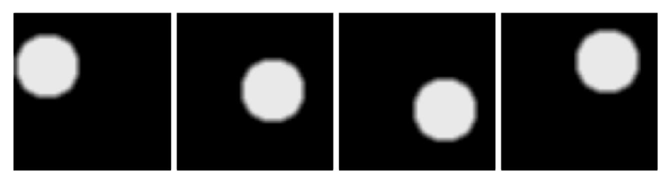

$$\mathcal{P}_{\mathcal{M}}(\mathbf{x}) = \mathbb{T}(\mathbf{x}, K)$$

  – Manifold of rank-*r* matrices



$$X = U\Sigma V$$
$$\mathcal{P}_{\mathcal{M}}(X) = U_r\Sigma_r V_r$$

  – Articulation Manifold (AM)



$$\widehat{\theta} = \arg\max \langle X, \theta(X_0)\rangle$$
$$\mathcal{P}_{\mathcal{M}} = \widehat{\theta}(X_0)$$

# Successive Projections onto Incoherent Manifolds (SPIN)

***Goal**:* given $y = \Phi(a^* + b^*) + e$, recover $(a^*, b^*)$

Initialize

$$a_0 = 0, \quad b_0 = 0$$

Iterate:

- $$a_{k+1} \leftarrow \mathcal{P}_{\mathcal{A}}(a_k + \eta \Phi^T (y - \Phi(a_k + b_k)))$$
- $$b_{k+1} \leftarrow \mathcal{P}_{\mathcal{B}}(b_k + \eta \Phi^T (y - \Phi(a_k + b_k)))$$

until convergence

[HB12]

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[HB12]

**"Projected Block-Coordinate Descent"**

# Convergence

Suppose that the *restricted isometry* constant of the sampling matrix relates to the manifold *incoherence* as:

$$0 \leq \delta < \frac{1 - 11\epsilon}{3 + 7\epsilon}$$

Then, for any precision parameter $\nu$, after a finite number of iterations of SPIN, we have:

$$\max\{\|\mathbf{a}_k - \mathbf{a}^*\|, \|\mathbf{b}_k - \mathbf{b}^*\|\} \leq \nu$$

*Proof approach:*

Define the residual error as $\psi(a, b) = \frac{1}{2}\|y - \Phi(a + b)\|^2$.

We can show the following iteration invariant:

$$\psi(a_{k+1}, b_{k+1}) \leq \alpha\psi(a_k, b_k) + C\|e\|^2$$

$\alpha < 1 \implies$ convergence!

# Example: Spikes and Sines

- SPIN (provably) recovers the components of the linear sum of $K_1$ spikes and $K_2$ sines provided:

$$K_1 + K_2 < \frac{1}{11\mu} \approx 0.091\sqrt{N}$$

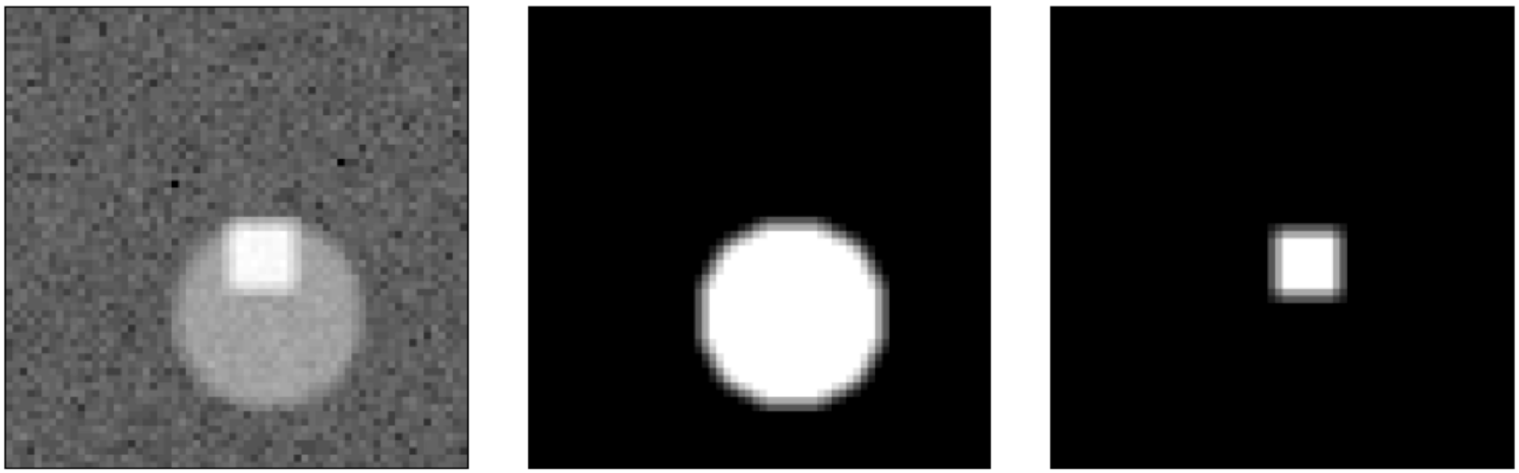
- Optimal bound for $\ell_1$-minimization

$$K_1 + K_2 < 0.91\sqrt{N}$$ [EB02,FN03]

***Constant can potentially be improved***

# Example: Articulation Manifolds

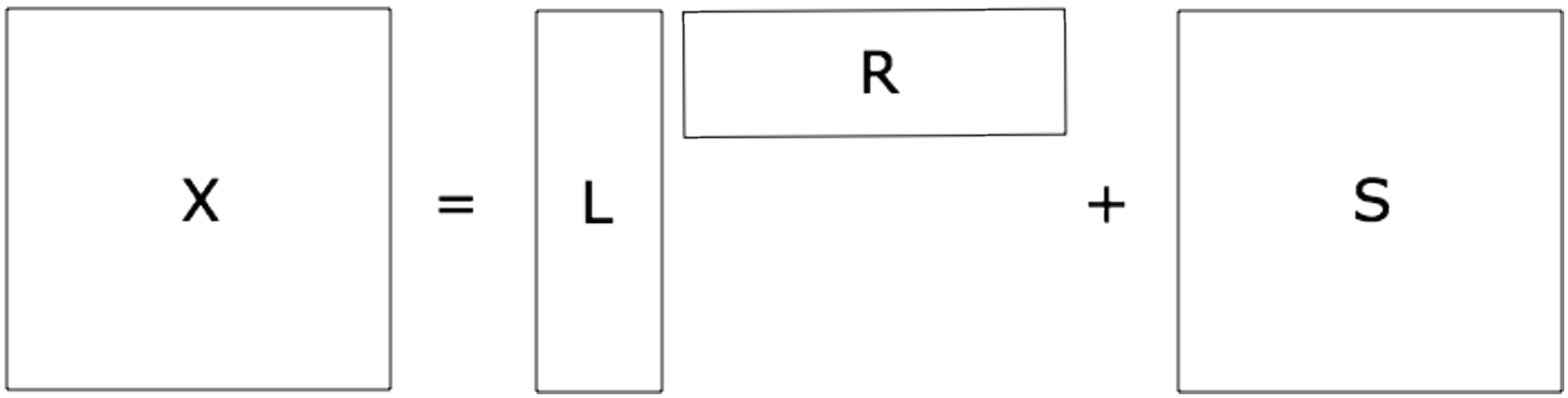
- Signal size: $N$ = 64 x 64 = 4096,
  number of measurements: $M$ = 50
- Noise added to the signal



- Near-perfect recovery for M/N = 1.2% meas.!

# Example: Matrix Decomposition

- Low-rank + sparse model

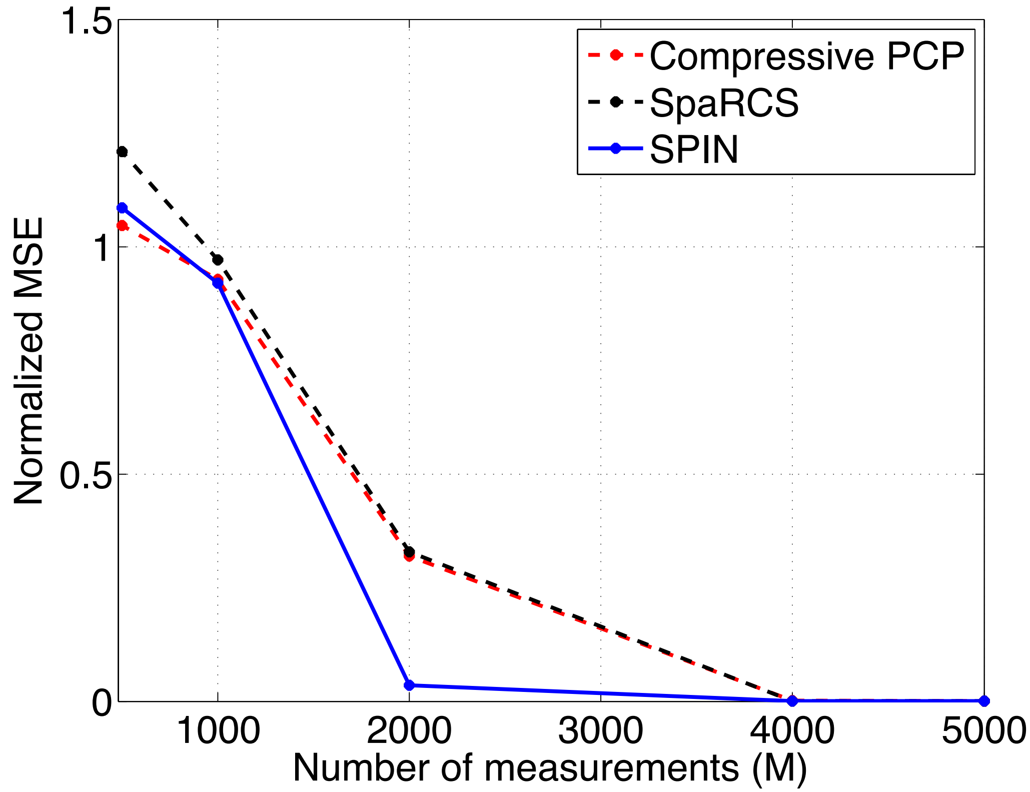



- Efficient projection operators exist for both manifolds

- But (global) incoherence assumption is **violated**
  - there exist low-rank matrices that are sparse + vice versa

# Numerical Comparison

- Parameters: $N = 128 \times 128$, $K = 164$, $r = 2$
- Test matrices generated at random (100 trials)




-SPIN outperforms Compressive PCP, SpaRCS [WGMM12,WSB11]

# Summary

- SPIN: **unified** framework for signal separation, multi-manifold recovery
- Highlights:
  - efficiency
  - provable convergence
  - conceptually simple: can be extended to more complex models, more than 2 manifolds, etc.
- Applications to MCA, signal/image denoising, background-foreground subtraction, etc.

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[HB12] C. Hegde, R. Baraniuk, “Signal Recovery on Incoherent Manifolds”, 2012.