Signal Recovery on Incoherent Manifolds

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Signal Separation

- Audio click removal
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- Morphological Components Analysis (MCA)
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- Background subtraction in video
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- Audio click removal
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- ...
- Numerous models have been explored
  - Spikes and sines
  - Points and lines
  - “Robust” compressive recovery
  - **Low-rank + sparse matrices**

**Goal:**

*Solve all such signal recovery problems in a unified, computationally efficient, stable framework*
Signal Recovery on Manifolds

Question asked (and answered) by several people
[CRT06, Don06, FRP07, BD09, BCDH10, ... , SC11]
Multi-Manifold Model

- Signal of interest:
  \[ x = a^* + b^*, \quad a^* \in \mathcal{A}, \quad b^* \in \mathcal{B} \]

- Noisy linear measurements:
  \[ y = \Phi x + e = \Phi(a^* + b^*) + e \]
Signal Separation
Geometric Ingredient #1

- **Manifold incoherence**

\[
\left\langle \frac{a_1 - a_2}{\|a_1 - a_2\|}, \frac{b_1 - b_2}{\|b_1 - b_2\|} \right\rangle \leq \epsilon
\]

- *Local vs. global* incoherence
Geometric Ingredient #2

- Restricted isometry

\[ \| \Phi (x_1 - x_2) \|^2 \leq \| x_1 - x_2 \|^2 \leq 1 + \delta \]
Geometric Ingredient #3

- Projections onto manifolds

\[ P_M(x) = \arg \min_{x' \in M} \| x' - x \|_2 \]
Geometric Ingredient #3

- **Projections onto manifolds**
  - Manifold of $K$-sparse signals
    $$\mathcal{P}_M(x) = \mathbb{T}(x, K)$$
  - Manifold of rank-$r$ matrices
    $$X = U\Sigma V$$
    $$\mathcal{P}_M(X) = U_r\Sigma_r V_r$$
  - Articulation Manifold (AM)
    $$\hat{\theta} = \arg \max \langle X, \theta(X_0) \rangle$$
    $$\mathcal{P}_M = \hat{\theta}(X_0)$$
Successive Projections onto Incoherent Manifolds (SPIN)

**Goal:** given \( y = \Phi(a^* + b^*) + e \), recover \( (a^*, b^*) \)

Initialize

\[
a_0 = 0, \quad b_0 = 0
\]

Iterate:

\[
\begin{align*}
    a_{k+1} &\leftarrow \mathcal{P}_A(a_k + \eta \Phi^T(y - \Phi(a_k + b_k))) \\
    b_{k+1} &\leftarrow \mathcal{P}_B(b_k + \eta \Phi^T(y - \Phi(a_k + b_k)))
\end{align*}
\]

until convergence

[HB12]
Successive Projections onto Incoherent Manifolds (SPIN)

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[HB12]

“Projected Block-Coordinate Descent”
Convergence

Suppose that the restricted isometry constant of the sampling matrix relates to the manifold incoherence as:

\[ 0 \leq \delta < \frac{1 - 11\epsilon}{3 + 7\epsilon} \]

Then, for any precision parameter \( \nu \), after a finite number of iterations of SPIN, we have:

\[ \max\{\|a_k - a^*\|, \|b_k - b^*\|\} \leq \nu \]

Proof approach:
Define the residual error as \( \psi(a, b) = \frac{1}{2} \|y - \Phi(a + b)\|^2 \).
We can show the following iteration invariant:

\[ \psi(a_{k+1}, b_{k+1}) \leq \alpha \psi(a_k, b_k) + C \|e\|^2 \]

\( \alpha < 1 \implies \text{convergence!} \)
Example: Spikes and Sines

- SPIN (provably) recovers the components of the linear sum of $K_1$ spikes and $K_2$ sines provided:

\[ K_1 + K_2 < \frac{1}{11\mu} \approx 0.091\sqrt{N} \]

- Optimal bound for $\ell_1$-minimization

\[ K_1 + K_2 < 0.91\sqrt{N} \]  \[\text{[EB02,FN03]}\]

*Constant can potentially be improved*
Example: Articulation Manifolds

- Signal size: $N = 64 \times 64 = 4096$, number of measurements: $M = 50$
- Noise added to the signal

- Near-perfect recovery for $M/N = 1.2\%$ meas.!
Example: Matrix Decomposition

- Low-rank + sparse model

\[ X = L + R + S \]

- Efficient projection operators exist for both manifolds

- But (global) incoherence assumption is **violated**
  - there exist low-rank matrices that are sparse + vice versa
Numerical Comparison

- Parameters: $N = 128 \times 128$, $K = 164$, $r = 2$
- Test matrices generated at random (100 trials)

- SPIN outperforms Compressive PCP, SpaRCS

[WGMM12, WSB11]
Summary

• SPIN: **unified** framework for signal separation, multi-manifold recovery

• Highlights:
  – efficiency
  – provable convergence
  – conceptually simple: can be extended to more complex models, more than 2 manifolds, etc.

• Applications to MCA, signal/image denoising, background-foreground subtraction, etc.
References