# Lecture 23



- So far: - implementing truthfully an affine maximizer
  - didn't care about prices paid to the auctioneer or from auctioneer to bidders
- This lecture: optimize revenue
  ↳ in single-parameter settings
- **General Single-Parameter Setting**
  - $n$ bidders, bidder $i$ has value $v_i$ for receiving service
  - auctioneer provides service to vector of agents $X = (X_1, .., X_n) \in \{0,1\}^n$
    ↳ possibly random
  - auctioneer has cost $c(x)$ for $x \in \{0,1\}^n$
- **Performance Measures of a Mechanism**

$$\mathbb{E}[\text{welfare}] = \sum_i v_i \cdot \mathbb{E}[X_i] - \mathbb{E}[c(X)]$$

$$\mathbb{E}[\text{Profit}] = \sum_i \mathbb{E}[P_i] - \mathbb{E}[c(x)]$$

where $X = X(b)$ : allocation rule of mechanism
and $p = p(b)$ : price rule

} both are fn's of bid vector $b$ and possibly randomized

E.g. single-item auction

$$c(x) = \begin{cases} 0, & \text{if } \sum x_i \le 1 \\ +\infty, & \text{o.w.} \end{cases}, \quad x \in \{0,1\}^n$$

Eg 2: single-minded combinatorial auction, known bundles

- let $S_i$ be the bundle desired by bidder $i$
- $$c(x) = \begin{cases} 0, & \text{if } (\forall i,j \;\; S_i \cap S_j \ne \emptyset \Rightarrow x_i x_j = 0) \\ +\infty, & \text{o.w.} \end{cases}$$

- **Bayesian Incentive Compatibility (BIC)**
  - Bayesian assumption: $v_i \sim F_i(\mathbb{R}_+)$, $F_i$ is known and $v_1, \dots, v_n$ independent
    (support of $F_i$ is non-negative reals)
  - a mechanism $X(b)$, $p(b)$ is BIC if

    for all $i$, $v_i$; $v_i'$:

expectation over the values of the other bidders & the randomness of the mechanism

$$\underbrace{\mathbb{E}_{v_{-i}}\left[v_i \cdot X(v_i; v_{-i}) - p(v_i, v_{-i})\right]}_{\text{expected utility if your true type is } v_i \text{ \& you bid } v_i \text{ assuming others report truthfully}} \ge \underbrace{\mathbb{E}_{v_{-i}}\left[v_i \cdot X(v_i', v_{-i}) - p(v_i'; v_{-i})\right]}_{\text{expected utility if you lie}}$$

- **Individual Rationality (ex-interim):**

$$(\text{IR}) \quad \text{for all } i, v_i: \quad \mathbb{E}_{v_{-i}}\left[v_i \cdot \chi(v_i; v_{-i}) - p(v_i; v_{-i})\right] \geq 0$$

(i.e. in expectation go home w/ non-negative utility)

- **No positive transfers:** $P_i \geq 0, \forall i$ (NPT)

- **Characterization of BIC Mechanisms**

Notation:

$$x_i(b_i) = \mathbb{E}_{v_{-i}}\left[\chi(b_i; v_{-i})\right]$$ : probability $i$ receives service if he bids $b_i$, assuming others report truthfully

$$P_i(b_i) = \mathbb{E}_{v_{-i}}\left[p_i(b_i; v_{-i})\right]$$

**Theorem:** A mechanism is BIC, IR, NPT, iff for all $i$:

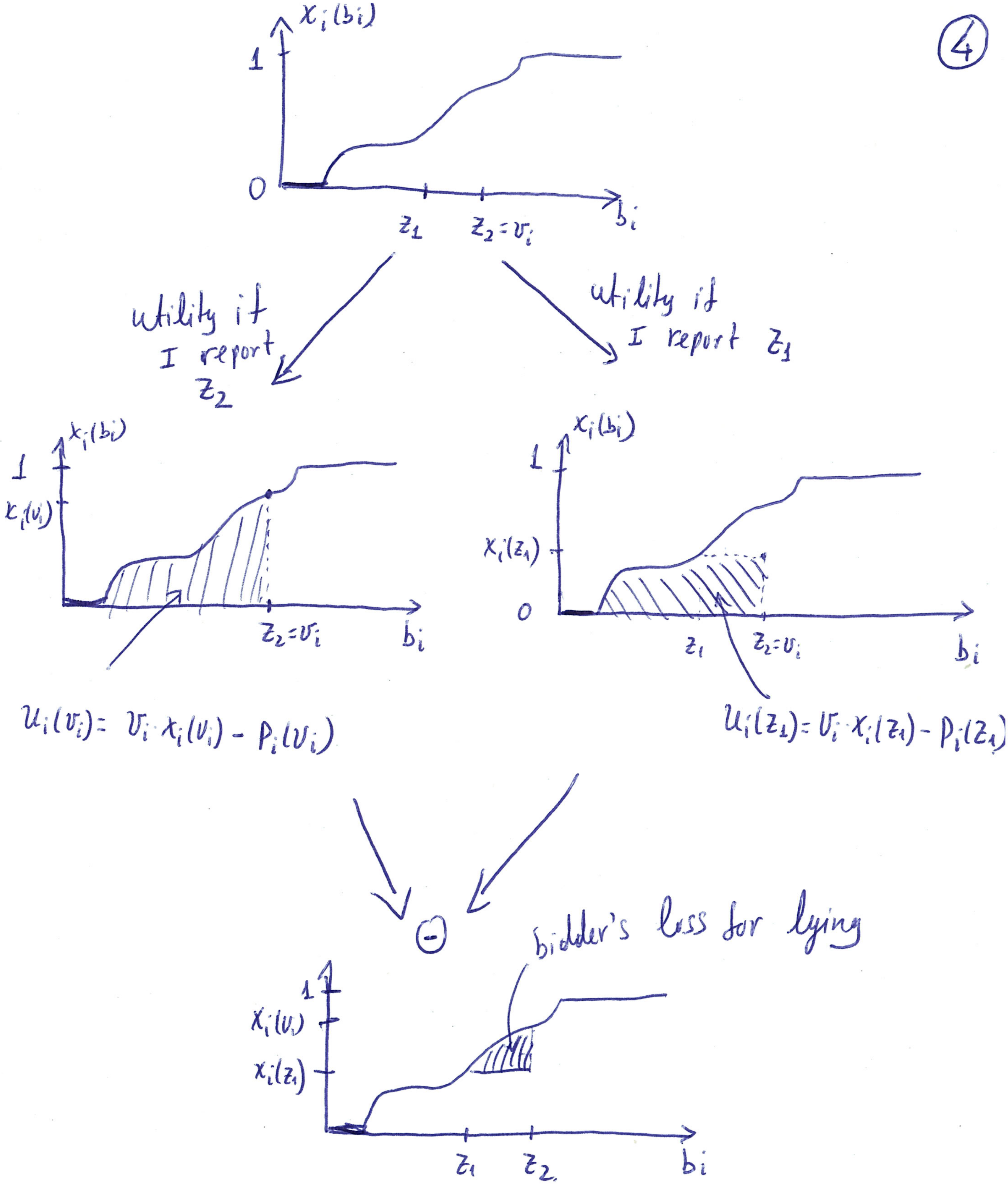
1> $x_i(b_i)$ is monotone non-decreasing

2> $P_i(b_i) = b_i \cdot x_i(b_i) - \int_0^{b_i} x_i(z)\,dz.$

**Proof:**

($\Leftarrow$) easy by picture

Suppose my true value is $z_2$ but I'm pondering lying $z_1$ where $z_1 < z_2$




- An analogous picture shows that for all $z_1, z_2$, $z_1 < z_2$ if a bidder's true value is $z_1$ she won't lie reporting $z_2$.

monotonicity

(⇒) • for all $i$, $v_i, b_i$:

(*)

$$v_i \cdot x_i(v_i) - p_i(v_i) \geq v_i \cdot x_i(b_i) - p_i(b_i), \text{ by BIC}$$

• For all $z_1, z_2$:

- Apply (*) for $v_i = z_1$, $b_i = z_2$

$$z_1 \cdot x_i(z_1) - p_i(z_1) \geq z_1 \cdot x_i(z_2) - p_i(z_2) \quad (1)$$

- Apply (*) for $v_i = z_2$, $b_i = z_1$

$$z_2 \cdot x_i(z_2) - p_i(z_2) \geq z_2 \, x(z_1) - p_i(z_1) \quad (2)$$

$\overset{+}{\Rightarrow}$

$$(z_1 - z_2)\big(x_i(z_1) - x_i(z_2)\big) \geq 0 \quad (**)$$

• Since (**) holds for all $z_1, z_2$ : $x_i(b_i)$ is non-decreasing in $b_i$

• (1), (2) ⇒

$$z_1\big(x_i(z_2) - x_i(z_1)\big) \leq p_i(z_2) - p_i(z_1) \leq z_2 \cdot \big(x_i(z_2) - x_i(z_1)\big)$$

take $z_2 = z_1 - \varepsilon$, divide by $\varepsilon$:

$$z_1 \cdot \frac{x_i(z_1+\varepsilon) - x_i(z_1)}{\varepsilon} \leq \frac{p_i(z_1+\varepsilon) - p_i(z_1)}{\varepsilon} \leq (z_1+\varepsilon) \cdot \frac{x_i(z_1+\varepsilon) - x_i(z_1)}{\varepsilon}$$

because $x_i(\cdot)$ is monotone its derivative exists almost everywhere hence

$$\frac{d\,p_i(z)}{dz} = z \cdot \frac{dx_i(z)}{dz} \quad \text{, almost everywhere.}$$

$$\int_{0}^{b_i} \frac{d\,p_i(z)}{dz}\,dz = \int_{0}^{b_i} z\,\frac{dx_i(z)}{dz}\,dz$$

$$p_i(b_i) - p_i(0) = \Big[z \cdot x_i(z)\Big]_{0}^{b_i} - \int_{0}^{b_i} x_i(z)\,dz$$

$$p_i(b_i) = \underbrace{p_i(0)}_{\substack{\parallel \\ 0 \text{ (using NPT, IR)}}} + b_i \cdot x_i(b_i) - \int_{0}^{b_i} x_i(z)\,dz$$

$\boxtimes$

- **Revenue of a BIC, IR, NPT mechanism**

Define $\varphi_i(x) = x - \dfrac{1-F_i(x)}{f_i(x)}$ : virtual value function

Theorem [Myerson]: The expected revenue of a BIC, NPT, IR mechanism is equal to its expected virtual welfare

$$\mathbb{E}_v\Big[\sum_i \varphi_i(v_i)\,x_i(v_i) - c(x(v))\Big]$$

**Lemma:** The expected payment of a bidder satisfies:

$$\mathbb{E}_{v_i}\left[p_i(v_i)\right] = \mathbb{E}_{v_i}\left[\varphi_i(v_i)\cdot x_i(v_i)\right]$$

**Proof:**

$$\mathbb{E}_{v_i}\left[p_i(v_i)\right] = \int_0^{+\infty} p_i(v_i)\, f_i(v_i)\, dv_i$$

$$= \int_0^{+\infty} v_i \cdot x_i(v_i)\, f_i(v_i)\, dv_i - \underbrace{\int_{v_i=0}^{+\infty}\int_{z=0}^{v_i} x_i(z)\, f_i(v_i)\, dv_i\, dz}_{\int_{z=0}^{+\infty} x_i(z)\cdot \underbrace{\int_{v_i=z}^{+\infty} f_i(v_i)\, dv_i}_{=\,1-F_i(z)}\, dz}$$

$$= \int_{z=0}^{+\infty} z\cdot x_i(z)\cdot f_i(z)\, dz - \int_{z=0}^{+\infty} x_i(z)\cdot\left(1-F_i(z)\right) dz$$

$$= \int_{z=0}^{+\infty}\left(z - \frac{1-F_i(z)}{f_i(z)}\right) x_i(z)\cdot f_i(z)\, dz \equiv \mathbb{E}_{v_i}\left[\varphi_i(v_i)\cdot x_i(v_i)\right] \quad \boxtimes$$

- Recall Single-Item Setting from last time:

  - 2 bidders, uniform values in $[0,1]$, independent
  - single item

<u>Claim:</u> VCG w/ reserve $1/2$ is revenue-optimal BIC mechanism.

<u>Proof:</u>
- if $F_i = U[0,1]$ then $\varphi_i(x) = 2x-1 \uparrow$ in $x$
- VCG w/ reserve $1/2$ is dominant strategy truthful
  ↳ in particular satisfies, BIC, NPT, IR
- Hence people w/ values $v_1, v_2$ submit bids

  $b_1 = v_1$

  $b_2 = v_2$
- virtual bids $2v_1 - 1$

  $2v_2 - 1$
- optimal virtual welfare is achieved by the allocation rule:
  - if $v_1, v_2 \geq \frac{1}{2}$ give item to highest $\arg\max_i \{v_i\}$
  - if $v_1, v_2 < 1/2$ don't give item to anyone
  - if exactly one is $\geq 1/2$, give the item to him

- notice that VCG w/ reserve 1/2 implements the above allocation rule
- so it optimizes virtual welfare
- so it's payment is the best among all BIC, IR, NPT mechanisms by Myerson's lemma

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