Homework will be collected every Tuesday in class. Below is a list of homework that has been assigned so far.

Due Tuesday, September 13th.
Assigned Thursday, September 8th:
Slides: `http://people.csail.mit.edu/costis/6853fa2011/Lecture01.pptx`

1. On slide 15, verify that the randomized strategy $(5/6, 1/6)$ is a Nash Equilibrium.
2. On slide 27, verify that the mixed strategies shown are in Nash Equilibrium.
3. On slide 38, verify that the 50-50 split shown is the unique Nash Equilibrium.

Due Tuesday, September 20th.
Assigned Tuesday, September 13th:
None. Do not do the problem assigned in the Notes for lecture 2.
Assigned Thursday, September 15th:
Notes: `http://people.csail.mit.edu/costis/6853fa2011/lec3.pdf`

1. Imagine the following three player game. Three grad students are choosing which free food event to attend. There are four events, catered by one star through four star restaurants. The utility of a grad student for attending an event is the number of stars divided by the number of attending students (IE: if two grad students attend the two star event, they each get utility 1). If each student is playing a mixed strategy where x_{ij} denotes the probability that grad student i attends the j-star event, write out the expected payoff for the first grad student.
   Tip: This will be shorter if you make use of the fact that the payoff only depends on how many other grad students attend your event, and not which other event they attend if they are somewhere else.
2. In class, we proved that the following linear program outputs a Nash Equilibrium for pairwise-separable zero-sum games (definition included in Notes):
   Variables:
   - w_p, for all players p.
   - x_{pi}, for all players p and strategies i, denoting the probability that player p plays strategy i.

   Constraints:
   - $w_p \geq \mathbb{E}[\text{Payoff}_p(i, \vec{x}_{-p})]$ (w_p is larger than the expected payoff of playing strategy i against $\vec{x}_{-p}$), for all p, i.
   - $\sum_i x_{pi} = 1$, for all p.
   - $x_{pi} \geq 0$, for all p, i.

   Minimizing:
   - $\sum_p w_p$

   The proof required using Nash's theorem to first show that the value of the LP was 0 before showing that any solution of the LP was a Nash Equilibrium. Prove that the value of the LP is 0 using only the strong LP duality theorem.
   Tip: Take the dual of the LP. It should look almost identical to the original but with a few small differences. Show that the primal LP has value ≥ 0, and the dual LP has value ≤ 0.

Due Tuesday, September 27th.
Assigned Tuesday, September 20th:
Notes: `http://people.csail.mit.edu/costis/6853fa2011/lec4.pdf`

1. Prove that when playing the "follow-the-leader" strategy when learning from expert advice yields loss L_t satisfying:

$$L_t \leq n(\min_i L_t(i) + 1)$$

   (In other words, prove Theorem 3 in the Notes).

2. Briefly describe how one could view fictitious play as two opposing players using "follow-the-leader" to choose their strategy (this should be at most a couple sentences, don't write out the math).

Assigned Thursday, September 22nd:
Notes: `http://people.csail.mit.edu/costis/6853fa2011/lec5.pdf`

1. Show that if both players of a zero-sum game use multiplicative weights to update their strategy profile that the average payoff to each player converges to the value of the game. The average payoff to a player is the average of the actual payoffs they receive each round, **not** the expected payoff of their choices against the average strategy of their opponent.

   This was shown in class as a consequence of Theorems 2 and 3 in the Notes. As Theorem 3 was also proved, you just need to prove Theorem 2 in the Notes.

Due Tuesday, October 4th.
Assigned Tuesday, September 27th:
Notes: `http://people.csail.mit.edu/costis/6853fa2011/Lecture06.pptx`
None

Assigned Thursday, September 29th:
Notes: `http://people.csail.mit.edu/costis/6853fa2011/Lecture07.pptx`

1. On slide 25 of the Notes, we claim that adding the envelope does not create any new pan-chromatic triangles. Prove this claim.

2. Prove Claim 2 on slide 29.

3. In class, we proved Nash's theorem by showing that a certain function had a fixed point. In this problem you will show that this function can be simple if the game has the right structure.

   Consider a separable game (**not necessarily zero-sum**) where each player has only two strategies (A separable game consists of a two-player game for every pair of players. Each player picks a strategy and uses that same strategy in every game. This is a special case where every player only has two strategies). Let $\vec{x}_p$ denote the strategy profile for player p. Then we saw in previous lectures that $U_p(i, \vec{x}_{-p})$ is a linear function. Show that there exists a *piece-wise linear* function f, such that:

   - f is a continuous mapping from a compact, convex space to itself.
   - Any fixed point of f represents a Nash Equilibrium.

   A n-valued function f is piece-wise linear if the single-valued function giving the i^{th} component is piece-wise linear for all i. A single-valued function is piece-wise linear if the domain can be broken up into finitely many compact subsets, and the restriction of f to those subsets is linear. Consult `http://en.wikipedia.org/wiki/Piecewise_linear_function` for some examples if this definition is giving you a hard time.

Due Tuesday, October 25th

1. Show that PPAD $\subseteq$ PPP. In other words, given the circuits P and N for an instance of a PPAD problem, construct a new circuit C, such that:

   (a) If $\exists x$ with $C(x) = 0^n$, then 0^n was not unbalanced in the original PPAD instance.

(b) Given $x \neq y$ with $C(x) = C(y)$, we can find in polynomial time a $z \neq 0^n$ (*hint:* possibly equal to x or y) that is unbalanced in the original PPAD instance.

Relevant lecture Notes: `http://people.csail.mit.edu/costis/6853fa2011/Lecture09.pptx`

2. In lecture, we saw how to construct a brittle comparison gadget. If the inequality was strict, the comparator was correct, but had undefined behavior when the two values were equal. Show that there does not exist a comparison gadget that is not brittle. In other words, there is no game such that:

   (a) There are three players, a, b, c each with two strategies, 0 and 1.

   (b) In any Nash Equilibrium, if $Pr[a \text{ plays } 1] \geq Pr[b \text{ plays } 1]$, then $Pr[c \text{ plays } 1] = 1$.

   (c) In any nash Equilibrium, if $Pr[a \text{ plays } 1] < Pr[b \text{ plays } 1]$, then $Pr[c \text{ plays } 1] = 0$.

   Relevant lecture Notes: `http://people.csail.mit.edu/costis/6853fa2011/Lecture10.pptx`
   Hint: Assume that such a game exists. Use this comparator as a gadget to construct a game with no Nash equilbrium, yielding a contradiction

3. Show that there exists a polynomial q, such that for any polymatrix game $\mathcal{GG}$ with payoffs that can be represented exactly using c bits, we can turn a $2^{-q(|\mathcal{GG}|c)}$-approximate Nash equilibrium into an exact Nash equilibrium. We'll break down the proof into a few steps.

   (a) Consider only a two player game between A and B. If an oracle gave you two subsets of strategies, S_A and S_B, and promised you that there was an exact Nash equilibrium where every strategy in S_A was a best response for A, every strategy in S_B was a best response for B, and neither player played any strategy outside of S_A or S_B, could you find it? *hint: Write a linear program*

   (b) Extend this result to polymatrix games. IE: If an oracle gave you a subset of strategies for every player, S_p, and promised you that there was an exact Nash where every strategy in S_p was a best response for player p, and no player played any strategy outside of S_p, could you find it?

   (c) Modify your result to solve the following problem instead: Given a subset of strategies, S_p, for every player in $\mathcal{GG}$, find the smallest ϵ such that there exists an ϵ-approximate Nash where every strategy in S_p is a best response for player p, and no player uses any strategy outside S_p.

   (d) The bit complexity of a LP is the largest number of bits needed for computation to find the minimizing feasible point (the bit complexity of a LP is polynomial in the number of constraints and number of bits per constant in the LP). Observe that the bit complexity of the LP you wrote is polynomial in $|\mathcal{GG}|$ and c, **regardless of the subsets S_p given as input**.

   (e) Denote by x an upper bound on the bit complexity of the LP you wrote, for any subsets S_p. Say that you have a 2^{-y}-approximate Nash equilibrium, with $y > x$. What is an obvious choice of S_p that would give your LP a value of at most 2^{-y}? Observe that this same LP must in fact have value 0, and therefore solving it will yield an exact Nash equilibrium.

Due Tuesday November 15

Notes: `http://people.csail.mit.edu/costis/6853fa2011/Lecture14.pptx`

(1) Show that the CES utility function (slide 30) is concave for all $\rho \leq 1$.

Due Tuesday November 22

Notes: `http://people.csail.mit.edu/costis/6853fa2011/Lecture15.pptx`

(1) For linear utility functions: (a) show that the Eisenberg-Gale program is a convex program; and (b) take the KKT conditions of the program, interpret Langrange multipliers as prices, and argue that the KKT conditions certify that a price equilibrium exists. Relevant slides: 13 and 14 of Lecture 15.

Due Tuesday November 22

Notes: `http://people.csail.mit.edu/costis/6853fa2011/6853-Lecture16.pdf`

(1) Show that any feasible solution of the program of Section 3.2 gives an equilibrium of the linear exchange model.

Due Tuesday December 6th

(1) **Public Project Problem:** A mayor wants to decide whether to build a bridge. The bridge has cost C and (private) value $v_i^* \geq 0$ for each citizen. The mayor wants to build the bridge if and only if $\sum_i v_i^* > C$. We designed an incentive compatible auction implementing this social choice function as follows: first, we introduced a virtual bidder who has value $-C$, if the bridge is built, and 0, if not; then we found the VCG auction with Clarke's pivot rule for this setting. This had the following form: (a) the bridge is built iff $\sum_i v_i^* > C$; (b) we only charge *pivotal bidders* i an amount equal to $p_i := C - \sum_{j \neq i} v_j^*$. (A bidder is called pivotal if $\sum_j v_j^* > C \geq \sum_{j \neq i} v_j^*$.)

Show that $\sum_i p_i < C$.

(2) **Reverse Auction:** The auctioneer wants to procure service from one of n agents. Every agent has a (private) cost c_i to provide service to the auctioneer. The auctioneer wants to be served by the bidder with the smallest cost. In class we provided the following mechanism: ask bidders to report their costs; choose the cheapest bidder, and pay him the cost of the second cheapest bidder; finally pay/charge 0 every other bidder. Is this auction a VCG auction? Is it using Clarke's payment rule? If not, what would be VCG with Clarke's payment rule?

Due Tuesday December 13th

Notes: `http://people.csail.mit.edu/costis/6853fa2011/Lecture22.pptx`

(1) Consider selling a single item to n bidders whose value is drawn independently and uniformly at random from $[0, 1]$ using the first price auction. What is the Bayes-Nash Equilibrium bidding strategies that the players will adopt? What is

the expected revenue of the auction? Is it the same as the expected revenue of the second price auction? why?

(2) Suppose we sell a single item to 2 bidders whose value is drawn independently and uniformly at random from $[0, 1]$ using the second price auction with reservation price $1/2$ (see Slide 26 for a description). Show that truth-telling is a dominant strategy for both bidders and that the expected revenue of the auction is $5/12$.