6.896: Topics in Algorithmic Game Theory

vol. 1: Spring 2010

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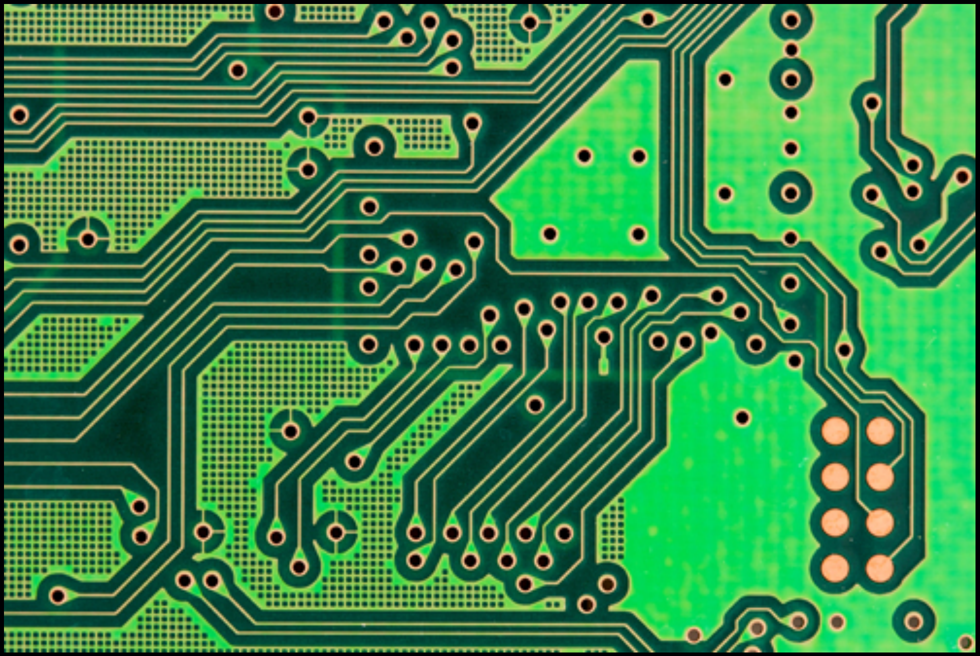
what we won't study in this class…



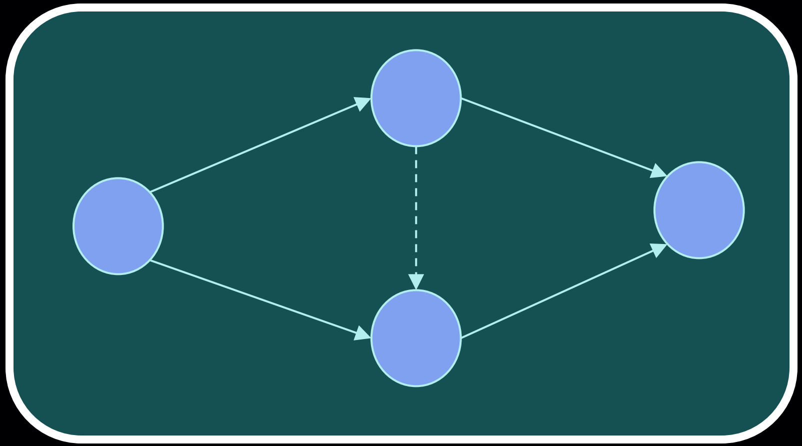
I only mean this as a metaphor of what we usually study in Eng.:

- *central design*
- *cooperative components*
- *rich theory*

what we will study in this class…

Markets

Routing in Networks



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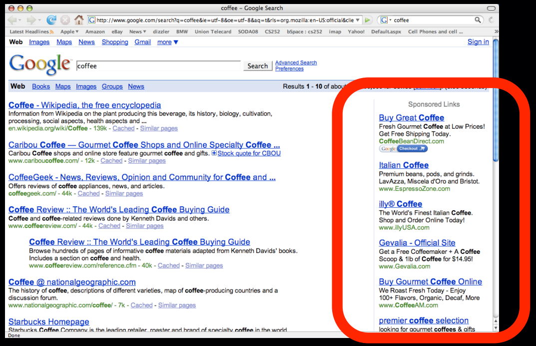
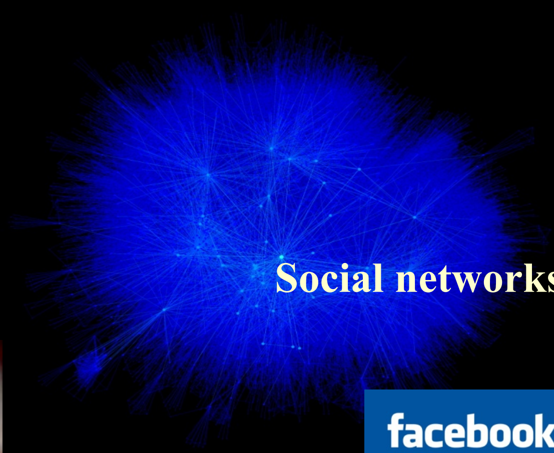


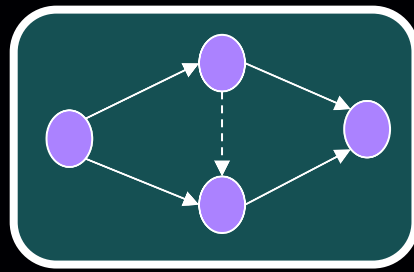
Evolution

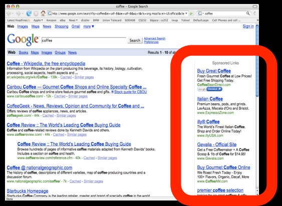
Elections

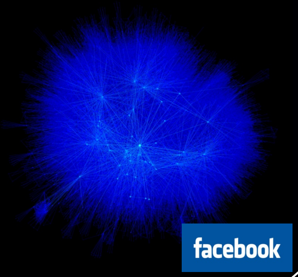
Social networks



facebook







Game Theory

- *central design ?*
- *cooperative components ?*
- *rich theory ?*

we will study (and sometimes question) the algorithmic foundations of this theory the column player

Game Theory

			
	0,0	-1,1	1,-1
	1,-1	0,0	-1 , 1
	-1,1	1 , -1	0,0

the row player

Game Theory

	 1/3	 1/3	 1/3
 1/3	0,0	-1,1	1,-1
 1/3	1,-1	0,0	-1 , 1
 1/3	-1,1	1 , -1	0,0

von Neumann '28:

exists in every 2-player zero-sum every game!

Equilibrium : *a pair of randomized strategies such that given what the column player is doing, the row player has no incentive to change his randomized strategy, and vice versa*

In this case also easy to find because of symmetry (and other reasons)

Algorithmic Game Theory

	 1/3	 1/3	 1/3
 1/3	0,0	-1,1	1,-1
 1/3	1,-1	0,0	-1 , 1
 1/3	-1,1	1 , -1	0,0

How can we design a system that will be launched and used by competitive users to optimize our objectives ?

?

Can we predict what will happen in a large system?

game theory says yes!

Can we efficiently predict what will happen in a large system?

Are the predictions of Game Theory likely to arise?

An overview of the class

- *Administration*
- *Solution Concepts*
- *Equilibrium Computation*
- *Price of Anarchy*
- *Mechanism Design*

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Administrativia

Everybody is welcome

If registered for credit (or pass/fail):

- Scribe two lectures
- Collect 20 points in total from problems given in lecture

  open questions will be 10 points, decreasing # of points for decreasing difficulty

- Project: Survey or Research (write-up + presentation)

If just auditing: - Consider registering in the class as listeners

this will increase the chance we'll get a TA for the class and improve the quality of the class

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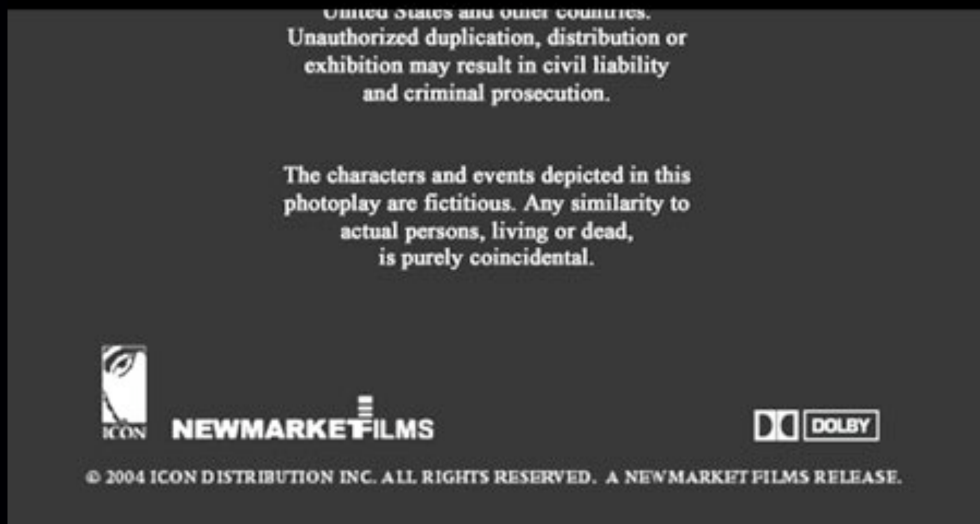
Battle of the Sexes

	Theater!	Football fine
Theater fine	1, 5	0, 0
Football!	0, 0	5, 1

Nash Equilibrium: A pair of strategies (deterministic or randomized) such that the strategy of the row player is a *Best Response* to the strategy of the column player and vice versa.

Disclaimer 1:

The Battle of the Sexes is a classical game in game theory.



That said, take the game as a metaphor of real-life examples.

Battle of the Sexes

	Theater!	Football fine
Theater fine	1, 5	0, 0
Football!	0, 0	5, 1

Nash Equilibria

Nash Equilibrium: A pair of strategies (deterministic or randomized) such that the strategy of the row player is a *Best Response* to the strategy of the column player and vice versa.

(Theater fine, Theater!)

(Football!, Football fine)

Disclaimer 2:

One-shot games intend to model repeated interactions provided that there are no strategic correlations between different occurrences of the game. If such correlations exist, we exit the realm of one-shot games, entering the realm of repeated games. Unless o.w. specified the games we consider in this class are one-shot.

How can repeated occurrences occur without inter-occurrence correlations?

Imagine a population of blue players (these are the ones preferring football) and orange players (these are those preferring theater). Members of the blue population meet randomly with members of the orange population and need to decide whether to watch football or theater.

What do the Nash equilibria represent?

The Nash equilibria predict what types of behaviors and (in the case of randomized strategies) at what proportions will arise in the two populations at the steady state of the game.

Battle of the Sexes

Suppose now that the blue player removes a strategy from his set of strategies and introduces another one:

	Theater!	Football fine
~~Theater fine~~	~~1, 5~~	~~0, 0~~
Football!	0, 0	5, 1
Theater great, I'll invite my mom	2, -1	0, 0

unique Equilibrium

(Football!, Football fine)

Moral of the story: The player who knows game theory managed to eliminate the unwanted Nash equilibrium from the game.

Rock-Paper-Scissors

	 1/3	 1/3	 1/3
 1/3	0,0	-1,1	1,-1
 1/3	1,-1	0,0	-1 , 1
 1/3	-1,1	1 , -1	0,0

The unique Nash Equilibrium is the pair of uniform strategies.

Contrary to the battle of the sexes, in RPS randomization is necessary to construct a Nash equilibrium.

Rock-Paper-Scissors

Rock-Paper-Scissors Competition:

- *one shot-games* are very different from *repeated games*

- the behavior observed in the RPS competition is very different from the pair of uniform strategies; in fact, the one-shot version of RPS does not intend to capture the repeated interaction between the same pair of players---recall Disclaimer 2 above; rather the intention is to model the behavior of a population of, say, students in a courtyard participating in random occurrences of RPS games

Two-Thirds of the Average game

- k teams of players $t_1, t_2, t_3, \ldots, t_k$

- each team submits a number in [0,100]

$$x_1, x_2, \ldots, x_k$$

- compute

$$\bar{x} := \frac{1}{k} \sum_{i=1}^{k} x_i$$

- find j, closest to $\frac{2}{3}\bar{x}$

- *j* wins \$100, -*j* lose

Let's Play!

Two-Thirds of the Average game

Is it rational to play above $\frac{2}{3} \cdot 100$?

A: no (why?)

Given that no rational player will play above $\frac{2}{3} \cdot 100$ is it rational to play above $(2/3)^2 \cdot 100$?

A: no (same reasons)

$\vdots$

All rational players should play 0.

The all-zero strategy is the only Nash equilibrium of this game.

Rationality versus common knowledge of rationality

historical facts: 21.6 was the winning value in a large internet-based competition organized by the Danish newspaper Politiken. This included 19,196 people and with a prize of 5000 Danish kroner.

Bimatrix Games

2 players: the row player & the column player

n strategies available to each player

game described by two payoff matrices

payoff to the column player for playing j when row player plays i

$$G = (R_{n \times n} \;,\; C_{n \times n})$$

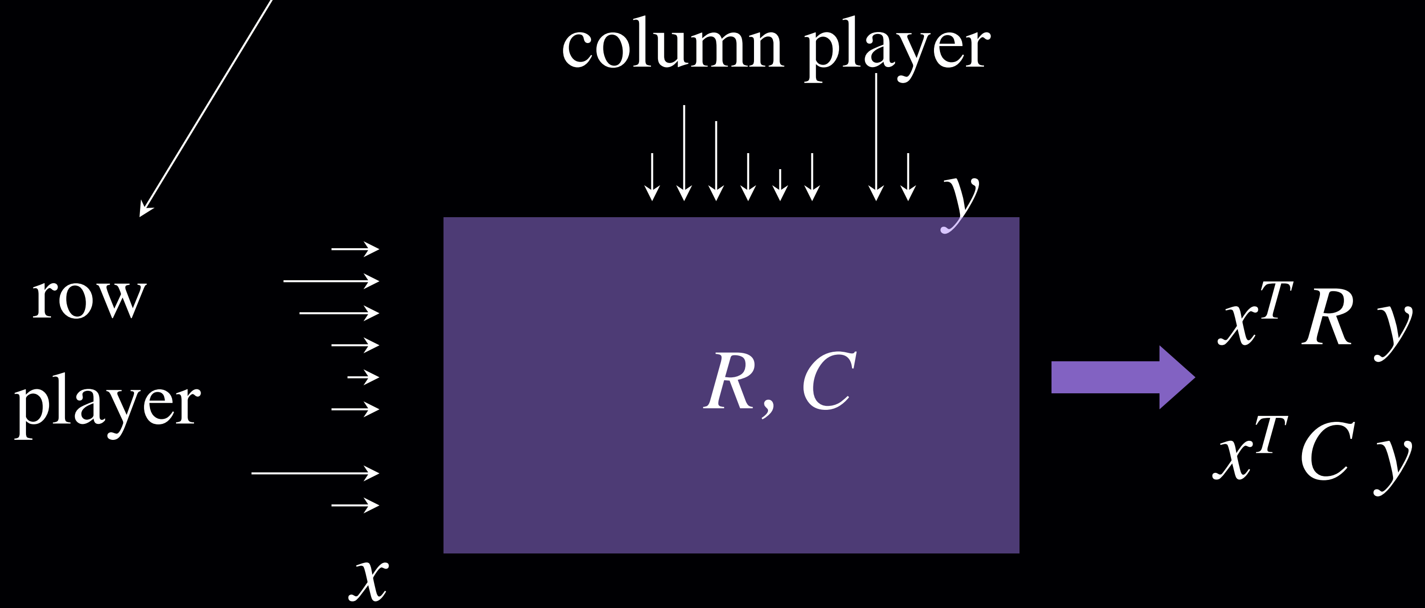
$$R_{ij}, C_{ij}$$

description size $O(n^2)$

payoff to the row player for playing i when column player plays j

Bimatrix Games

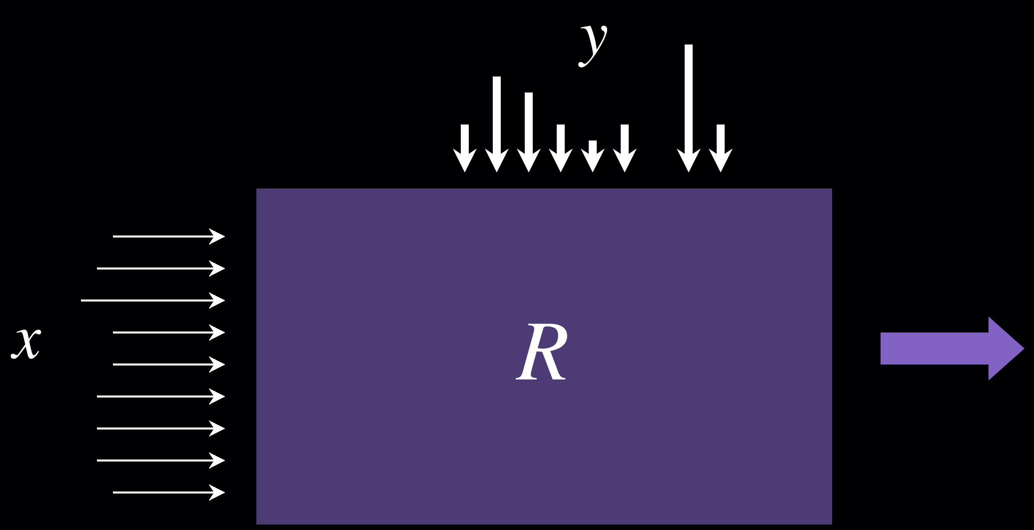
game $\quad G = (R_{n \times n} \ , \ C_{n \times n})$



Nash Equilibrium

(x, y) is a Nash Equilibrium iff

row player: $\forall\, x' \,.\; x^T R\, y \geq x'^T R\, y$
and same for column player

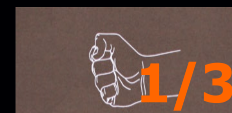


x maximizes utility of row player

OK, Nash equilibrium is stable, but does it always exist?

2-player Zero-Sum Games

$R + C = 0$

	 1/3	 1/3	 1/3
 1/3	0,0	-1,1	1,-1
 1/3	1,-1	0,0	-1 , 1
 1/3	-1,1	1 , -1	0,0

von Neumann '28:
For two-player zero-sum games, it always exists.

[original proof uses analysis]

Danzig '47

$\Longleftrightarrow$

LP duality

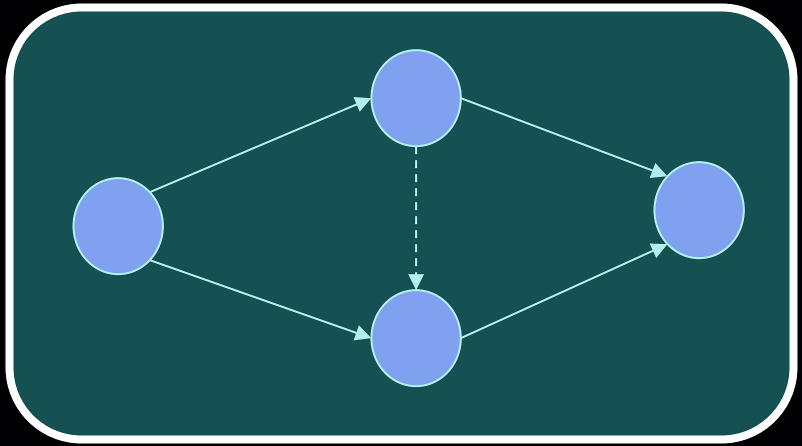
Poker

von Neuman's predictions are in fact accurate in predicting players' strategies in two-player poker.

But what about larger systems (more than 2 players) or systems where players do not have directly opposite interests?

Markets

Routing in Networks



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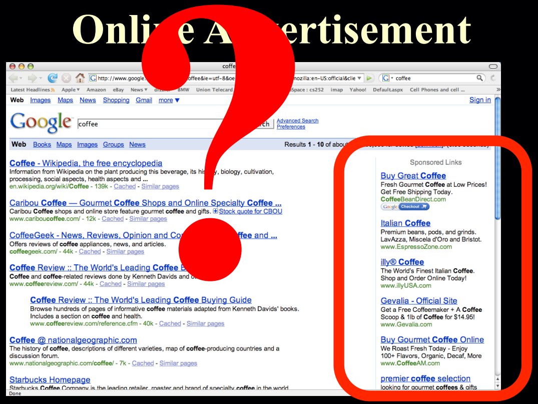
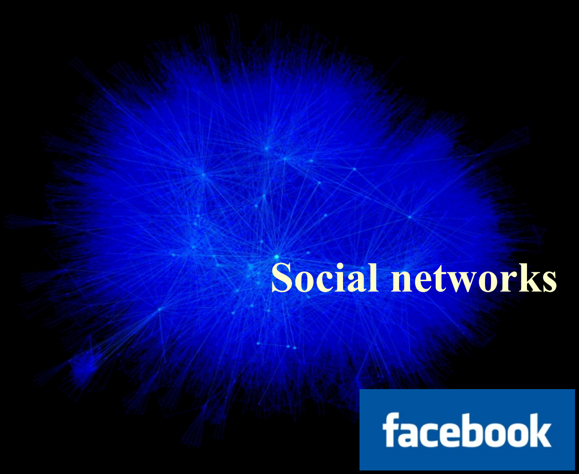


Evolution

Elections

Social networks



Modified Rock Paper Scissors

Not zero-sum any more

	25%	50%	25%
33%	0,0	-1, 1	2,-1
33%	1,-1	0,0	- 1 , 1
33%	- 2, 1	1 , -1	0,0

Is there an equilibrium now?

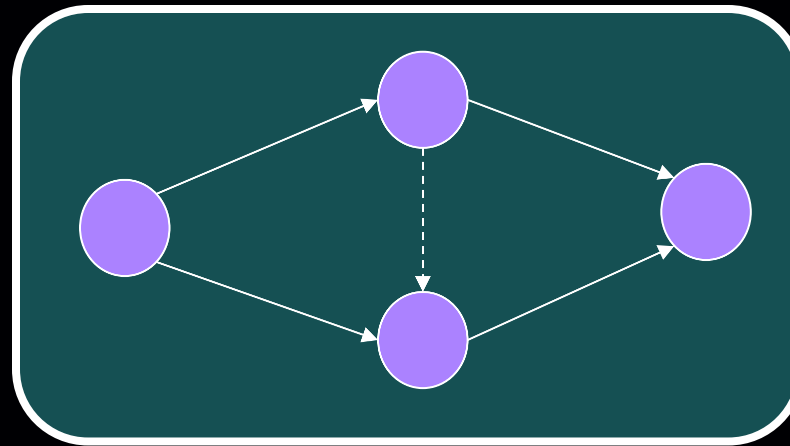
[that is a pair of randomized strategies so that no player has incentive to deviate given the other player's strategy ?]

John Nash '51:
There always exists a Nash equilibrium, regardless of the game's properties.

Nobel 1994, due to its large influence in understanding systems of competitors…

Markets

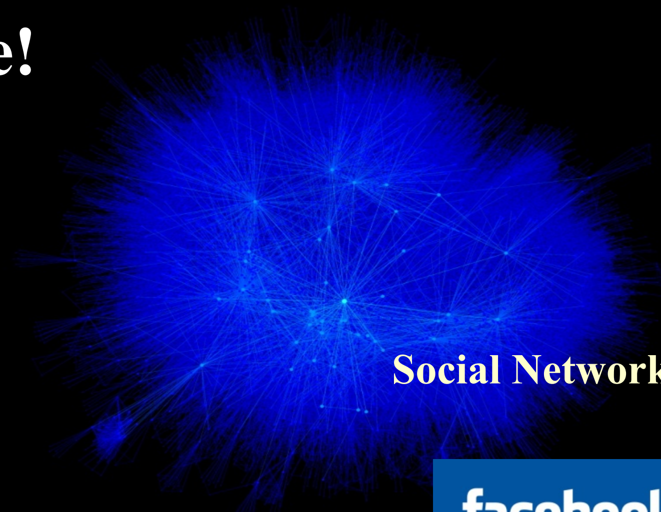
Routing in Networks



and every other game!

Evolutionary Biology

Elections



Social Networks

facebook

Applications…

game =

market → price equilibrium

Internet → packet routing

roads → traffic pattern

facebook,
hi5, myspace, … → structure of the social network

Modified Rock Paper Scissors

Not zero-sum any more

	25%	50%	25%
33%	0,0	-1, 1	2,-1
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33%	- 2, 1	1 , -1	0,0

Highly Non-Constructive

Is there an equilibrium now?

[that is a pair of randomized strategies so that no player has incentive to deviate given the other player's strategy ?]

John Nash '51:
There always exists a Nash equilibrium, regardless of the game's properties.

Brouwer's Fixed Point Theorem

Nobel 1994

How can we compute a Nash equilibrium?

- if we had an algorithm for equilibria we could predict what behavior will arise in a system, before the systems is launched

- if a system is at equilibrium we can verify this efficiently

- in this case, we can easily compute the equilibrium, thanks to gravity!

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- *Mechanism Design*

2-player zero-sum vs General Games

1928 Neumann:

- existence of min-max equilibrium in *2-player*, *zero-sum* games;

- proof uses analysis;

+ Danzig '47: equivalent to LP duality;

+ Khachiyan'79: poly-time solvable;

+ a multitude of distributed algorithms converge to equilibria.

1950 Nash:

- existence of an equilibrium in *multiplayer*, *general-sum* games;

- Proof uses Brouwer's fixed point theorem;

- intense effort for equilibrium computation algorithms:

  Kuhn '61, Mangasarian '64, Lemke-Howson '64, Rosenmüller '71, Wilson '71, Scarf '67, Eaves '72, Laan-Talman '79, etc.

- Lemke-Howson: simplex-like, works with LCP formulation;

no efficient algorithm is known after 50+ years of research.

hence, also no efficient dynamics …

Robert Aumann, 1987:

> *"Two-player zero-sum games are one of the few areas in game theory, and indeed in the social sciences, where a fairly sharp, unique prediction is made."*

the Pavlovian reaction

"Is it NP-complete to find a Nash equilibrium?"

Why should we care about the complexity of equilibria?

• First, if we believe our equilibrium theory, efficient algorithms would enable us to make predictions:

Herbert Scarf writes…

> *"[Due to the non-existence of efficient algorithms for computing equilibria], general equilibrium analysis has remained at a level of abstraction and mathematical theoretizing far removed from its ultimate purpose as a method for the evaluation of economic policy."*

The Computation of Economic Equilibria, 1973

• More importantly: If equilibria are supposed to model behavior, computational tractability is an important modeling *prerequisite*.

> *"If your laptop can't find the equilibrium, then how can the market?"*

Kamal Jain, Microsoft Research

N.B. computational intractability implies the non-existence of efficient dynamics converging to equilibria; how can equilibria be universal, if such dynamics don't exist?

the Pavlovian reaction

"Is it NP-complete to find a Nash equilibrium?"

two answers

1. probably not, since the problem is very different than the typical NP-complete problem (here the solution is guaranteed to exist by Nash's theorem)

2. moreover, it is NP-complete to solve harder problems than finding a Nash equilibrium; e.g., the following problems are NP-complete:

- find two Nash equilibria, if more than one exist
- find a Nash equilibrium with a certain property, if any

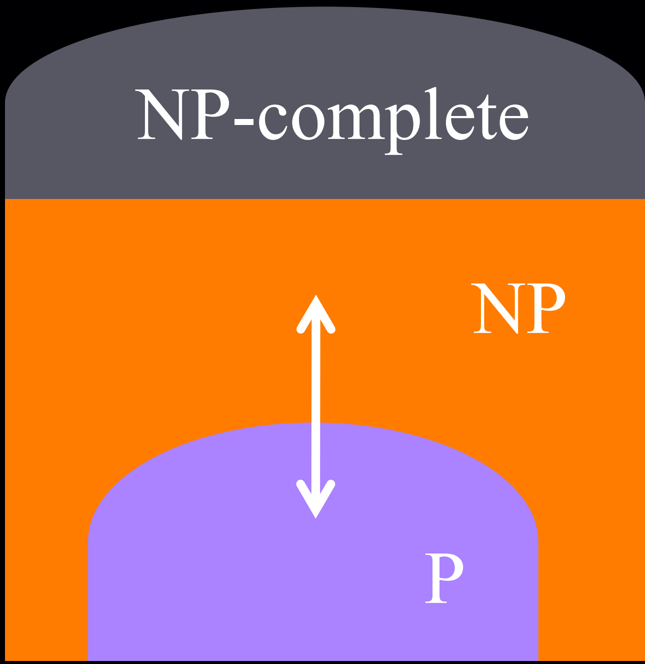
[Gilboa, Zemel '89; Conitzer, Sandholm '03]

so, how hard is it to find a single equilibrium?

- the theory of NP-completeness does not seem appropriate;

- in fact, NASH seems to lie below NP-complete;

- Stay tuned! we are going to answer this question later this semester



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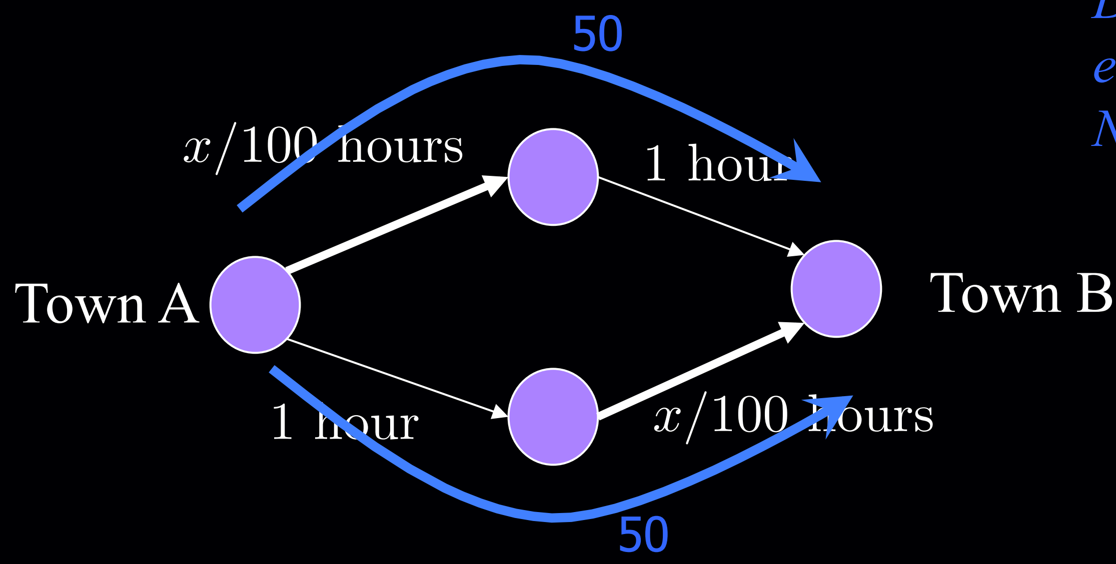
Traffic Routing



Delay is 1.5 hours for everybody at the unique Nash equilibrium

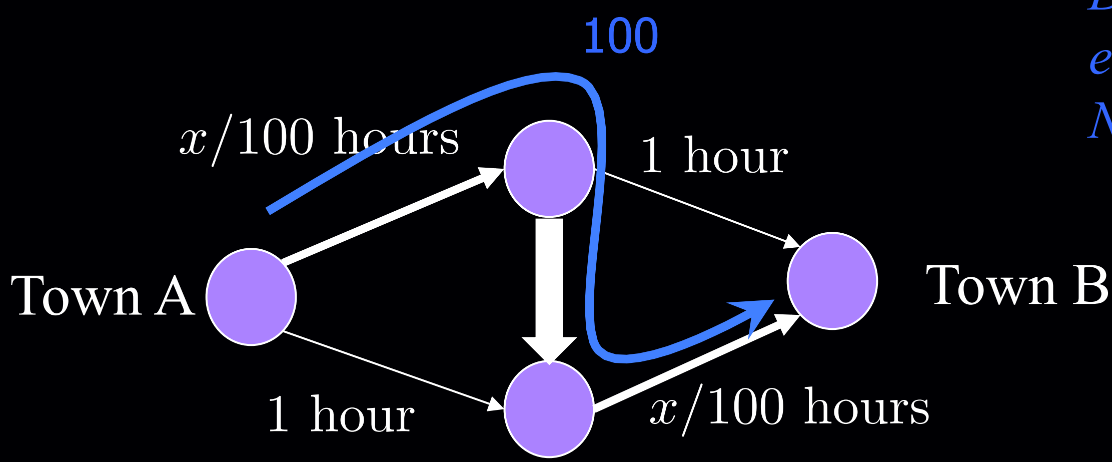
Suppose 100 drivers leave from town A driving towards town B.

Every driver wants to minimize his own travel time.

What is the traffic on the network?

In any unbalanced traffic pattern, all drivers on the most loaded path have incentive to switch their path.

Traffic Routing



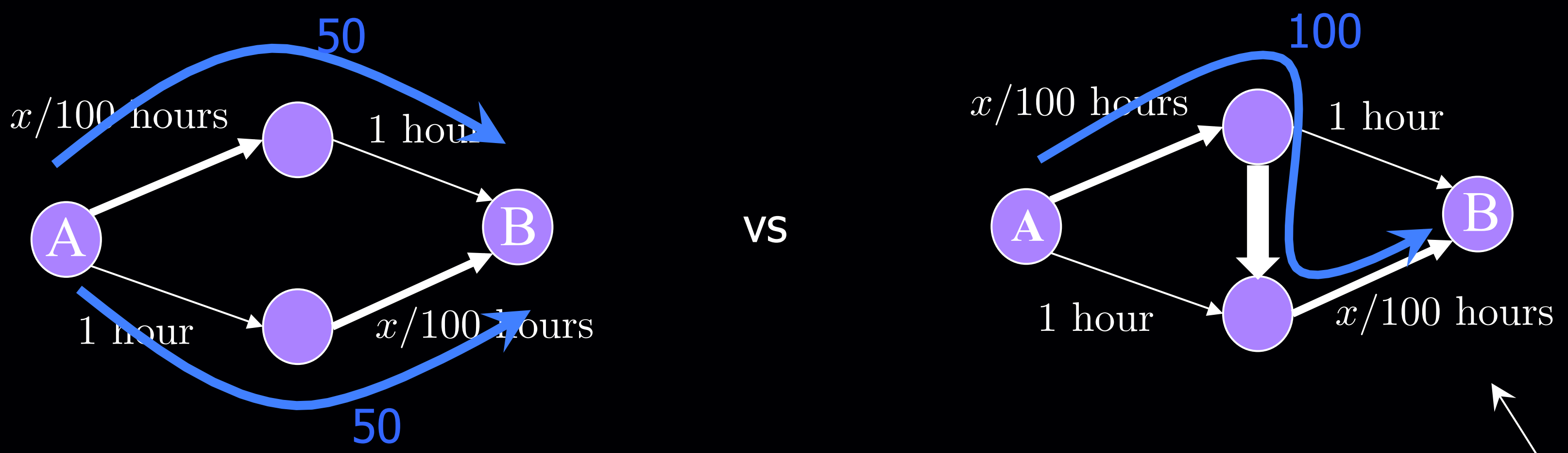
Delay is 2 hours for everybody at the unique Nash equilibrium

A benevolent mayor builds a superhighway connecting the fast highways of the network.

What is now the traffic on the network?

No matter what the other drivers are doing it is always better for me to follow the zig-zag path.

Traffic Routing



Adding a fast road on a road-network is not always a good idea!

Braess's paradox

4/3

In the RHS network there exists a traffic pattern where all players have delay 1.5 hours.

$$\mathrm{PoA} = \frac{\text{performance of system in worst Nash equilibrium}}{\text{optimal performance if drivers did not decide on their own}}$$

Price of Anarchy: measures the lost in system performance due to free-will

Traffic Routing

Obvious Questions:

What is the worst-case PoA in a system?

How do we design a system whose PoA is small?

In other words, what incentives can we provide to induce performance that is close to optimal?

E.g. tolls?

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- *Price of Anarchy*
- *Mechanism Design*

Auctions

- We have one item for sale.

- k parties (or *bidders*) are interested in the item.

- party i has value u_i for the item, which is private, and we won't to give the item to the party with the largest value for the item (alternatively make as much as possible from the sale).

- we ask each party for its value for the item, and based on the *declared values* $b_1, b_2, \ldots, b_k$ we decide who gets the item and how much she pays

-if bidder i gets the item and pays price p, her total payoff is $b_i - p$

Auctions

First Price Auction: Give item to bidder with largest b_i, and charge him b_i

clearly a bad idea to bid above your value (why?)

but you may bid below your value (and you will!)

e.g. two bidders with values $u_1 = \$5$, $u_2 = \$100$

Nash equilibrium = $(b_1, b_2) = (\$5, \$5.01)$

non truthful!

- bidders place different bids, depending on opponents hence cycling etc,
- non-obvious how to play
- auctioneer does not learn people's true values

Auctions

Second Price Auction:

Give item to bidder with highest bid and charge him the second largest bid.

e.g. if the bids are $(b_1, b_2) = (\$5, \$10)$, then second bidder gets the item and pays $5

bidding your value is a dominant strategy, regardless of what others are doing

truthful!

Auctions

Second Price Auction:

Give item to bidder with highest bid and charge him the second largest bid.

e.g. if the bids are $(b_1, b_2) = (\$5, \$10)$, then second bidder gets the item and pays \$5

bidding your value is a dominant strategy, regardless of what others are doing

truthful!

In conclusion

- *We are going to study and question the algorithmic foundations of Game Theory*

- *Complexity of finding equilibria*

  NP-completeness theory not relevant, new theory below NP…

- *Models of strategic behavior*

  dynamics of player interaction:
  e.g. best response, exploration-exploitation,…

- *System Design*

  robustness against strategic entities, e.g., routing

- *Theory of Networks with incentives*

  information, graph-structure, dynamics…