6.896: Topics in Algorithmic Game Theory

Lecture 15

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Recap

Exchange Market Model

n traders

k divisible goods

trader *i* has:

- utility function $u_i \; : \; \mathcal{X}_i \subseteq \mathbb{R}^k_+ \longrightarrow \mathbb{R}_+$

non-negative reals

consumption set for trader *i*

specifies trader *i*'s utility for bundles of goods

- endowment of goods $e_i \in \mathcal{X}_i$

amount of goods trader comes to the marketplace with

Fisher Market Model

n traders with: money m_i, and utility function u_i

k divisible goods owned by seller; seller has q_j units of good j

can be obtained as a special case of an exchange market, when endowment vectors are parallel:

$$e_i = m_i \cdot e, \quad m_i > 0, \quad m_i : \text{ scalar}, e : \text{ vector}$$

in this case, relative incomes of the traders are independent of the prices.

Competitive (or Walrasian) Market Equilibrium

Def: A price vector $p \in \mathbb{R}^k_+$ is called a *competitive market equilibrium* iff there exists a collection of optimal bundles $x_i(p)$ of goods, for all traders $i = 1,\ldots, n$, such that the total supply meets the total demand, i.e.

$$\underbrace{\sum_{i=1}^{n} x_i(p)}_{\text{total demand}} \leq \underbrace{\sum_{i=1}^{n} e_i}_{\text{total supply}}$$

[For Fisher Markets: $\sum_{i=1}^{n} x_i(p) \leq q$]

Arrow-Debreu Theorem 1954

Theorem [Arrow-Debreu 1954]: Suppose

(i) $\mathcal{X}_i$ is closed and convex

(ii) $e_i >> 0$, for all i (all coordinates positive)

(iii a) u_i is continuous

(iii b) u_i is quasi-concave

$$u_i(x) > u_i(y) \implies u_i(\lambda x + (1-\lambda)y) > u_i(y),\ \forall \lambda \in (0,1)$$

(iii c) u_i is nonsatiated

$$\forall y \in \mathcal{X}_i,\quad \exists x \in \mathcal{X}_i \text{ s.t. } u_i(x) > u_i(y)$$

Then a competitive market equilibrium exists.

Utility Functions

CES (Constant Elasticity of Substitution) utility functions:

$$u_i(x) = \left(\sum_j u_{ij} \cdot x_j^\rho \right)^{\frac{1}{\rho}}, \quad -\infty < \rho \leq 1$$

$\rho = 1$ $\longrightarrow$ linear utility form $u_i(x) = \sum_j a_{ij} x_j$

$\rho \to -\infty$ $\longrightarrow$ Leontief utility form $u_i(x) = \min_j \{a_{ij} x_j\}$

$\rho \to 0$ $\longrightarrow$ Cobb-Douglas form $u_i(x) = \prod_j x_j^{a_{ij}}, \quad \text{where } \sum_j a_{ij} = 1$

Eisenberg-Gale's Convex Program for Fisher Model

$$\max \quad u_1^{m_1} \cdot u_2^{m_2} \cdot \ldots \cdot u_n^{m_n}$$

$$\text{s.t} \quad u_i = \left(\sum_j u_{ij} x_{ij}^{\rho} \right)^{\frac{1}{\rho}}$$

$$\sum_i x_{ij} \leq q_j$$

$$x_{ij} \geq 0$$

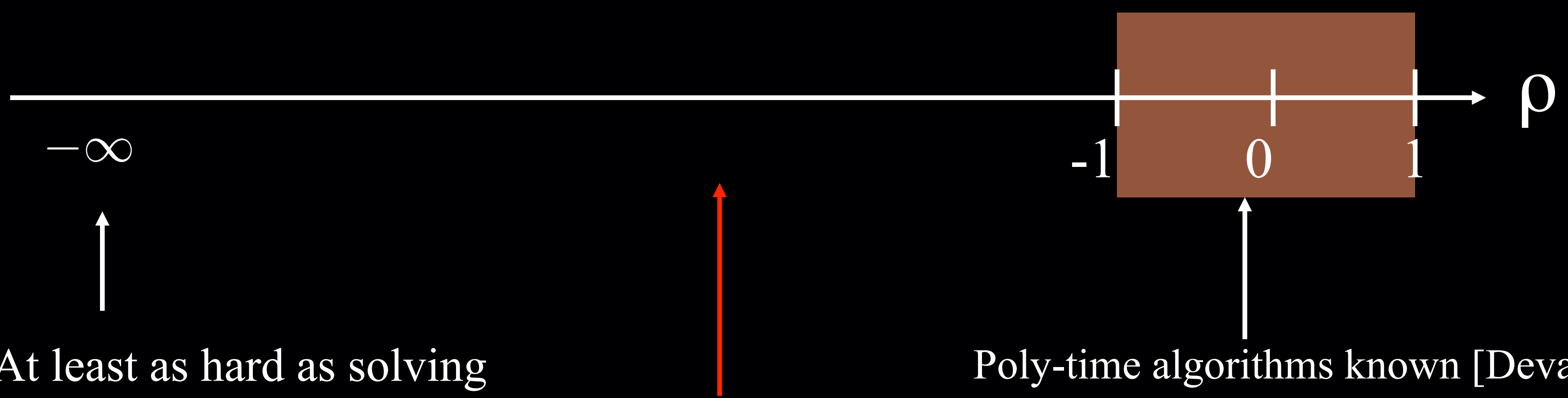
Remarks:

- No budgets constraint!
- Optimal Solution is a market equilibrium (alternative proof of existence)
- It is not necessary that the utility functions are CES; the program also works if they are concave, and homogeneous

Complexity of the Exchange Model

Back to the Exchange Model

Complexity of market equilibria in CES exchange economies.



At least as hard as solving Nash Equilibria [CVSY '05]

OPEN!!

Poly-time algorithms known [Devanur, Papadimitriou, Saberi, Vazirani '02], [Jain '03], [CMK '03], [GKV '04],…

Hardness of Leontief Exchange Markets

Theorem [Codenotti, Saberi, Varadarajan, Ye ’05]:

Finding a market equilibrium in a Leontief exchange economy is at least as hard as finding a Nash equilibrium in a two-player game.

Corollary: Leontief exchange economies are PPAD-hard.

Proof Idea:

Reduce a 2-player game to a Leontief exchange economy, such that given a market equilibrium of the exchange economy one can obtain a Nash equilibrium of the two-player game.

Gross-Substitutability Condition

Excess Demand at prices *p*

suppose there is a unique demand at a given price vector *p* and its is continuous (see last lecture)

$$f(p) := \sum_i x_i(p) - \sum_i e_i$$

$$f_j(p) := \sum_i x_{ij}(p) - \sum_i e_{ij}, \quad \forall j$$

We already argued that under the Arrow-Debreu Thm conditions:

(H) f is homogeneous, i.e. $f(a \cdot p) = f(p), \; \forall a > 0$

(WL) f satisfies Walras's Law, i.e. $p^{\mathrm{T}} \cdot f(p) = 0, \; \forall p$

(we argued that the last property is true using nonsatiation + quasi-concavity, see next slie)

Justification of (WL) under Arrow-Debreu Thm conditions

Nonsatiation + quasi-concavity ➔ local non-satiation

➔ at equilibrium every trader spends all her budget, i.e. if $x_i(p)$ is an optimal solution to Program$_i(p)$ then

$$p \cdot x_i(p) = p \cdot e_i$$

$$\implies p \cdot \left(\sum_i x_i(p) - \sum_i e_i \right) = 0$$

i.e. every good with positive price is fully consumed

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Gross-Substitutability (GS)

Def: The excess demand function satisfies Gross Substitutability iff for all pairs of price vectors *p* and *p'*:

$$\left(\begin{array}{c} p_i \leq p'_i, \ \forall i \\ p_j < p'_j, \ \text{for some } j \end{array}\right) \Longrightarrow f_k(p) < f_k(p'), \ \forall k \text{ s.t. } p_k = p'_k$$

In other words, if the prices of some goods are increased while the prices of some other goods are held fixed, this can only cause an increase in the demand of the goods whose price stayed fixed.

Differential Form of Gross-Substitutability (GS_D)

Def: The excess demand function satisfies the Differential Form of Gross Substitutability iff for all r, s the partial derivatives $\frac{\partial f_r}{\partial p_s}$ exist and are continuous, and for all p :

$$\frac{\partial f_r}{\partial p_s} > 0, \text{ for all } r \neq s.$$

Clearly: (GS) $\Rightarrow$ (GS_D)

Not all goods are free (Pos)

Def: The excess demand function satisfies (Pos) if not all goods are free at equilibrium. I.e. there exists at least one good in which at least one trader is interested.

Properties of Equilibrium

Lemma 1 [Arrow-Block-Hurwicz 1959]:

Suppose that the excess demand function of an exchange economy satisfies (H), (GS_D) and (Pos). Then if $\bar{p}$ is an equilibrium price vector

$$\bar{p}_r > 0, \forall r.$$

Call this property (E_+)

Lemma 2 [Arrow-Block-Hurwicz 1959]:

Suppose that the excess demand function of an exchange economy satisfies (H), (GS) and (E_+). Then if $\bar{p}$ and $\bar{p}'$ are equilibrium price vectors, there exists $\lambda > 0$ such that

$$\bar{p} = \lambda \cdot \bar{p}'.$$

i.e. we have uniqueness of the equilibrium ray

Weak Axiom of Revealed Preferences (WARP)

Theorem [Arrow-Hurwicz 1960's]:

Suppose that the excess demand function of an exchange economy satisfies (H), (WL), and (GS). If $\bar{p}>0$ is any equilibrium price vector and $p>0$ is any non-equilibrium vector we have

$$\bar{p}^{\mathrm{T}} \cdot f(p) > 0.$$

Proof on the board

Computation of Equilibria

Corollary 1 (of WARP): If the excess demand function satisfies (H), (WL), and (GS) and it can be computed efficiently, then a positive equilibrium price vector (if it exists) can be computed efficiently.

proof sketch: W. l. o. g. we can restrict our search space to price vectors in $[0,1]^k$, since any equilibrium can be rescaled to lie in this set (by homogeneity of the excess demand function). We can then run ellipsoid, using the separation oracle provided by the weak axiom of revealed preferences. In particular, for any non-equilibrium price vector *p*, we know that the price equilibrium lies in the half-space

$$S = \{x \mid x^{\mathrm{T}} \cdot f(p) > 0\}.$$

Tatonnement

Corollary 2: If the excess demand function satisfies continuity, (H), (WL), (GS_D), and (Pos), then the tatonnement process (price-adjustment mechanism) described by the following differential equation converges to a market equilibrium

$$\left.\begin{array}{l} \dfrac{dp_j}{dt} = f_j(p), \ \forall j. \\ p(t=0) > 0. \end{array}\right\}$$

proof sketch: Because of continuity, the above system has a solution. Moreover, because of the initial condition, it can be shown (…) that the solution stays positive, for all t, and remains within the box $B=[\min p(0), \max p(0)]^k$.

To show convergence to a price equilibrium, let us pick *an arbitrary* price equilibrium vector $\bar{p}$ and consider the following potential function

$$V(p) = \frac{1}{2}\sum_{i=1}^{k}(p_i - \bar{p}_i)^2.$$

Corollaries

proof sketch (cont): We have

$$\begin{aligned}\frac{dV}{dt} &= \textstyle\sum_{i=1}^{k} (p_i - \bar{p}_i) \frac{dp_i}{dt} \\ &= \textstyle\sum_{i=1}^{k} (p_i - \bar{p}_i) f_i(p) \\ &= p^{\mathrm{T}} \cdot f(p) - \bar{p}^{\mathrm{T}} \cdot f(p) = -\bar{p}^{\mathrm{T}} \cdot f(p)\end{aligned}$$

Observe that if, for some t_0, $p(t_0)$ is a price equilibrium vector, then

$$f(p(t_0)) = 0 \quad \text{(by lemma 1)}$$

$$\implies p(t) = p(t_0), \forall t \geq t_0$$

On the other hand, as long as $p(t)$ is not an equilibrium, WARP implies that

$$\frac{dV}{dt} = -\bar{p}^{\mathrm{T}} \cdot f(p(t)) < 0$$

implying that the L2 distance from $\bar{p}$ is monotonically decreasing for all t.

Corollaries

proof sketch (cont):

On the other hand, as long as $p(t)$ is not an equilibrium, WARP implies that

$$\frac{dV}{dt} = -\bar{p}^{\mathrm{T}} \cdot f(p(t)) < 0$$

implying that the L2 distance from $\bar{p}$ is monotonically decreasing for all t.

To show convergence to a price equilibrium vector, assume for a contradiction that the $p(t)$ stays at distance $\geq \epsilon$ from the equilibrium ray for all t.

The continuity of $\frac{dV}{dt}$ and compactness of B can be used to show that in this case the absolute value of $\frac{dV}{dt}$ remains bounded away from 0. This leads to a contradiction since $V(t) \geq 0$. ∎