6.896: Topics in Algorithmic Game Theory

Lecture 8

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2 point Exercise

5. NASH ➔ BROUWER (cont.):

- Final Point:

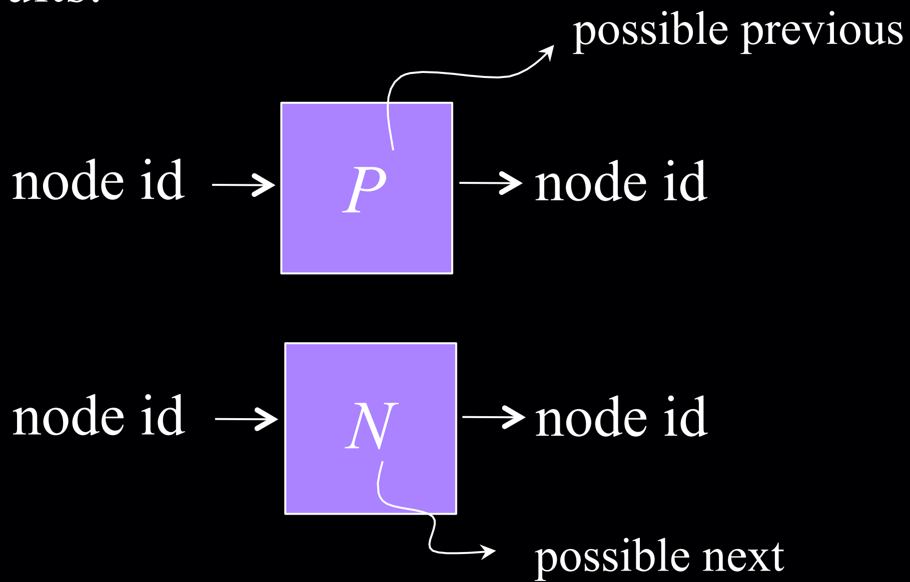
We defined BROUWER for functions in the hypercube. But Nash's function is defined on the product of simplices. Hence, to properly reduce NASH to BROUWER we first embed the product of simplices in a hypercube, then extend Nash's function to points outside the product of simplices in a way that does not introduce approximate fixed points that do not correspond to approximate fixed points of Nash's function.

Last Time…

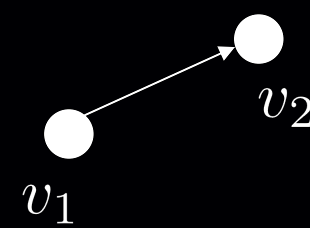
The PPAD Class [Papadimitriou '94]

"A directed graph with an unbalanced node (indegree ≠ outdegree) must have another unbalanced node"

Suppose that an exponentially large graph with vertex set $\{0,1\}^n$ is defined by two circuits:



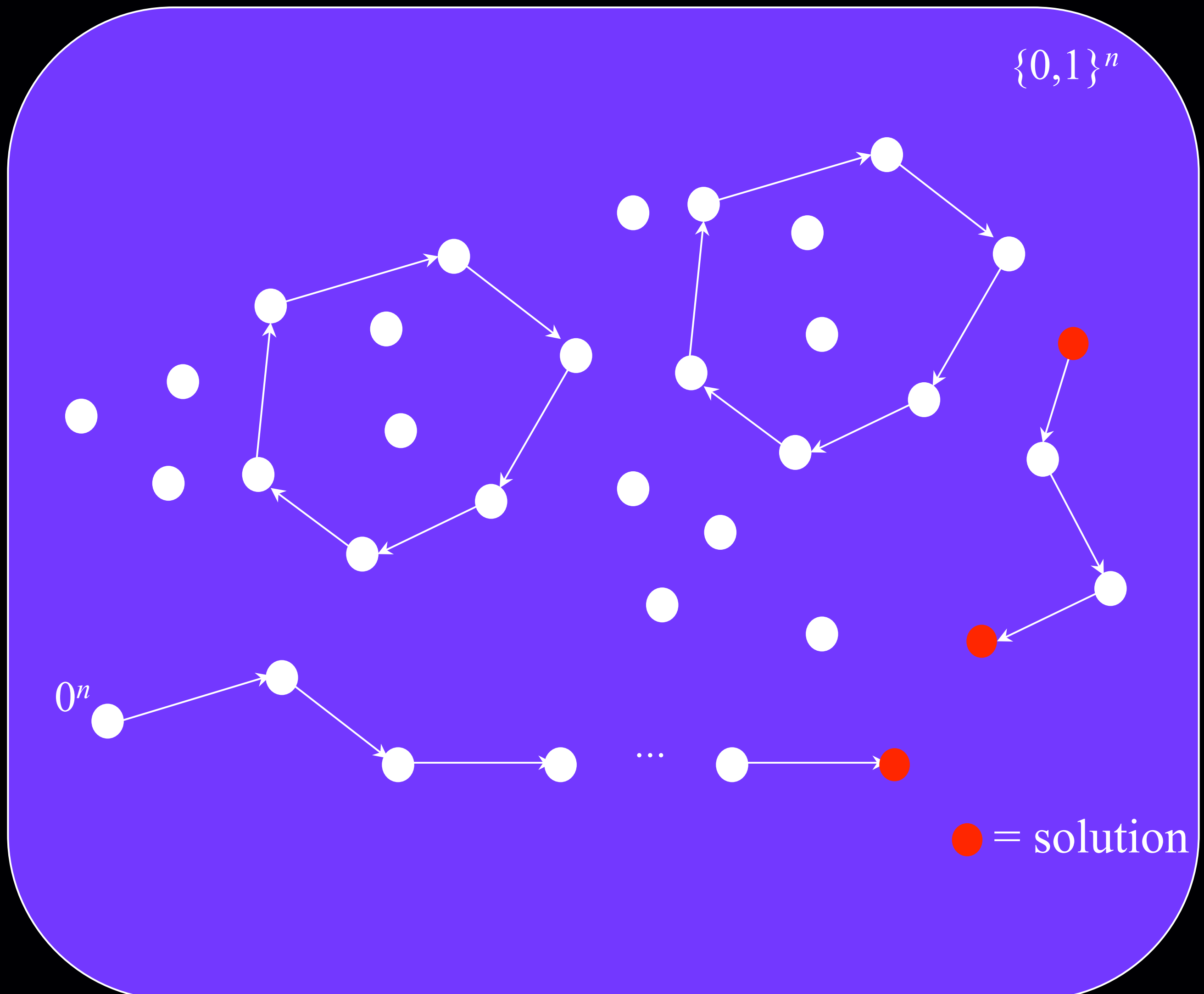
$$P(v_2) = v_1 \wedge N(v_1) = v_2$$

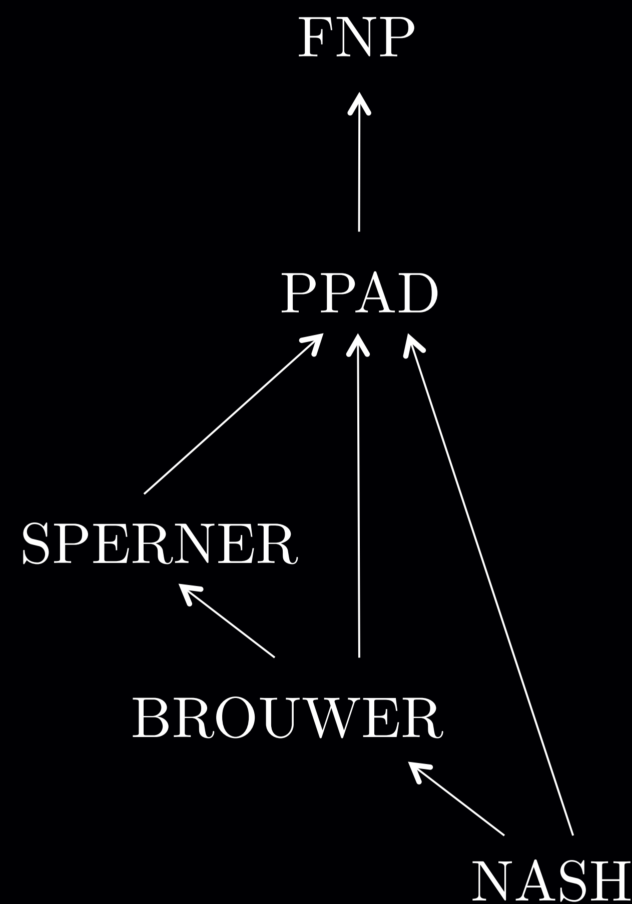


END OF THE LINE: Given P and N: If 0^n is an unbalanced node, find another unbalanced node. Otherwise say "yes".

PPAD = *{ Search problems in FNP reducible to END OF THE LINE}*

The Directed Graph




FNP
PPAD
SPERNER
BROUWER
NASH

Other Combinatorial Arguments of Existence

four arguments of existence

"If a directed graph has an unbalanced node it must have another."

PPAD

"If a graph has a node of odd degree, then it must have another."

PPA

"Every directed acyclic graph must have a sink."

PLS

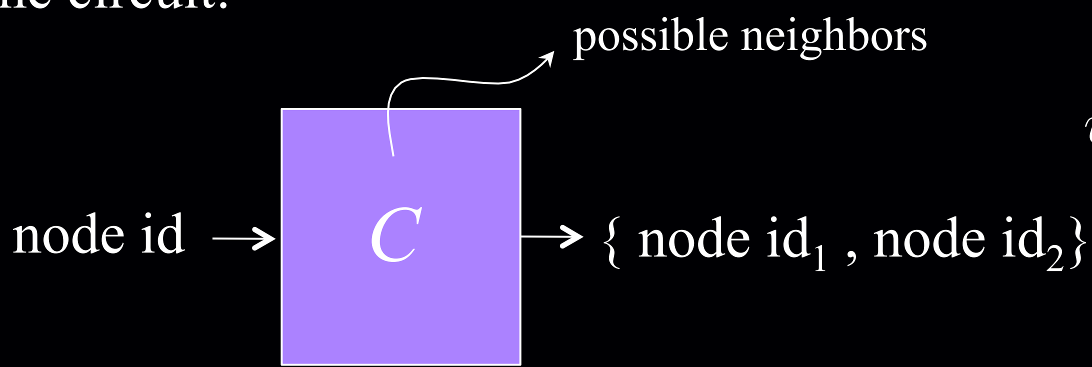
"If a function maps n elements to n-1 elements, then there is a collision."

PPP

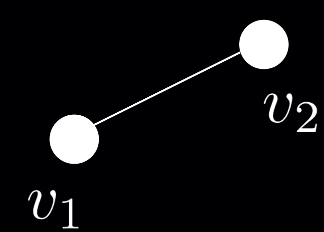
The Class PPA [Papadimitriou '94]

"If a graph has a node of odd degree, then it must have another."

Suppose that an exponentially large graph with vertex set $\{0,1\}^n$ is defined by one circuit:



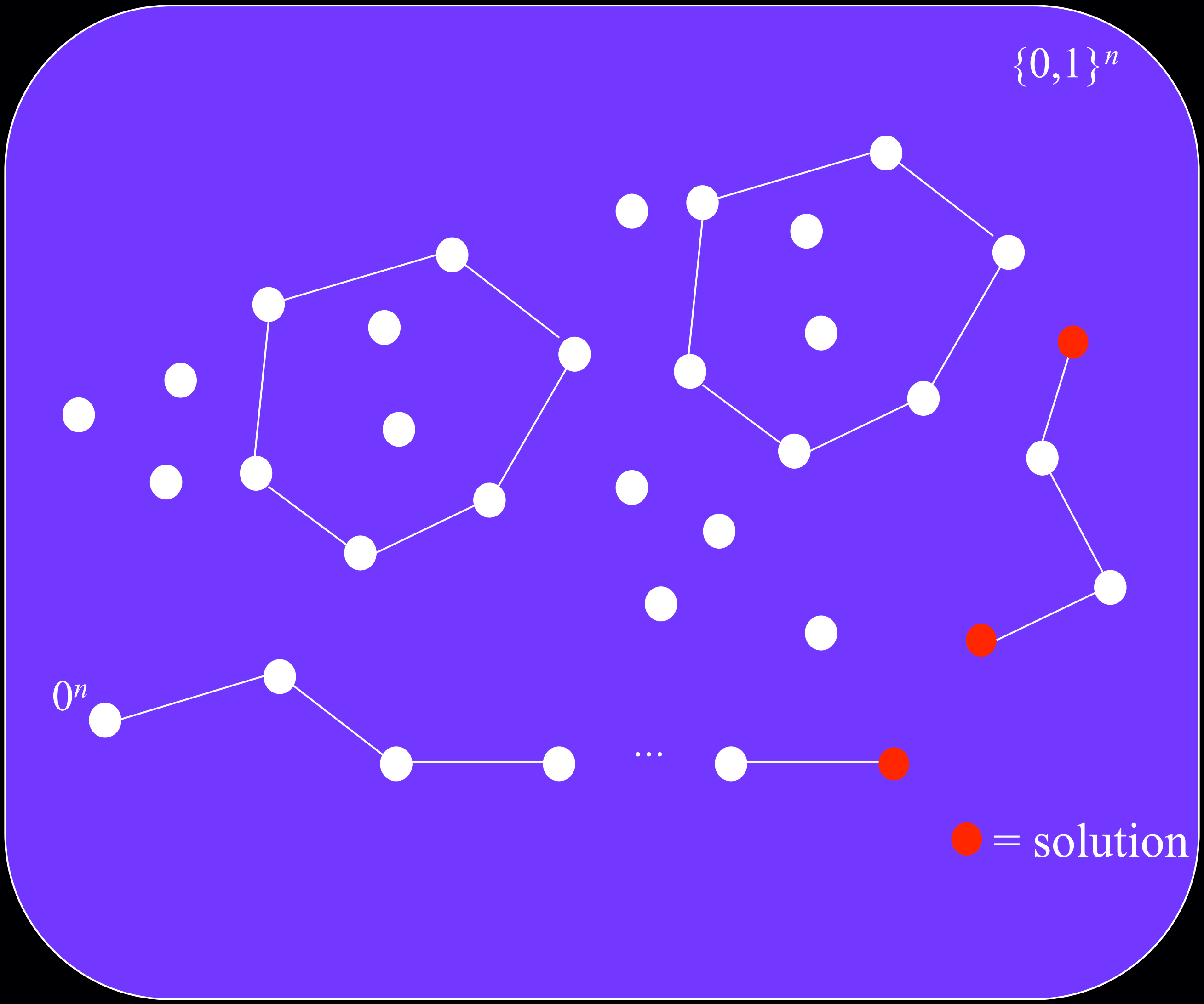
$$v_1 \in C(v_2) \quad \wedge \quad v_2 \in C(v_1)$$



ODD DEGREE NODE: Given C: If 0^n has odd degree, find another node with odd degree. Otherwise say "yes".

PPA = $\{$ *Search problems in FNP reducible to* ODD DEGREE NODE $\}$

The Undirected Graph



The Class PLS [JPY '89]

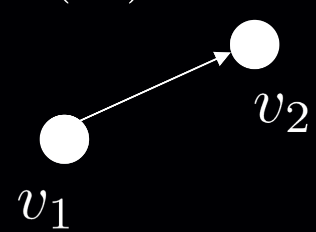
"Every DAG has a sink."

Suppose that a DAG with vertex set $\{0,1\}^n$ is defined by two circuits:

node id $\longrightarrow$ C $\longrightarrow$ {node id$_1$, …, node id$_k$}

$$v_2 \in C(v_1) \quad \wedge \quad F(v_2) > F_{(}v_1)$$

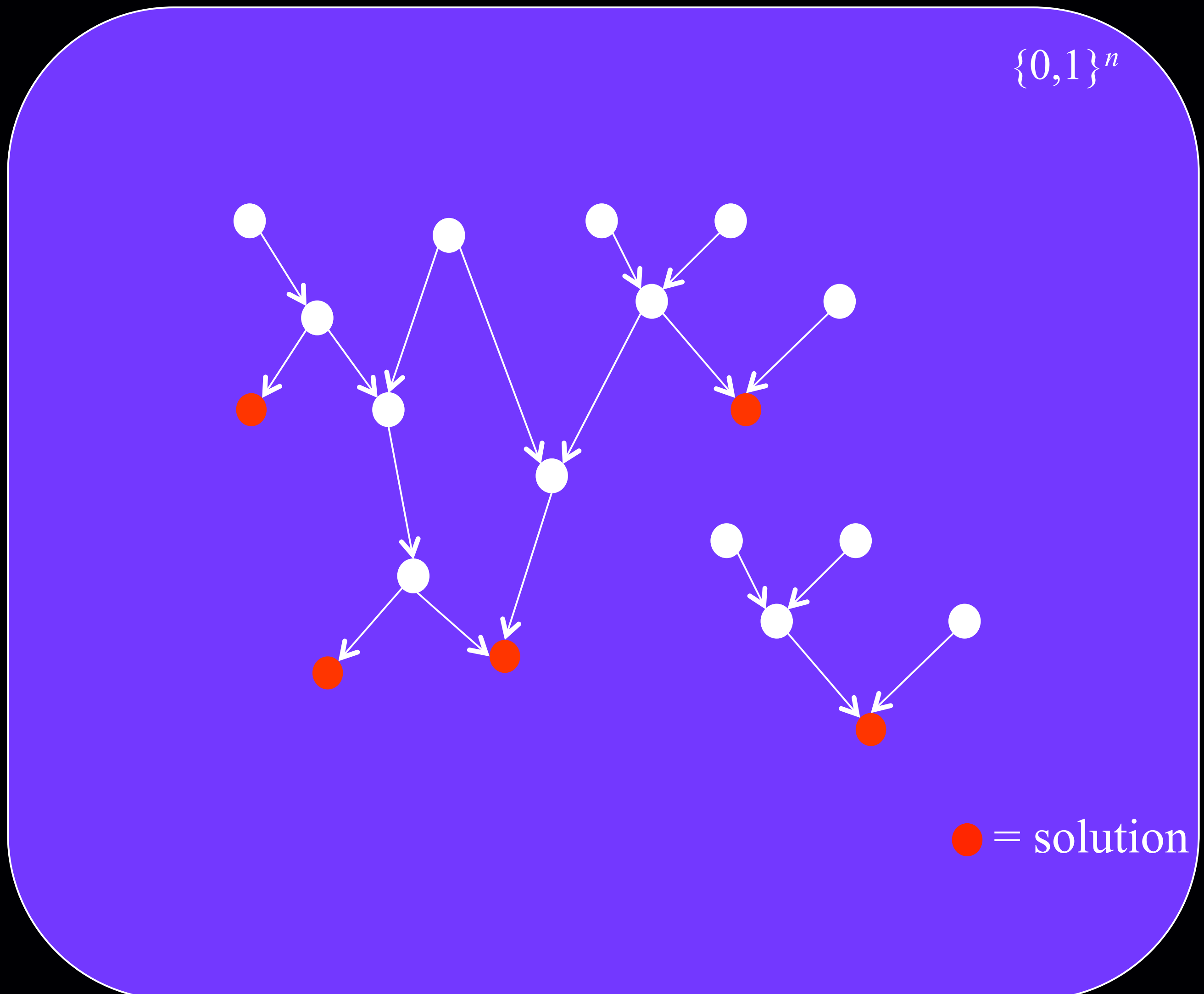
node id $\longrightarrow$ F $\longrightarrow$ $\mathbb{R}$



FIND SINK: Given *C, F*: Find x s.t. $F(x) \geq F(y)$, for all $y \in C(x)$.

PLS = $\{$ *Search problems in FNP reducible to* FIND SINK $\}$

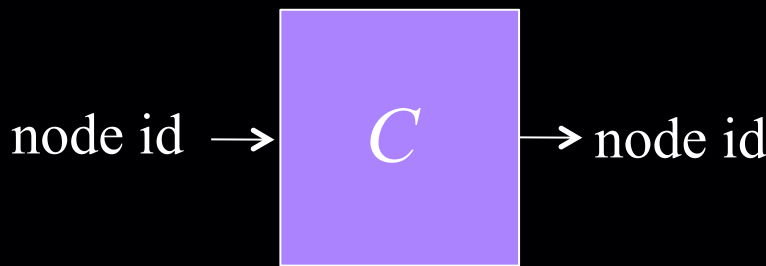
The DAG



The Class PPP [Papadimitriou '94]

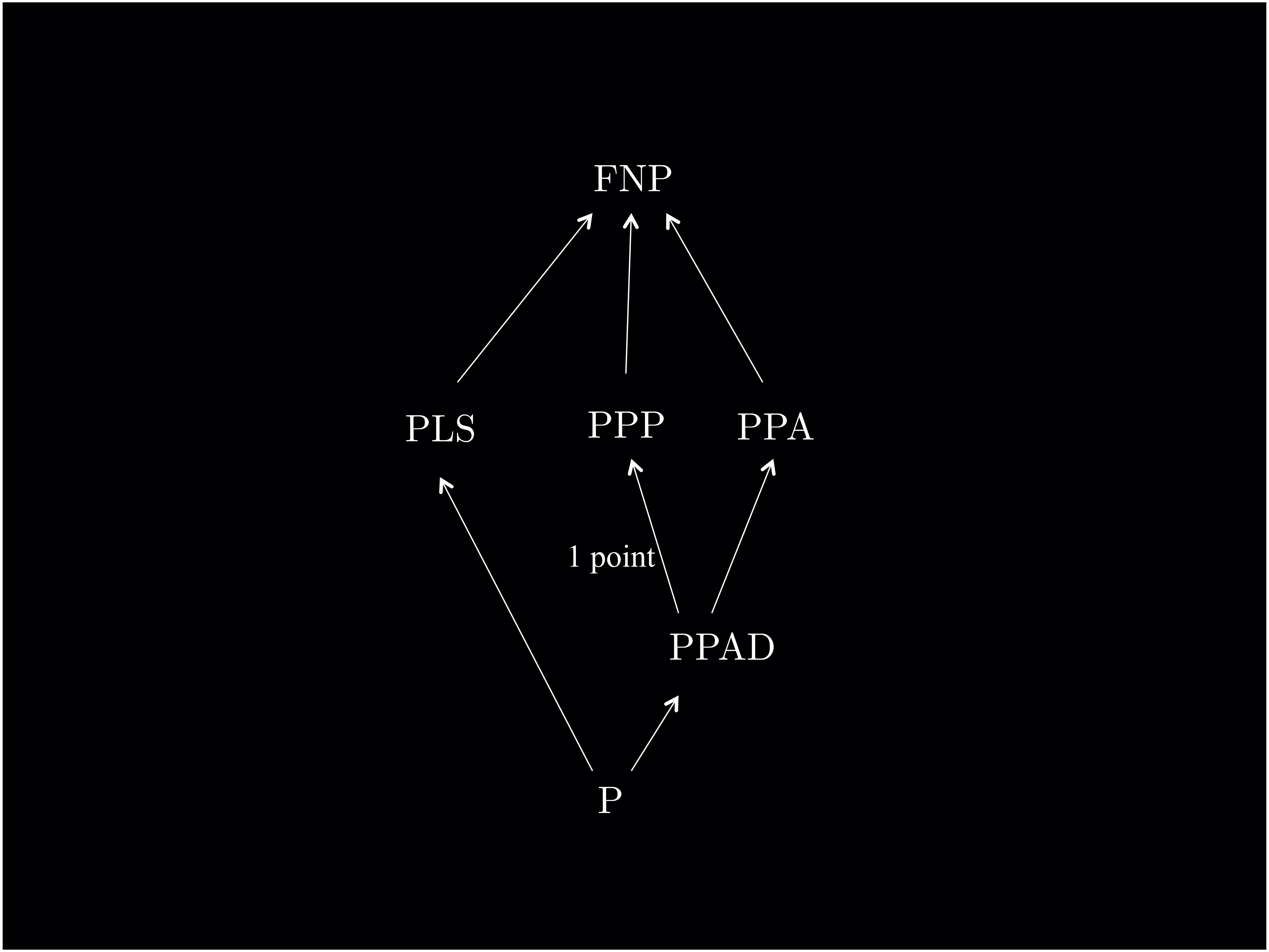
"If a function maps n elements to n-1 elements, then there is a collision."

Suppose that an exponentially large graph with vertex set $\{0,1\}^n$ is defined by one circuit:



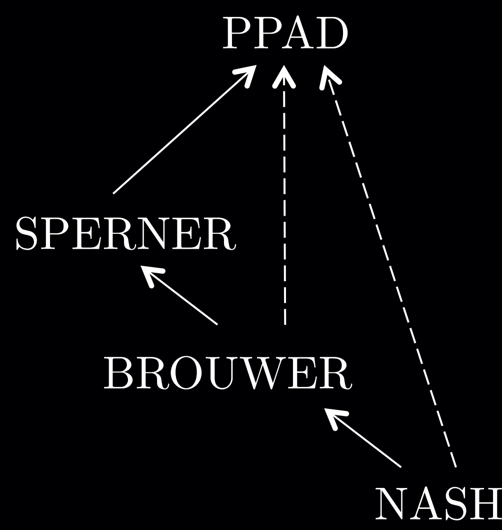
COLLISION: Given C: Find x s.t. $C(x) = 0^n$; or find $x \neq y$ s.t. $C(x)=C(y)$.

PPP = $\{$ *Search problems in FNP reducible to* COLLISION $\}$

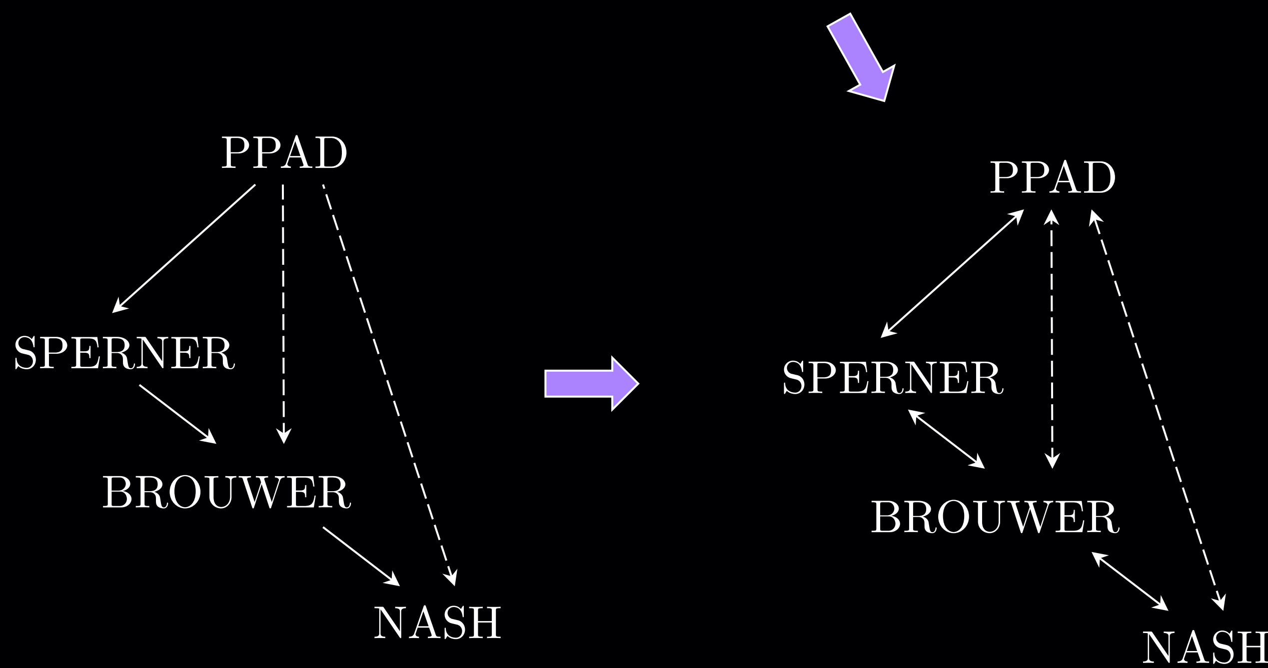

FNP
PLS
PPP
PPA
1 point
PPAD
P

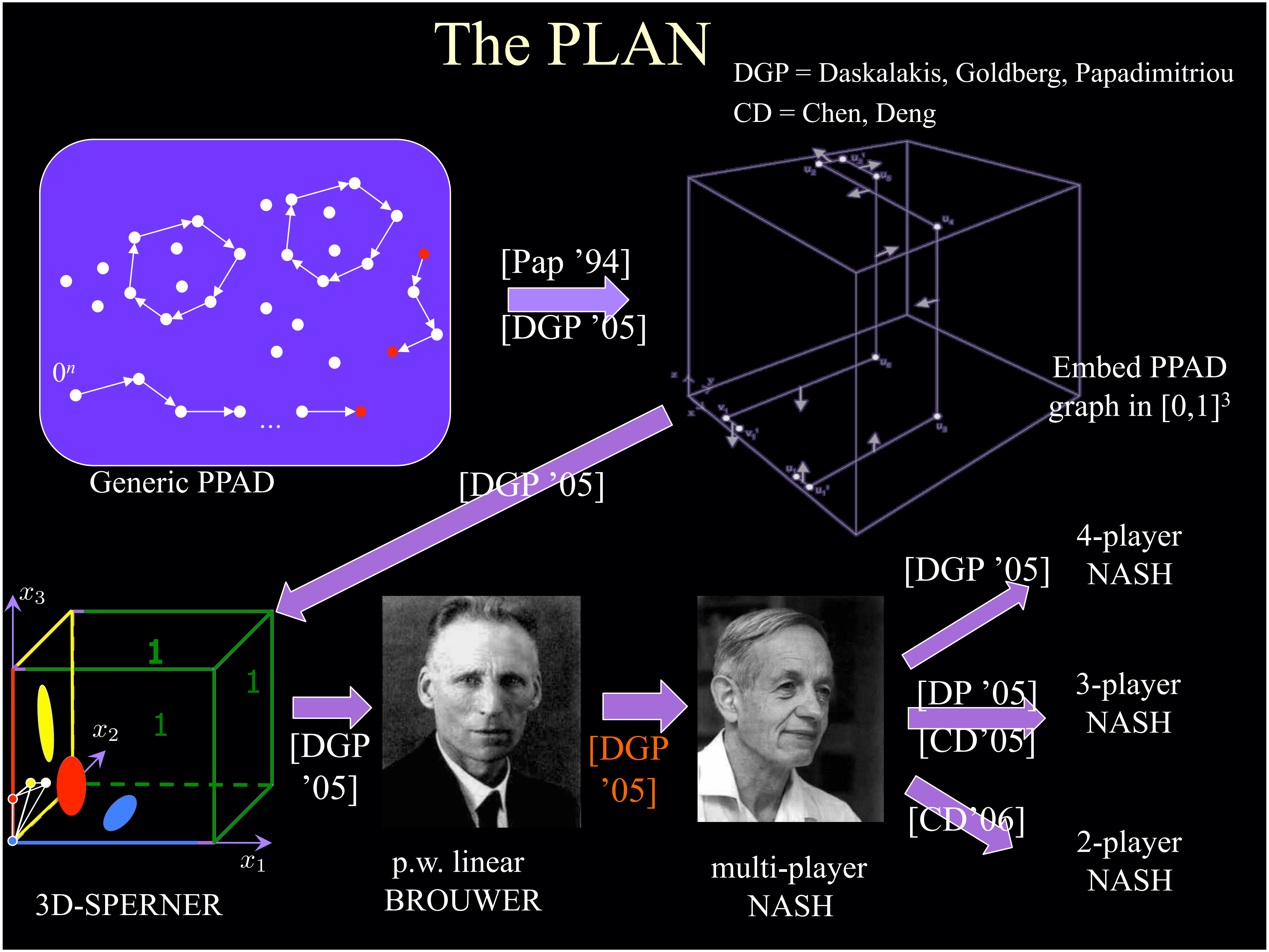
Hardness Results

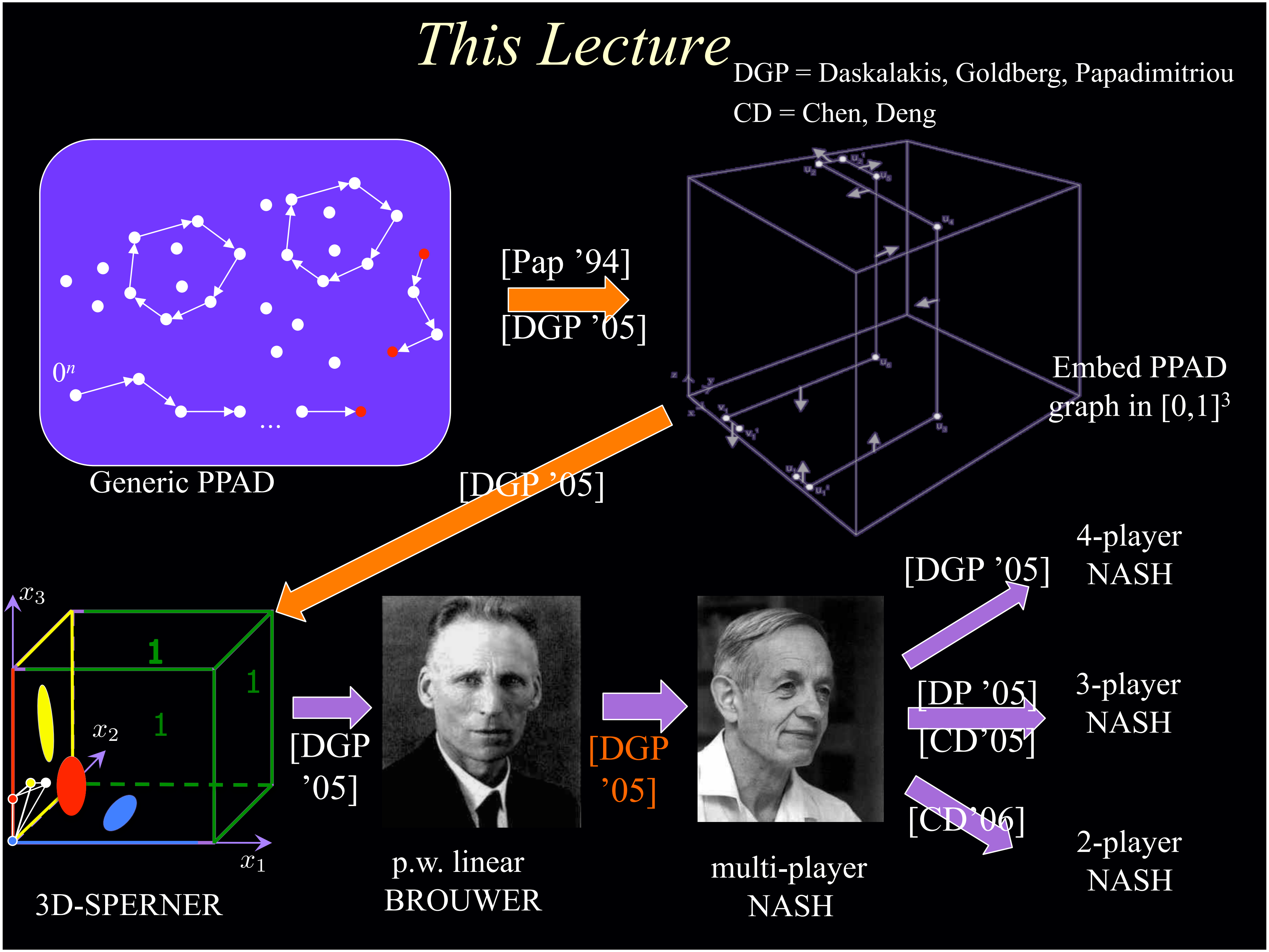
Inclusions we have already established:



Our next goal:




The PLAN
DGP = Daskalakis, Goldberg, Papadimitriou
CD = Chen, Deng
0^n
...
Generic PPAD
[Pap '94]
[DGP '05]
Embed PPAD graph in $[0,1]^3$
[DGP '05]
x_3
x_2
x_1
1
1
1
3D-SPERNER
[DGP '05]
p.w. linear BROUWER
[DGP '05]
multi-player NASH
[DGP '05]
4-player NASH
[DP '05]
[CD'05]
3-player NASH
[CD'06]
2-player NASH


This Lecture
DGP = Daskalakis, Goldberg, Papadimitriou
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x3
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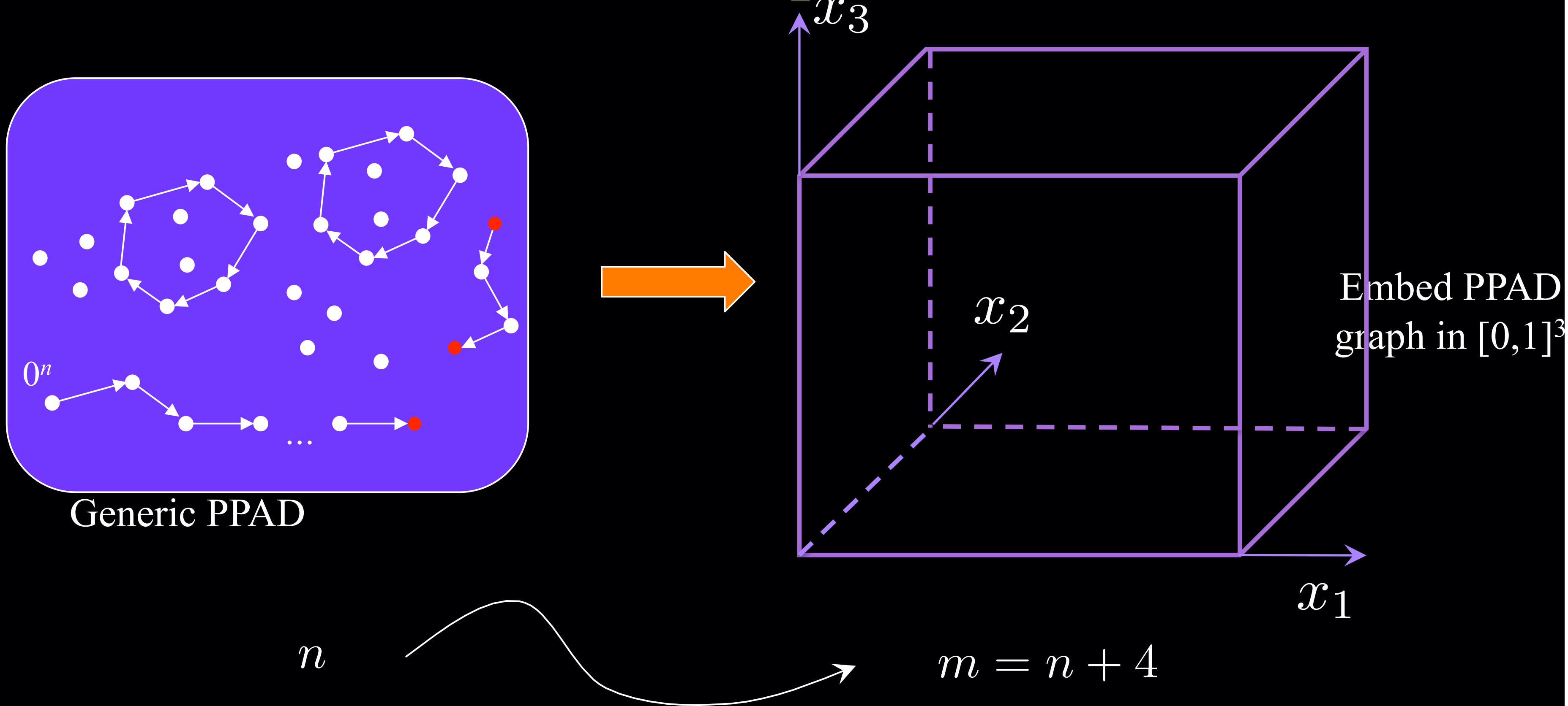
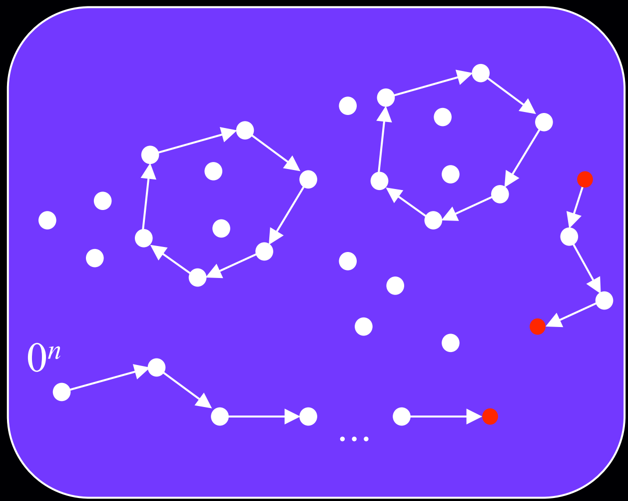
[DGP '05]
multi-player NASH
[DGP '05]
4-player NASH
[DP '05]
[CD'05]
3-player NASH
[CD'06]
2-player NASH

First Step

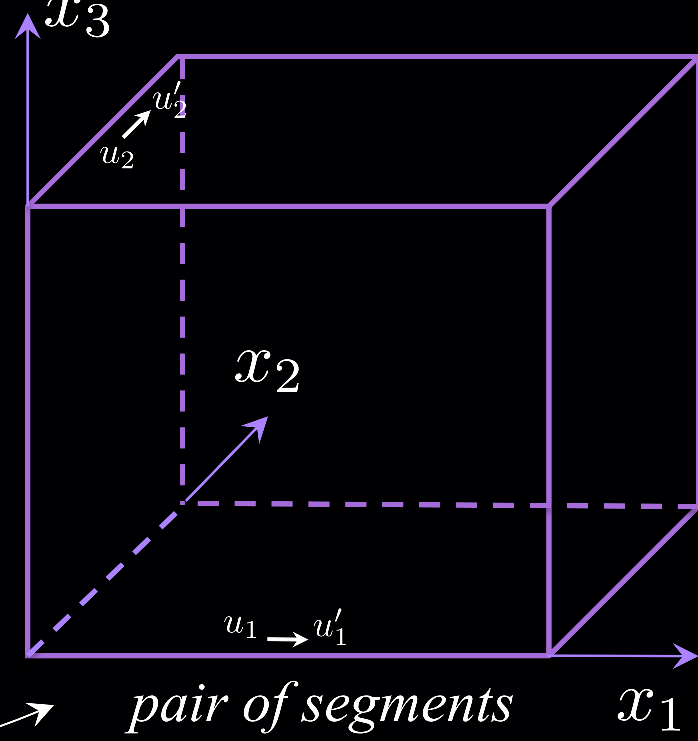


*our goal is to identify a piecewise linear, single dimensional subset of the cube, corresponding to the PPAD graph; we call this subset **L***

Non-Isolated Nodes map to pairs of segments



Generic PPAD



Non-Isolated Node

pair of segments

$$u \in \{0,1\}^n$$

$$u_1 = (8\langle u \rangle + 2, 3, 3) \qquad u'_1 = (8\langle u \rangle + 6, 3, 3)$$

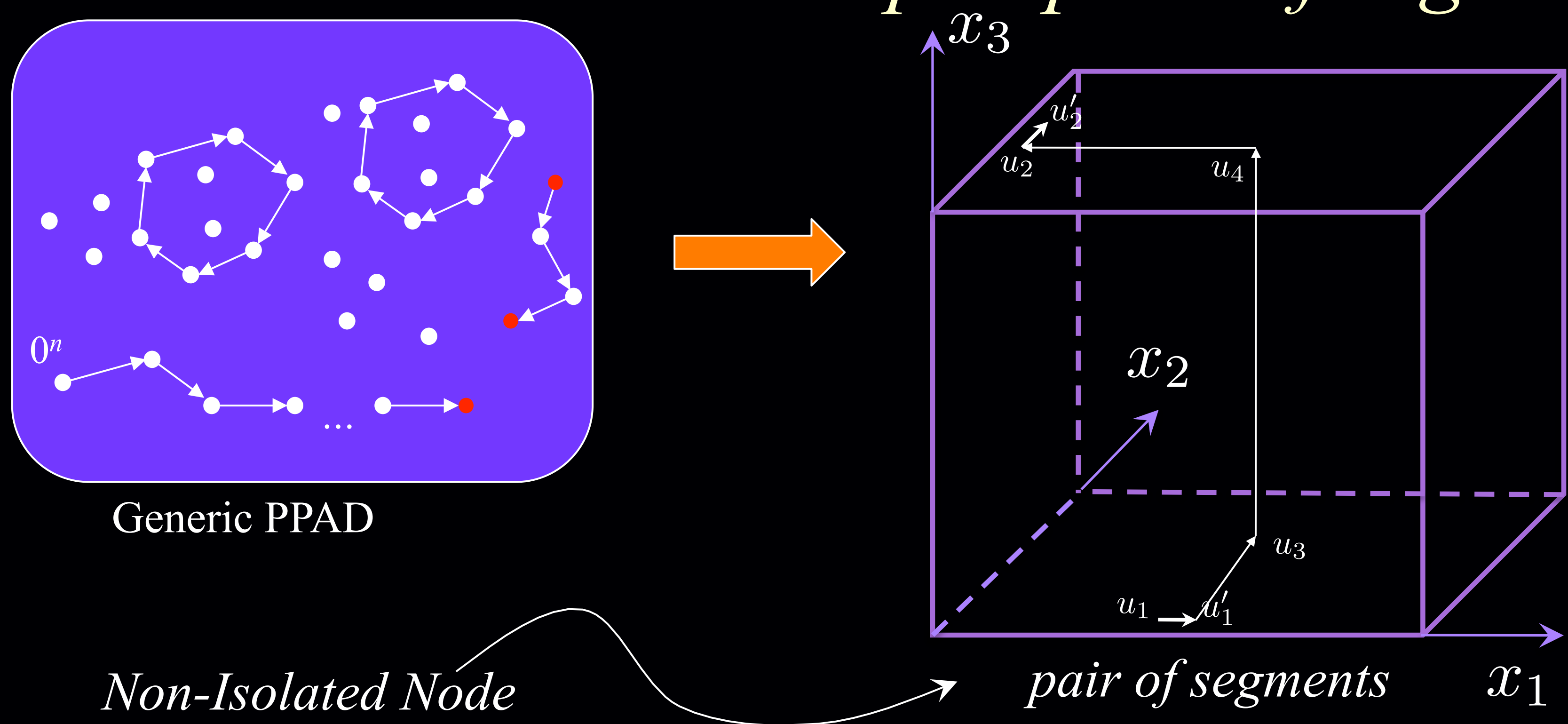
main

$$u_2 = (3, 8\langle u \rangle + 6, 2^m - 3)$$

auxiliary

$$u'_2 = (3, 8\langle u \rangle + 10, 2^m - 3)$$

Non-Isolated Nodes map to pairs of segments

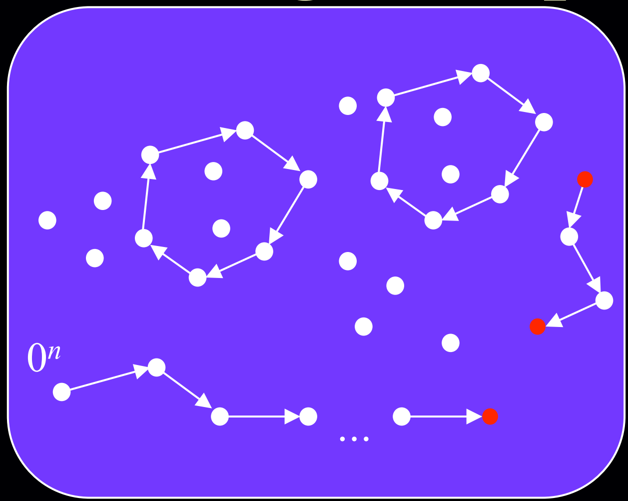


also, add an orthonormal path connecting the end of main segment and beginning of auxiliary segment

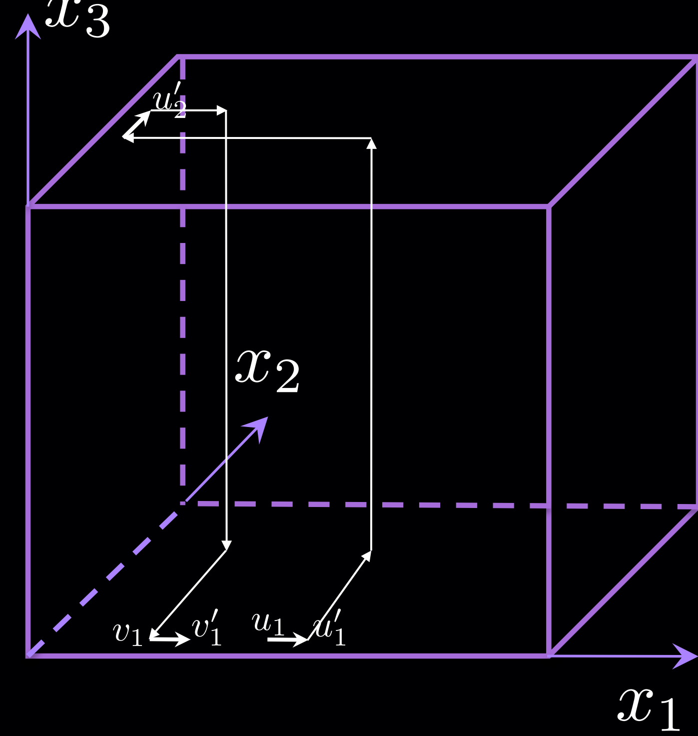
breakpoints used:

$$u_3 = (8\langle u\rangle + 6, 8\langle u\rangle + 6, 3)$$
$$u_4 = (8\langle u\rangle + 6, 8\langle u\rangle + 6, 2^m - 3)$$

Edges map to orthonormal paths



Generic PPAD



Edge between
u *and* v

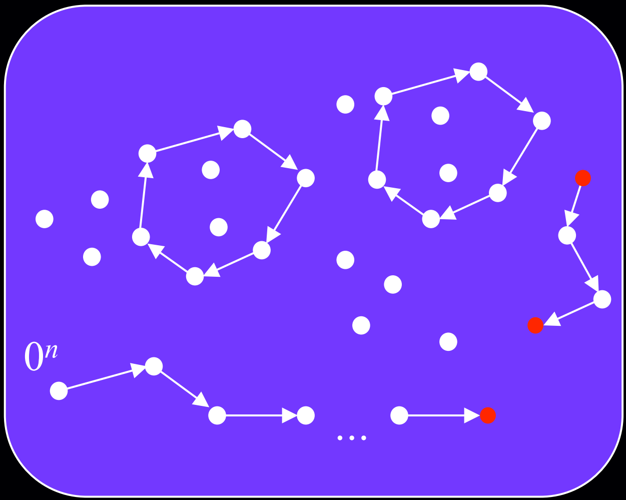
orthonormal path connecting the end of the auxiliary segment of u with beginning of main segment of v

breakpoints used:

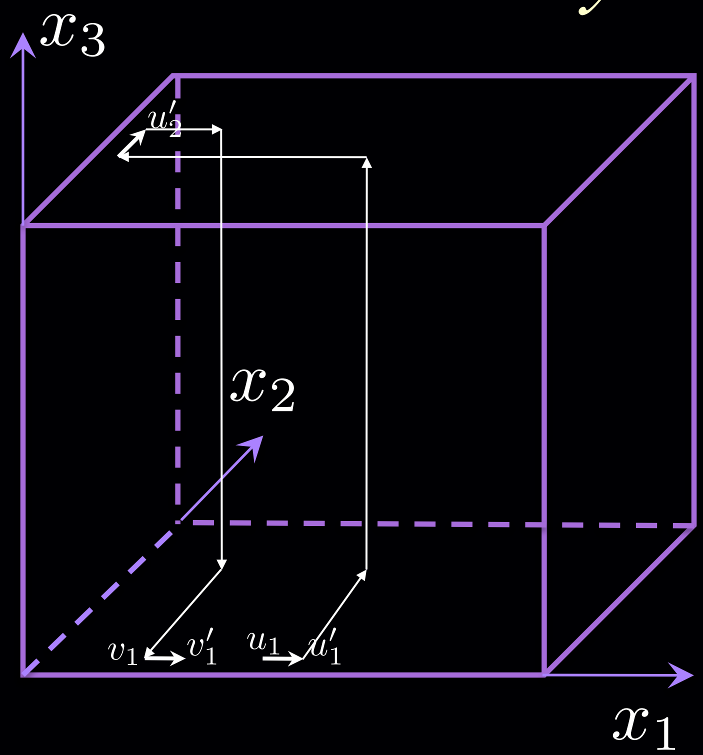
$$(8\langle v\rangle + 2, 8\langle u\rangle + 10, 2^m - 3)$$

$$(8\langle v\rangle + 2, 8\langle u\rangle + 10, 3)$$

Exceptionally 0^n is closer to the boundary…



Generic PPAD



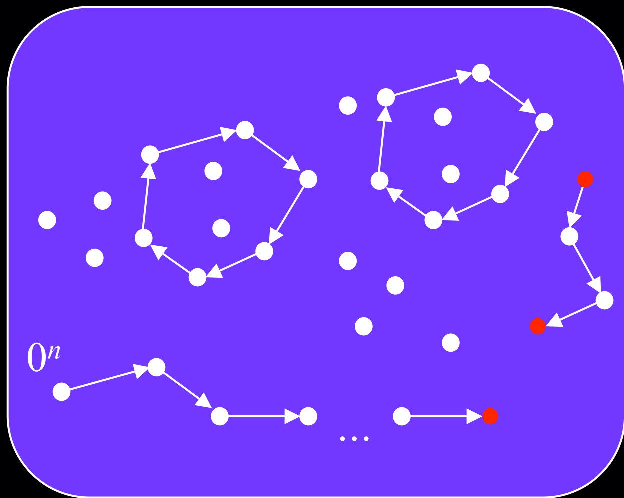
$$0_1 = (2, 2, 2)$$

$$0_1' = (6, 2, 2)$$

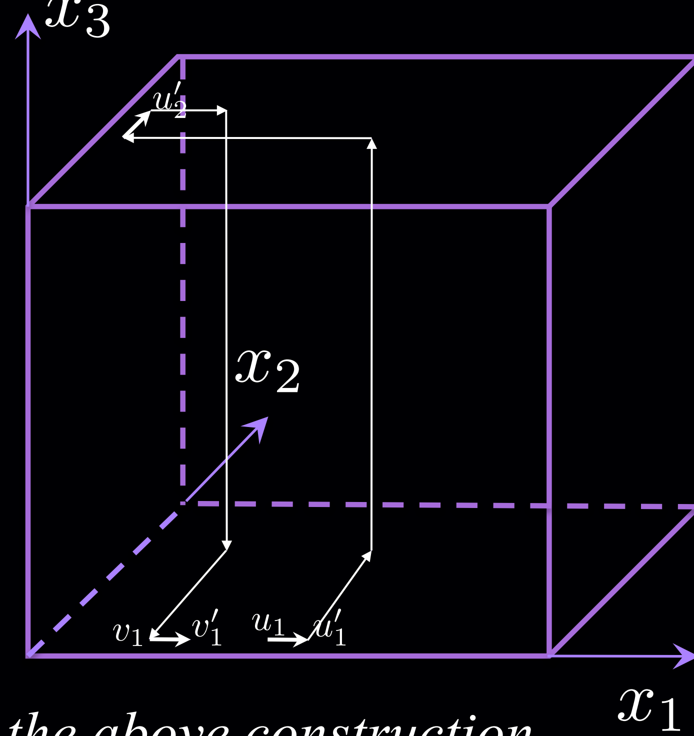
$$0_3 = (6, 6, 2)$$

This is not necessary for the embedding of the PPAD graph, but will be useful later in the definition of the Sperner instance…

Finishing the Embedding



Generic PPAD



Call L the orthonormal line defined by the above construction.

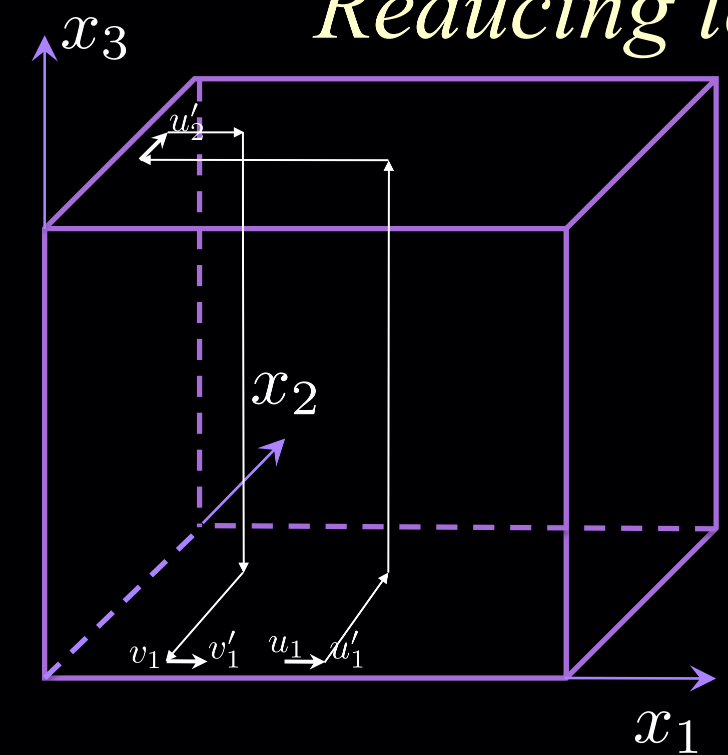
Claim 1: Two points p, p' of L are closer than $3 \cdot 2^{-m}$ in Euclidean distance only if they are connected by a part of L that has length $8 \cdot 2^{-m}$ or less.

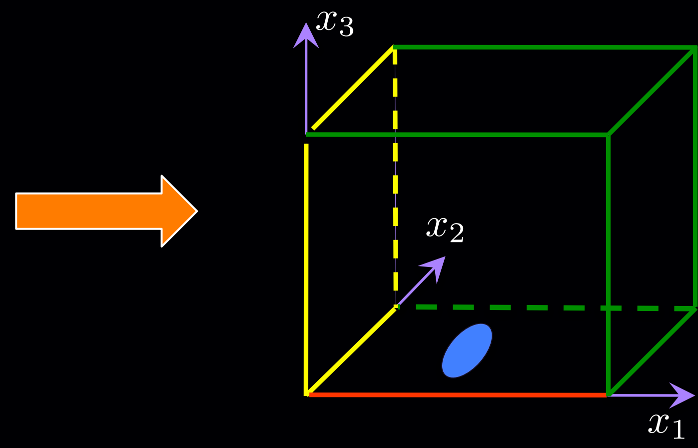
Claim 2: Given the circuits P, N of the END OF THE LINE instance, and a point x in the cube, we can decide in polynomial time if x belongs to L.

Claim 3: u is a sink in PPAD graph $\Leftrightarrow$ L is disconnected at u_2'

u is a source in PPAD graph $\Leftrightarrow$ L is disconnected at u_1

Reducing to 3-d Sperner

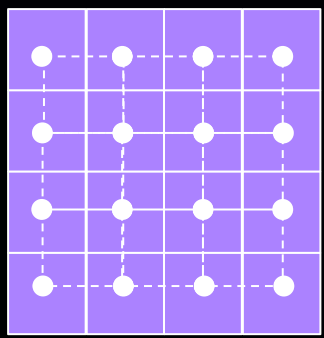




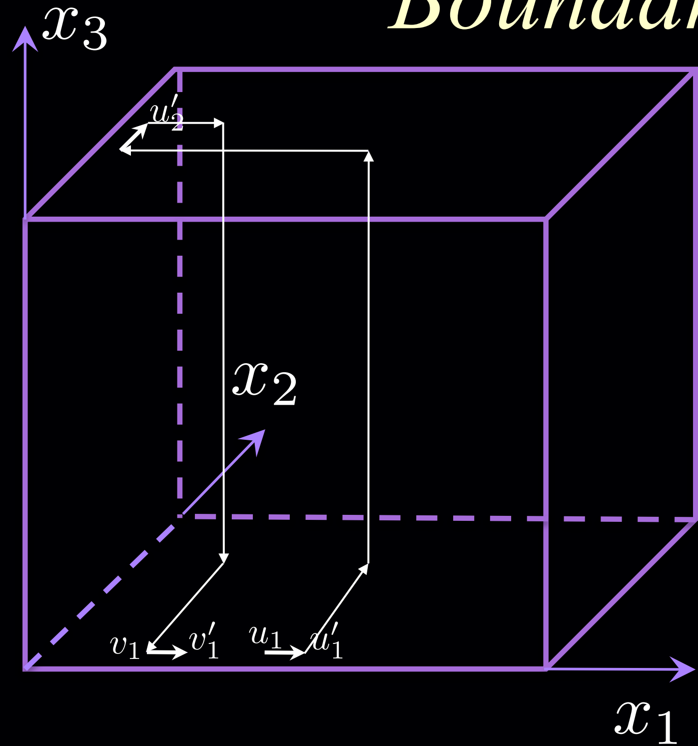
3-d Sperner

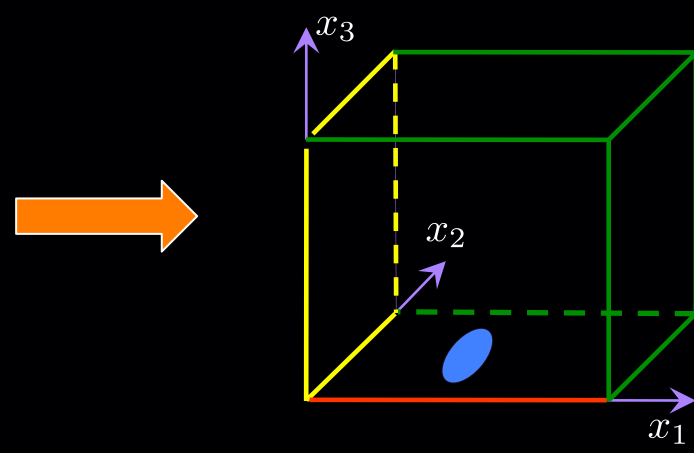
Instead of coloring vertices of the triangulation (the points of the cube whose coordinates are integer multiples of 2^{-m}), color the centers of the cubelets; i.e. work with the dual graph.

K_{ijk} : center of cubelet whose least significant corner has coordinates $(i, j, k) \cdot 2^{-m}$



Boundary Coloring



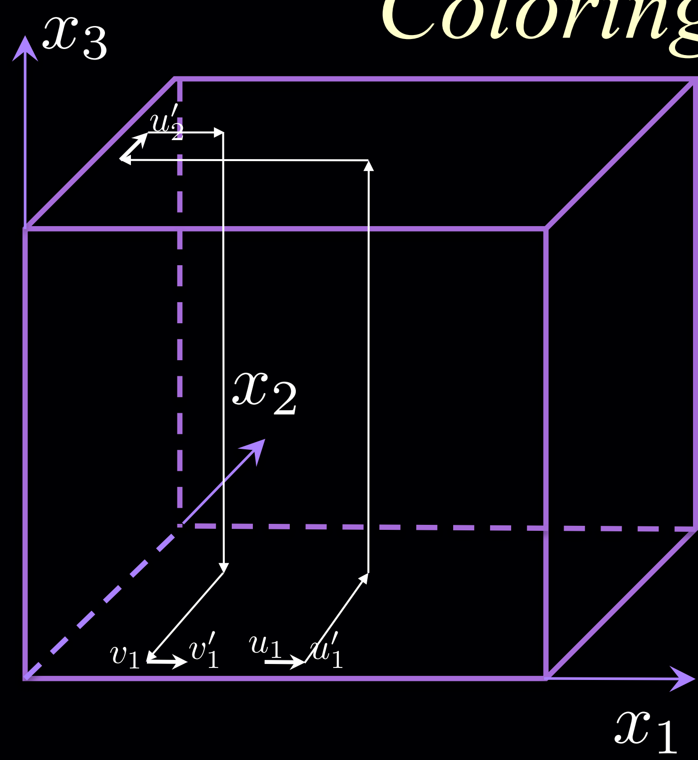


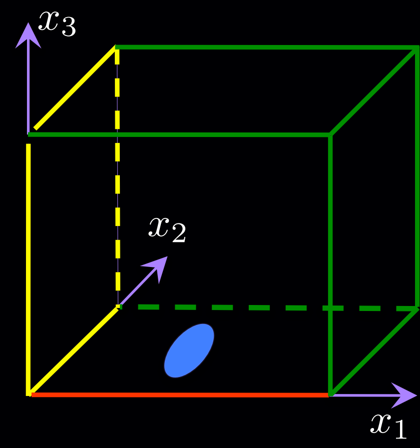
$$\left.\begin{array}{ll} K_{ijk} \leftarrow 0, & \text{if any of } i,\, j,\, k \text{ is } 2^m - 1 \\ K_{ijk} \leftarrow 1, & \text{if } i = 0 \\ K_{ijk} \leftarrow 2, & \text{if } j = 0 \\ K_{ijk} \leftarrow 3, & \text{if } k = 0 \end{array}\right\}$$

legal coloring for the dual graph (on the centers of cubelets)

N.B.: this coloring is not the envelope coloring we used earlier; also color names are permuted

Coloring of the Rest

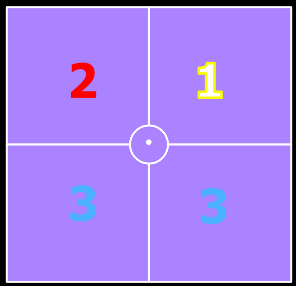




Rest of the coloring: *All cubelets get color **0**, unless they touch line L.*

*The cubelets surrounding line L at any given point are colored with colors **1**, 2, 3 in a way that "protects" the line from touching color 0.*

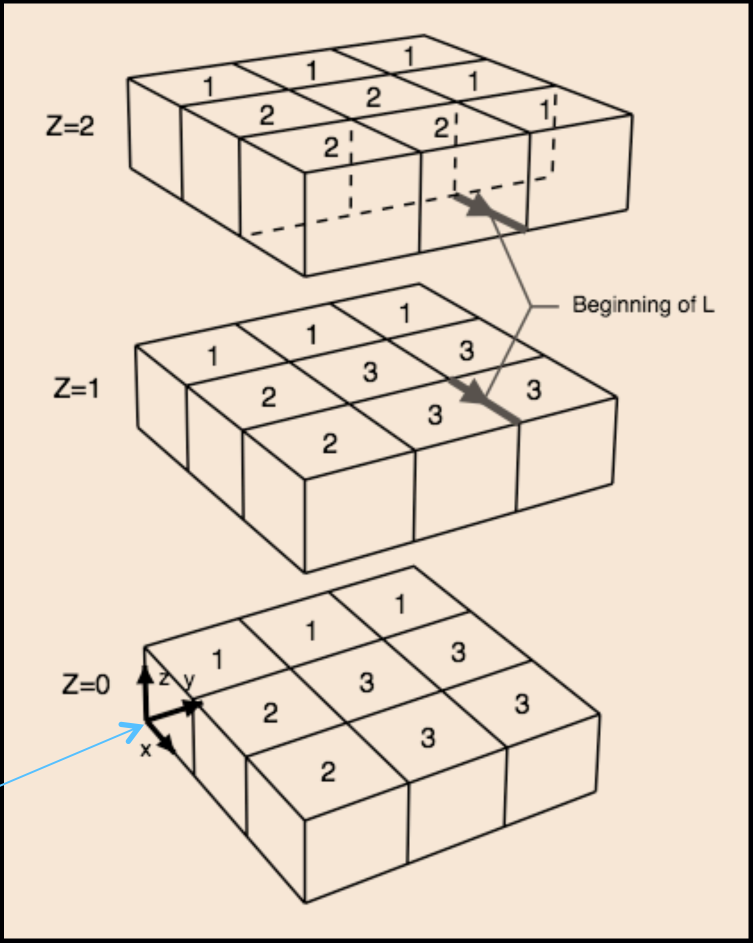
Coloring around L



*colors **1**, 2, 3 are placed in a clockwise arrangement for an observer who is walking on L*

two out of four cubelets are colored 3, one is colored 1 and the other is colored 2

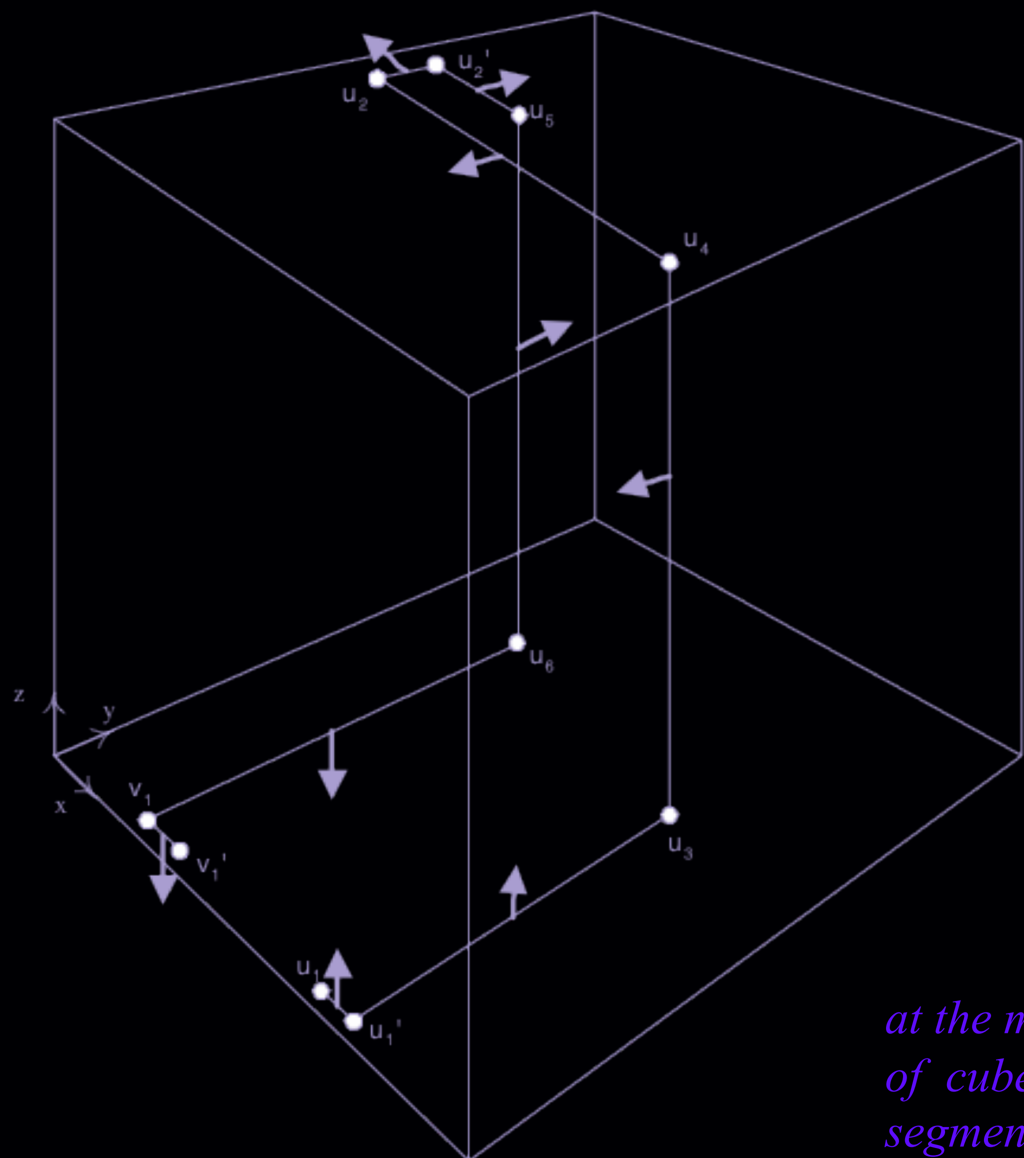
The Beginning of L at 0^n



$(0, 0, 0)$

notice that given the coloring of the cubelets around the beginning of L (on the left), there is no point of the subdivision in the proximity of these cubelets surrounded by all four colors…

Color Twisting



out of the four cubelets around L which two are colored with color 3 ?

- in the figure on the left, the arrow points to the direction in which the two cubelets colored 3 lie

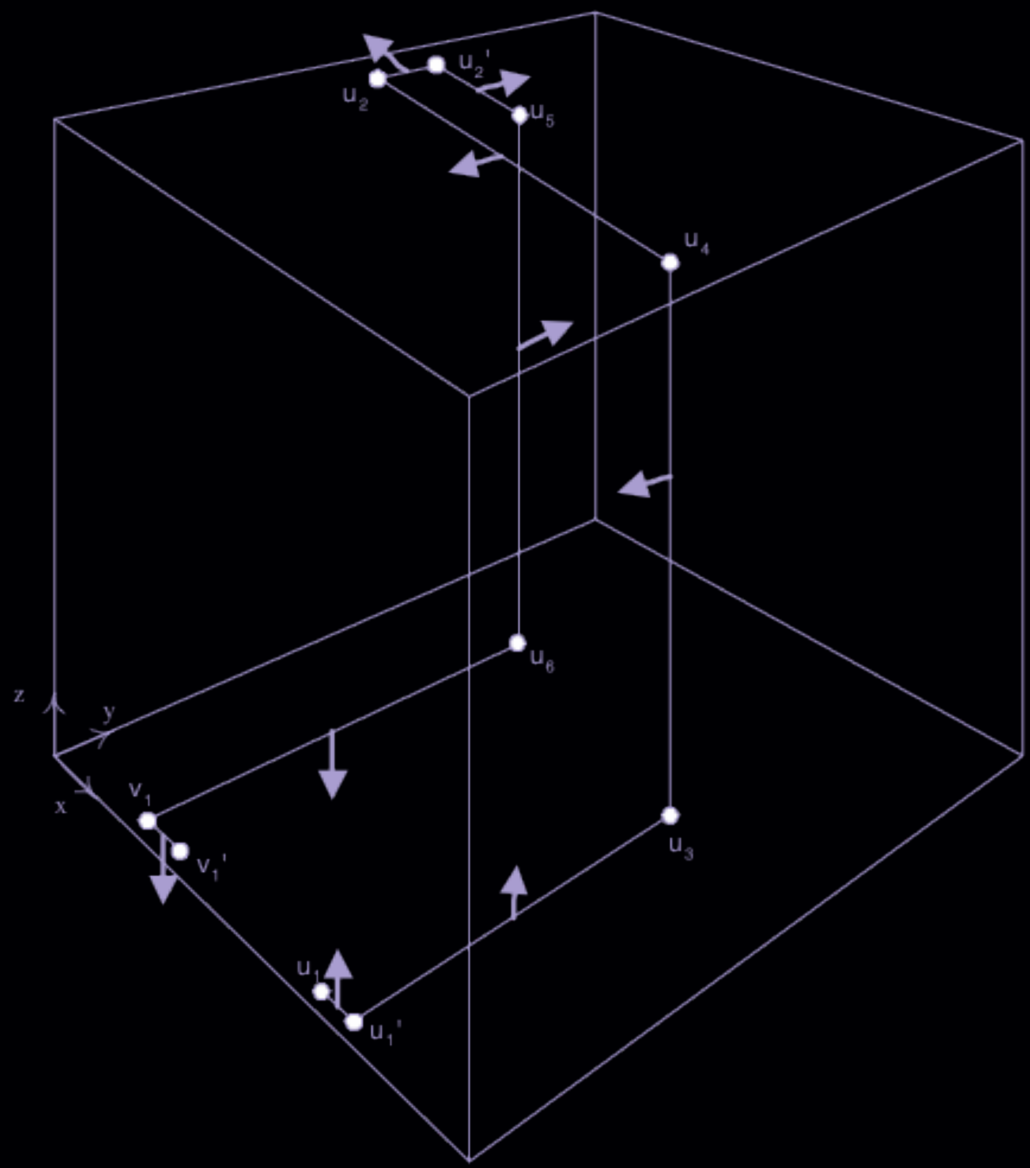
- observe also the way the twists of L affect the location of these cubelets with respect to L

IMPORTANT directionality issue:

the picture on the left shows the evolution of the location of the pair of colored 3 cubelets along the subset of L corresponding to an edge (u, v) of the PPAD graph…

at the main segment corresponding to u the pair of cubelets lies above L, while at the main segment corresponding to v they lie below L

Color Twisting



the flip in the location of the cubelets makes it impossible to locally decide where the colored 3 cubelets should lie!

Claim1: This is W.L.O.G.

to resolve this we assume that all edges (u,v) of the PPAD graph join an odd u (as a binary number) with an even v (as a binary number) or vice versa

for even u's we place the pair of 3-colored cubelets below the main segment of u, while for odd u's we place it above the main segment

convention agrees with coloring around main segment of 0^n

Proof of Claim of Previous Slide

- Duplicate the vertices of the PPAD graph

$$u \Longrightarrow \begin{matrix} u0 \\ \\ u1 \end{matrix}$$

- If node u is non-isolated include an edge from the 0 to the 1 copy

$$\text{non-isolated } u \Longrightarrow \begin{array}{c} u0 \\ \downarrow \\ u1 \end{array}$$

- Edges connect the 1-copy of a node to the 0-copy of its out-neighbor

$$\begin{tikzcd} u \arrow[d] & & u0 \arrow[d] & v0 \arrow[d] \\ v \arrow[r, phantom, "\Longrightarrow"] & {} & u1 \arrow[ur] & v1 \end{tikzcd}$$

Finishing the Reduction

A point in the cube is panchromatic iff it is the corner of some cubelet (i.e. it belongs to the subdivision of mutliples of 2^{-m}), and all colors are present in the cubelets containing this point.

Claim 1: *A point in the cube is panchromatic in the described coloring iff it is:*

- an endpoint u_2' of a sink vertex u of the PPAD graph, or
- an endpoint u_1 of a source vertex $u \neq 0^n$ of the PPAD graph.

Claim 2: *Given the description P, N of the PPAD graph, there is a polynomial-size circuit computing the coloring of every cubelet* K_{ijk}.