Lecture 13

- Finish analysis of Jerrum-Sinclair MC for sampling matchings (see notes of Lecture 12).

- Recall
  - monomer-dimer model: $\pi_\lambda(M) = \frac{1}{Z(\lambda)} \cdot \lambda^{|M|}$
  - sampling time $O\left(\lambda^2 \cdot |E|^2 \cdot |V|\right)$, if starting from maximum matching.

- **Estimating the partition function $Z(\lambda)$**

  → #P-complete for all $\lambda > 0$

  → so want to approximate

  → **Idea:** define ~~smooth~~ sequence of smooth subproblems:

$$Z(\lambda) = \frac{Z(\lambda_r)}{Z(\lambda_{r-1})} \cdot \frac{Z(\lambda_{r-1})}{Z(\lambda_{r-2})} \cdots \frac{Z(\lambda_2)}{Z(\lambda_1)} \cdot Z(\lambda_1) \qquad (*)$$

  where $\lambda_1 < \lambda_2 < \ldots < \lambda_r = \lambda$.

  → Choose $\lambda_1 = \frac{\varepsilon}{|E|}$, $\lambda_i = \left(1 + \frac{1}{n}\right)\lambda_{i-1}$, for $i = 2, \ldots, r-1$

  for appropriate r so that $\lambda_r = \lambda \leq \left(1 + \frac{1}{n}\right)\lambda_{r-1}$.

  $r = O\left(n \cdot \left(\log \lambda + \log |E| + \log \frac{1}{\varepsilon}\right)\right)$, so there are polynomially many factors in $(*)$

  → Now, easy to see that $Z(\lambda_1) = 1 + O(\varepsilon)$.

→ Moreover,

$$1 \leq \frac{Z(\lambda_i)}{Z(\lambda_{i-1})} = \frac{\sum_k m_k \lambda_i^k}{\sum_k m_k \lambda_{i-1}^k} \leq \left(1+\frac{1}{n}\right)^n \leq e.$$

at the same time:

claim: $$\frac{Z(\lambda_i)}{Z(\lambda_{i-1})} = \mathbb{E}_{\pi_{\lambda_{i-1}}}\left[\left(\frac{\lambda_i}{\lambda_{i-1}}\right)^{|M|}\right], \quad \forall\ i \geq 2.$$

Proof:

$$\mathbb{E}_{\pi_{\lambda_{i-1}}}\left[\left(\frac{\lambda_i}{\lambda_{i-1}}\right)^{|M|}\right] = \frac{1}{Z(\lambda_{i-1})} \sum_k m_k \lambda_{i-1}^k \left(\frac{\lambda_i}{\lambda_{i-1}}\right)^k$$

$$= \frac{1}{Z(\lambda_{i-1})} \sum_k m_k \lambda_i^k = \frac{Z(\lambda_i)}{Z(\lambda_{i-1})}. \quad \boxtimes$$

→ So can try to estimate $\frac{Z(\lambda_i)}{Z(\lambda_{i-1})}$ via sampling from $\pi_{\lambda_{i-1}}$;

to be more accurate we sample from $\hat{\pi}_{\lambda_{i-1}}$ s.t. $\|\hat{\pi}_{\lambda_{i-1}} - \pi_{\lambda_{i-1}}\|_{TV}$ is small, where $\hat{\pi}_{\lambda_{i-1}}$ is the distn obtained by running the Markov Chain for t^* steps.

→ Easy to see: $$\left|\mathbb{E}_{\pi_{\lambda_{i-1}}}\left[\left(\frac{\lambda_i}{\lambda_{i-1}}\right)^{|M|}\right] - \mathbb{E}_{\hat{\pi}_{\lambda_{i-1}}}\left[\left(\frac{\lambda_i}{\lambda_{i-1}}\right)^{|M|}\right]\right| \leq (e-1)\cdot \|\pi_{\lambda_{i-1}} - \hat{\pi}_{\lambda_{i-1}}\|_{TV},$$

because $\left(\frac{\lambda_i}{\lambda_{i-1}}\right)^{|M|} \in [1, e]$.

→ Choose $t^* = O\left(\lambda^2 \cdot |E|^2 \cdot |V| \cdot \log\frac{r}{\varepsilon}\right)$ so that $\|\hat{\pi}_{\lambda_{i-1}} - \pi_{\lambda_{i-1}}\|_{TV} \leq \frac{\varepsilon}{r}$

→ Taking $\frac{r^2}{\varepsilon^2}\log\frac{1}{\delta}$ samples from MC, and averaging the resulting $\left(\frac{\lambda_i}{\lambda_{i-1}}\right)^{|M|}$ guarantees: $\underset{\text{estimator}}{\widehat{\left(\frac{Z(\lambda_i)}{Z(\lambda_{i-1})}\right)}} = \frac{Z(\lambda_i)}{Z(\lambda_{i-1})}\left(1 \pm O\left(\frac{\varepsilon}{r}\right)\right)$ with probability $\geq 1-\delta$

Hence overall, if we set:

$$\widehat{Z}(\lambda) = \left(\frac{\widehat{Z(\lambda_r)}}{Z(\lambda_{r-1})}\right)\cdot\left(\frac{\widehat{Z(\lambda_{r-1})}}{Z(\lambda_{r-2})}\right)\cdot\ \ldots\ \cdot\left(\frac{\widehat{Z(\lambda_2)}}{Z(\lambda_1)}\right),$$

We have that:

$$\frac{\widehat{Z}(\lambda)}{Z(\lambda)} = \left(1 \pm O\left(\tfrac{\varepsilon}{r}\right)\right)\left(1 \pm O\left(\tfrac{\varepsilon}{r}\right)\right)\ldots\left(1 \pm O\left(\tfrac{\varepsilon}{r}\right)\right)\cdot\left(1 \pm O(\varepsilon)\right)$$

$$= 1 \pm O(\varepsilon) \qquad , \text{ with probability } \geq 1 - r\cdot\delta.$$

The overall running time is

$$O\left(\underset{\text{number of factors in } (*) \text{ we need to estimate}}{r} \cdot \underbrace{\left(\frac{r^2}{\varepsilon^2}\log\frac{1}{\delta}\right)}_{\text{\#samples we need to take from MC for estimating each factor in } (*)} \cdot \underbrace{\lambda^2 |E|^2 \cdot |V| \cdot \log\frac{r}{\varepsilon}}_{\text{time we need to run MC to obtain a good enough sample}}\right)$$

$$\leq O\left(\frac{\lambda^2}{\varepsilon^2}|V|^4 \cdot |E|^2 \cdot \text{poly}\left(\log\frac{1}{\delta}, \log\frac{1}{\varepsilon}, \log n, \log\lambda, \log|E|\right)\right).$$